HH0110 Gauss' Law

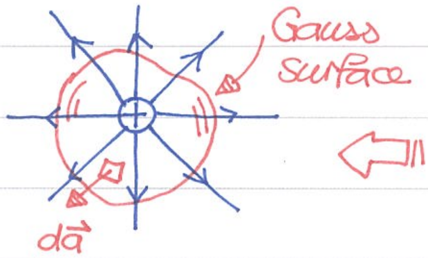
In previous notes, we learned that the Maxwell equations for static charge distribution are:

If ρ = point charge density, the solution turns out to be the Coulomb's law.

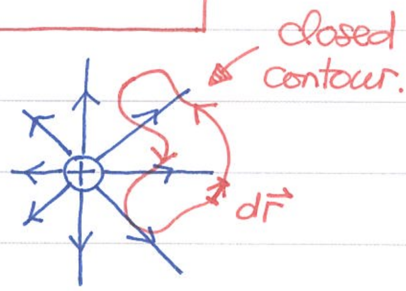
$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \frac{\rho}{\epsilon_0} \\ \vec{\nabla}\times\vec{E} &= 0 \end{aligned}}$$

The above differential equations can be cast into integral form.

$$\boxed{\oint \vec{E}\cdot d\vec{a} = \frac{Q}{\epsilon_0}} \quad \text{Gauss' law} \qquad\qquad \boxed{\oint \vec{E}\cdot d\vec{r} = 0}$$

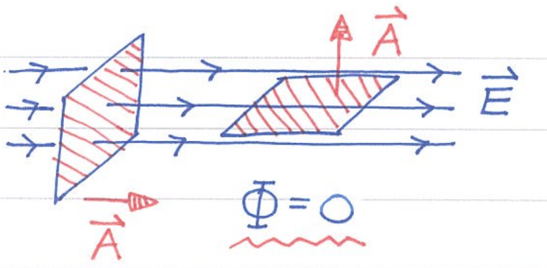


The flux of $\vec{E}$ field through a closed surface is proportional to charges inside!



⊘ Flux of vector field.

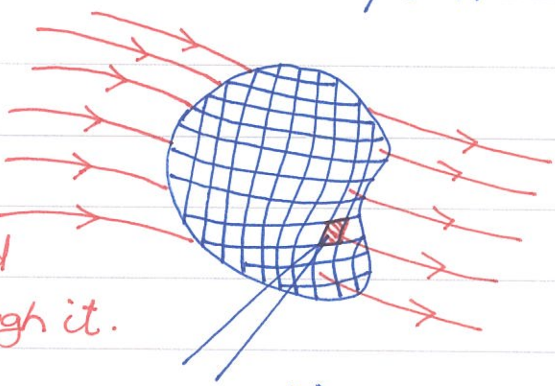
Consider a constant $\vec{E}$ field. One can introduce the FLUX Φ



$\Phi = EA$

$$\boxed{\Phi \equiv \vec{E}\cdot\vec{A} = |\vec{E}|\cdot|\vec{A}|\cos\theta}$$

The notion of flux can be easily generalized to arbitrary surfaces.



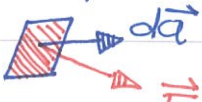
The total flux through the surface is the integral

$$\boxed{\Phi = \int \vec{E}\cdot d\vec{a}} \propto \text{\# of field lines through it.}$$

If the surface is a closed one,

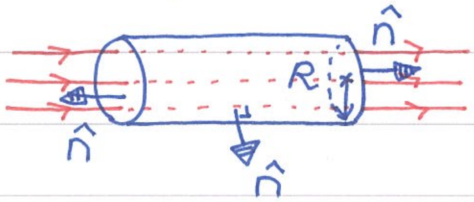
$$\Phi = \oint \vec{E}\cdot d\vec{a}.$$

$\oint$ means integral over the closed surface.



$d\Phi = \vec{E}\cdot d\vec{a}$

Let's compute the flux through a closed cylinder in the presence of the constant $\vec{E}$ field. The total flux contains three parts:



$$\boxed{\begin{aligned}\Phi &= \oint \vec{E}\cdot d\vec{a} \\ &= -E\cdot\pi R^2 + 0 + E\cdot\pi R^2\end{aligned}}$$

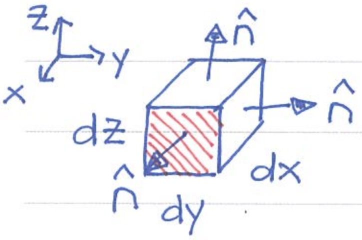
The total flux $\Phi = 0$. In fact, you don't need fancy field theory to understand the above result. As is clear in the picture, # of field lines in = # of field lines out. The net # is zero $\longrightarrow$ no flux.

