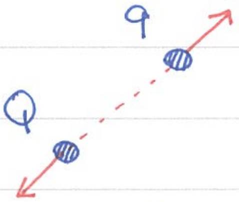
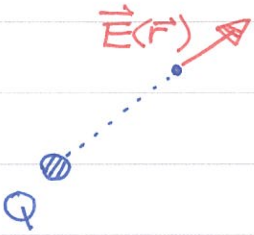
HH0109 Electric Field.

Consider the electric interaction between two static charges $\longrightarrow$ Coulomb's law.

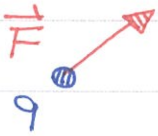


$$\boxed{\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\hat{r}}$$ charge $\leftrightarrow$ charge.

However, we can look at the same problem in different perspective: charge $\leftrightarrow$ field $\leftrightarrow$ charge.



① The charge Q establishes the electric field.



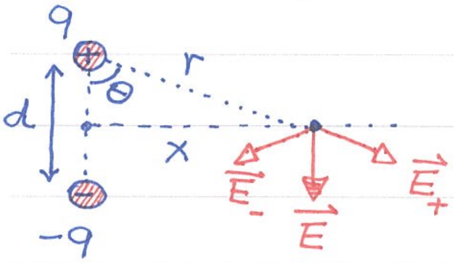
② The charge q feels the force from the local field.

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\hat{r} \longrightarrow \boxed{\vec{E}(r) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r} \ \ \& \ \ \vec{F} = q\vec{E}}$$

The equivalence is trivial when the charges are static. However, charges do move in real life. $\longrightarrow$ It turns out that the field view is more natural and in fact easier ☺

Ø Electric dipole.

The configuration of two equal but opposite charges separated by a distance is called an electric dipole.



$\vec{E} = \vec{E}_+ + \vec{E}_-$ By symmetry, the vertical components survive ☺

$E_+ = E_-$

$$E = E_+\cos\theta + E_-\cos\theta$$

$$= \frac{2q}{4\pi\epsilon_0 r^2}\cos\theta \longleftarrow \cos\theta = \frac{d/2}{\sqrt{x^2+(d/2)^2}}$$

Thus, the total electric field from the charges is

$$\boxed{E = \frac{qd}{4\pi\epsilon_0}\frac{1}{(x^2+d^2/4)^{3/2}}} \propto qd \equiv p \ \text{ dipole moment}$$

We often observe a dipole with distance $x \gg d$. The binomial expansion comes in handy,

$$\boxed{(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \cdots\cdots}$$

Rewrite the electric field in power series,

$$E = \frac{p}{4\pi\epsilon_0 x^3}\left[1+\left(\frac{d}{2x}\right)^2\right]^{-\frac{3}{2}} \approx \frac{p}{4\pi\epsilon_0 x^3}\left[1 - \frac{3}{8}\left(\frac{d}{x}\right)^2 + \cdots\right]$$

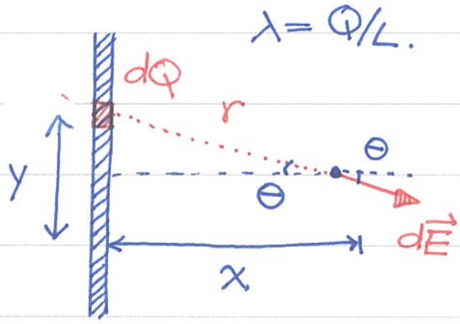
In the long-distance limit, $d/x \ll 1$. We can drop the higher order terms $\longrightarrow$ The electric field of an electric dipole is

$$\boxed{E \approx \frac{1}{4\pi\epsilon_0}\frac{p}{x^3}}$$

only depends on qd together. And, the power law is x^{-3}!

② Uniform line of charge. The linear charge density is λ.