⊘ Maxwell equations in different forms. In the following, I would like to show that the differential form is

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} \Leftrightarrow \oint \vec{E}\cdot d\vec{a} = \frac{Q}{\epsilon_0}$$

equivalent to the integral form. ∇-form is good for algebraic calculations while $\oint$-form gives simple geometric picture.

Let us recall our old friend: the tiny cube



We would like to compute the total flux through the tiny cube

$$\begin{aligned}\Phi = \oint \vec{E}\cdot d\vec{a} &= \left[E_x(x+dx) - E_x(x)\right] dy\,dz \\ &+ \left[E_y(y+dy) - E_y(y)\right] dx\,dz \\ &+ \left[E_z(z+dz) - E_z(z)\right] dx\,dy\end{aligned}$$

note that

$$E_x(x+dx) - E_x(x) = \frac{\partial E_x}{\partial x}dx$$
$$E_y(y+dy) - E_y(y) = \frac{\partial E_y}{\partial y}dy$$
$$E_z(z+dz) - E_z(z) = \frac{\partial E_z}{\partial z}dz$$

The total flux can be written as

$$\Phi = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dx\,dy\,dz$$

$$\longrightarrow \quad \boxed{\Phi = \nabla\cdot\vec{E}\; dV}$$

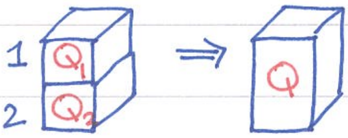
The total flux is directly related to its divergence

Making use of the Maxwell equation $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$

$$\Phi \equiv \oint \vec{E}\cdot d\vec{a} = \int \vec{\nabla}\cdot\vec{E}\, dV = \frac{1}{\epsilon_0}\int \rho\, dV = \frac{Q_{in}}{\epsilon_0}$$

The flux is proportional to the charge inside! COOL!

Suppose we stack two tiny cubes together,



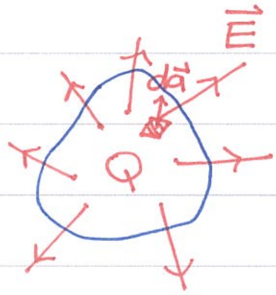
$$\oint \vec{E}\cdot d\vec{a} = \oint_1 \vec{E}\cdot d\vec{a} + \oint_2 \vec{E}\cdot d\vec{a}$$

$Q = Q_1 + Q_2$

$$= \frac{Q_1}{\epsilon_0} + \frac{Q_2}{\epsilon_0} = Q/\epsilon_0$$

still correct

Arbitrary closed surfaces can be decomposed into MANY tiny cubes → We expect that the relation is still true.

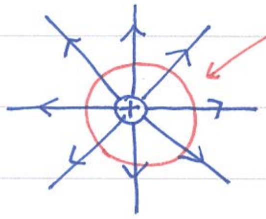


$$\boxed{\oint \vec{E}\cdot d\vec{a} = \frac{Q}{\epsilon_0}}$$

For curious cats, what happens if the charge is right on the surface?

The above integral equation is the Gauss' law. Because it can be derived from the Maxwell equation, their equivalence is obvious.

⊘ Point, line, surface charges revisited. Let us go back to the point charge again. Choose a spherical Gauss surface.



$$\oint \vec{E}\cdot d\vec{a} = \frac{Q}{\epsilon_0}$$

By symmetry → $E \cdot 4\pi r^2 = Q/\epsilon_0$

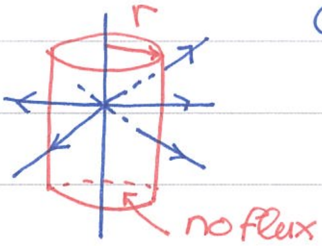
$$\boxed{E = \frac{Q}{4\pi\epsilon_0 r^2} \propto \frac{1}{r^2}}$$

The Gauss' law provides a simple geometric picture why Coulomb's law is inverse square.

→ $E \propto \dfrac{1}{r^{d-1}}$ where $d=3$ is the spatial dimension.

Apply Gauss' law to the line of uniform charge.

Choose the cylindrical Gauss surface



$\lambda = Q/L$

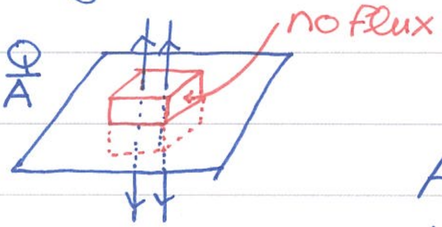
$$E \cdot 2\pi r \cdot \cancel{L} = \frac{\lambda \cancel{L}}{\epsilon_0}$$

$$\rightarrow \boxed{E = \frac{\lambda}{2\pi\epsilon_0 r} \propto \frac{1}{r}}$$

Now we see that $E \propto \frac{1}{r}$ arises from the geometric property of the cylindrical Gauss surface.

Finally, let us study the $\vec{E}$ field by the surface of uniform charge. Choose the following Gauss surface.

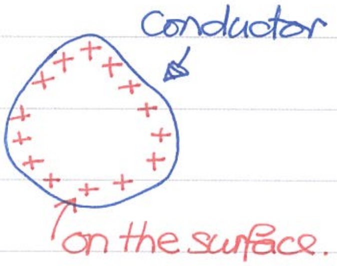
$\sigma = \frac{Q}{A}$



$$E \cdot 2\cancel{A} = \frac{\sigma \cancel{A}}{\epsilon_0} \rightarrow \boxed{E = \frac{\sigma}{2\epsilon_0}}$$

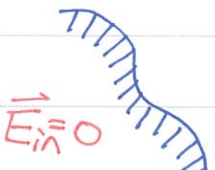
Again, $E \propto r^0$ can be explained by the geometry of the Gauss surface.

Ⓓ Electric field in conductors. We can apply Gauss' law to show several interesting properties in conductors.



An excess charge placed on an isolated conductor moves entirely to the outer surface of the conductor.

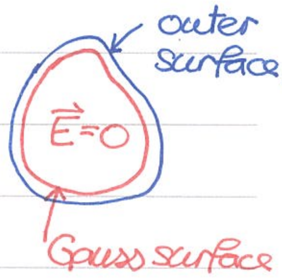
First of all, we would like to show that $\vec{E} = 0$ inside the conductor. Why? If $\vec{E} \neq 0$, electrons inside will move until the electrostatic equilibrium is reached.



So, $\vec{E} = 0$ inside a conductor.

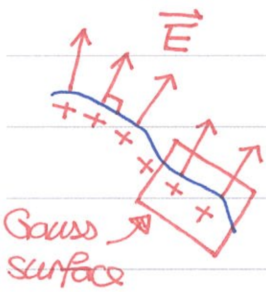
What about on the surface?

Let us choose the Gauss surface just inside the outer surface. Because $\vec{E}=0$ inside, the total flux is zero.



$$\Phi = \oint \vec{E}\cdot d\vec{a} = 0$$ ← no charge inside.

Thus, all excess charge must show up on the outer surface.

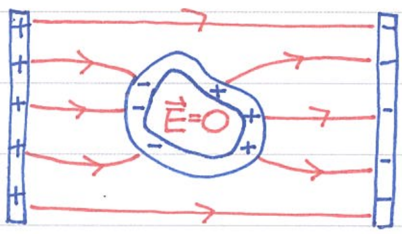


In addition, the electric field is not necessarily zero on the surface. As long as $\vec{E}$ field is perpendicular to the surface, static equilibrium can be reached.

Apply the Gauss' law → $E\cdot A + 0\cdot A = \dfrac{\sigma A}{\epsilon_0}$ → $\boxed{E = \dfrac{\sigma}{\epsilon_0}}$

The electric field on the surface of a conductor is porprotional to the surface charge density. Note that $E = \sigma/\epsilon_0$ is twice larger because $E=0$ inside the conductor.

Finally, it is fun to comment on the electrical shielding inside a conductor. Place a conductor inside the external electric field as shown on the left.



The induced charges will adjust to make the electric field vanish inside the shell. This is "electrical shielding"!

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2014.0227