$\lambda = Q/L.$



Thus, the infinitesimal charge $dQ = \lambda\, dy$. By symmetry, we know the vertical components cancel out $\longrightarrow$ only need to keep the horizontal one.

$$\boxed{dE = \frac{1}{4\pi\epsilon_0}\frac{dQ}{r^2}\cdot\cos\theta}$$

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

Express everything in terms of y and write down the integral.

$$E = \int dE = \int_{-\infty}^{+\infty} \frac{\lambda}{4\pi\epsilon_0}\frac{x}{(x^2+y^2)^{\frac{3}{2}}}\,dy = \frac{\lambda x}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{dy}{(y^2+x^2)^{\frac{3}{2}}}$$

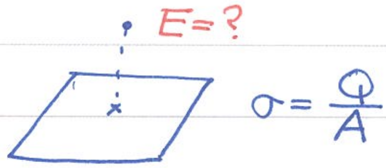
Look up the integral table,

$$\int \frac{dz}{(z^2+a^2)^{\frac{3}{2}}} = \frac{z}{a^2\sqrt{z^2+a^2}} \longrightarrow E = \frac{\lambda x}{4\pi\epsilon_0}\cdot\frac{1}{x^2}\left.\frac{y}{\sqrt{y^2+x^2}}\right|_{-\infty}^{+\infty}$$

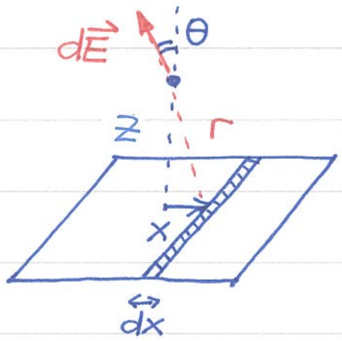
$$\longrightarrow E = \frac{\lambda}{4\pi\epsilon_0}\frac{1}{x}\left(1-(-1)\right) \qquad \boxed{E = \frac{\lambda}{2\pi\epsilon_0 x}} \propto \frac{1}{x}$$

The electric field decays in the power law x^{-1}.

⑪ Uniform surface of charge. Now we turn to a uniformly charged surface.



The surface can be viewed as a collection of infinite uniform wires (just calculated before). The linear charge density λ is related to the surface density



$$\boxed{\lambda = \sigma\, dx}$$ ← Think it over ?

Again, the horizontal component vanishes due to symmetry argument.

$$dE = \frac{\lambda}{2\pi\epsilon_0 r} \cdot \cos\theta = \frac{\sigma dx}{2\pi\epsilon_0 r} \cdot \frac{z}{r} = \frac{\sigma z}{2\pi\epsilon_0} \frac{1}{x^2+z^2} dx$$

The total electric field can be obtained by integration,

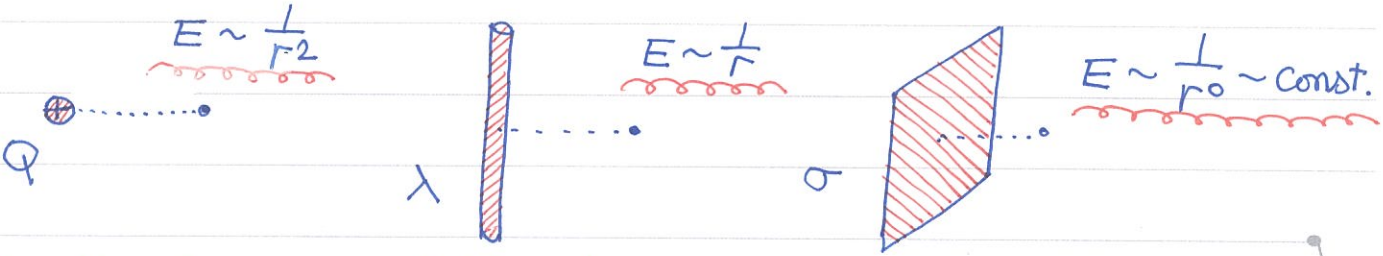
$$E = \int dE = \int_{-\infty}^{+\infty} \frac{\sigma z}{2\pi\epsilon_0} \frac{dx}{x^2+z^2}$$ ← change variable $x = z\tan\theta$

$$= \frac{\sigma \cancel{z}}{2\pi\epsilon_0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cancel{z \sec^2\theta}\, d\theta}{\cancel{z^2 \sec^2\theta}} = \frac{\sigma}{2\pi\epsilon_0} \int_{-\pi/2}^{\pi/2} d\theta = \frac{\sigma}{2\pi\epsilon_0} \cdot \pi$$

$$\rightarrow \boxed{E = \frac{\sigma}{2\epsilon_0}}$$

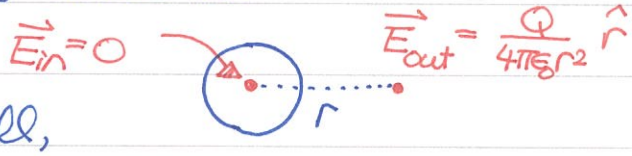
It is quite remarkable that the electric field is constant, independent of the distance to the surface!

It's inspiring to compare the electric fields by different geometries.

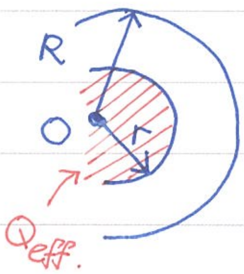


As the dimension of the charge increases, the power law of the electric field changes accordingly.

⊘ Uniform sphere of charge. Finally, we come to the case with uniform charge density $\rho = Q/V$. Need the important "Shell Theorem" discussed in the previous notes HH0095.



For a uniformly charged shell,
the electric field is zero inside the shell and the field outside is the same as if placing the total charge at the center of the sphere.



Consider a uniformly charged sphere of radius R and total charge Q. The electric field outside $\vec{E}_{out}$ is described by the Coulomb's law ← simple. We focus on the spatial dependence of $E(r)$ when $r \le R$ ← more interesting.

$$\boxed{\rho = \frac{Q}{V} = \frac{Q}{\frac{4}{3}\pi R^3}}$$

$$\boxed{E(r) = \frac{1}{4\pi\epsilon_0}\frac{Q_{eff}}{r^2}} \quad \text{where } Q_{eff} = \rho \cdot \frac{4}{3}\pi r^3$$

$$\rightarrow E(r) = \frac{1}{\cancel{4\pi}\epsilon_0}\cdot\frac{1}{\cancel{r^2}}\cdot \rho\,\frac{\cancel{4}}{3}\cancel{\pi} r^{\cancel{3}} \rightarrow \boxed{E(r) = \frac{\rho}{3\epsilon_0} r}$$

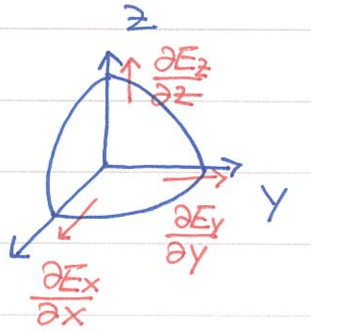
The electric field grows linearly in the presence of uniform charge density ρ. (Compare with Q, λ, σ cases ☺).

⊘ Relation to Maxwell equations. When all charges are static, the Maxwell equations for $\vec{E}$ and $\vec{B}$ fields become decoupled ...

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho/\epsilon_0 \\ \vec{\nabla}\times\vec{E} &= 0 \end{aligned}}$$

→ the electric part ☺

It turns out that the Coulomb's law is nothing but the solution for these two equations in the presence of a point charge ☺



Look at the uniformly charged sphere again and try to solve the Maxwell equation.

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \Longrightarrow \boxed{\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}}$$

ρ is const here.

By spherical symmetry, we expect the derivatives in the divergence are the same $\frac{\partial E_x}{\partial x} = \frac{\partial E_y}{\partial y} = \frac{\partial E_z}{\partial z}$

$$\Longrightarrow 3\frac{\partial E_x}{\partial x} = \frac{\rho}{\epsilon_0}, \quad \frac{\partial E_x}{\partial x} = \frac{\rho}{3\epsilon_0}, \quad E_x = \frac{\rho}{3\epsilon_0}x + \text{const.}$$

Because $\vec{E}=0$ at the center, $E_x(x=0)=0$ and the const is zero. The other two components can be found in a similar fashion and the electric field is

$$\boxed{\vec{E} = (E_x, E_y, E_z) = \left(\frac{\rho x}{3\epsilon_0}, \frac{\rho y}{3\epsilon_0}, \frac{\rho z}{3\epsilon_0}\right) = \frac{\rho}{3\epsilon_0}\vec{r}}$$

The above result is the same as that obtained by the shell theorem. You should have more confidence in Maxwell eqs now ☺ For curious cats, verify that $\vec{\nabla}\times\vec{E}=0$ for the above solution $\vec{E}=\frac{\rho}{3\epsilon_0}\vec{r}$!

2014.0225
堉大東院