HH0112 Electrostatic Potential Energy

When the charge distribution is static, the electric field is decoupled from the magnetic field:

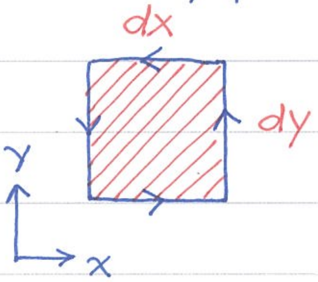
$$\boxed{\begin{aligned}\vec{\nabla}\cdot\vec{E} &= \rho/\epsilon_0\\ \vec{\nabla}\times\vec{E} &= 0\end{aligned}}$$

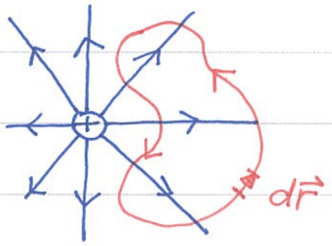
We learned that the first equation leads to Gauss' law. Here we would concentrate on the second equation. The vanishing curl ensures that the $\vec{E}$ field is conservative $\rightarrow$ the existence of the electric potential V.

⊘ Electric potential. We would like to show that $\vec{\nabla}\times\vec{E}=0$ is equivalent to zero circulation ☺

$$\boxed{\oint \vec{E}\cdot d\vec{r} = 0}$$

Let us consider a tiny square on the x-y plane. The circulation consists of 4 parts:





$$\oint \vec{E}\cdot d\vec{r}$$

$$= \left[E_y(x+dx) - E_y(x)\right]dy \leftarrow E_y(x+dx) - E_y(x) = \frac{\partial E_y}{\partial x}dx$$

$$+ \left[-E_x(y+dy) + E_x(y)\right]dx$$

The circulation of the $\vec{E}$ field around the tiny square is

$$\oint \vec{E}\cdot d\vec{r} = \frac{\partial E_y}{\partial x}dxdy - \frac{\partial E_x}{\partial y}dydx = \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)dxdy.$$

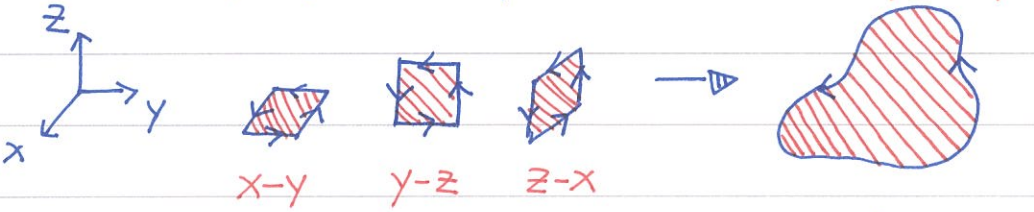
Note that the tiny square defines a tiny area $d\vec{a} = dxdy\,\hat{z}$

$$\oint \vec{E}\cdot d\vec{r} = (\vec{\nabla}\times\vec{E})_z\,dxdy \rightarrow \boxed{\oint \vec{E}\cdot d\vec{r} = (\vec{\nabla}\times\vec{E})\cdot d\vec{a}}$$

Because the curl vanishes in electrostatics, the circulation is

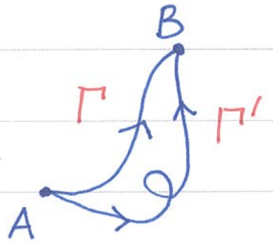
$$\boxed{\oint \vec{E}\cdot d\vec{r} = 0} \leftarrow \text{The integral form of the Maxwell equation } \vec{\nabla}\times\vec{E}=0$$ ☺

Obviously, the above calculation can be generalized to other tiny square loops.



As explained in HH0090, zero circulation of the $\vec{E}$ field enables us to define the corresponding electric potential V,

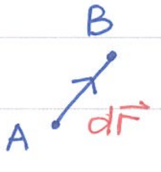
$$\boxed{\Delta V = V(\vec{r}_B) - V(\vec{r}_A) = -\int_\Gamma \vec{E}\cdot d\vec{r}}$$



The potential difference ΔV is indepent of the path Γ as long as the end points are the same. Thus, we can choose the easiest path to compute the potential difference.

Γ, Γ' give the same ΔV.

Now we know how to build a scalar field V (up to a constant) from the vector field $\vec{E}$. Can we achieve the opposite? In general, "Nope" because a vector field contains more information. But, when $\vec{\nabla}\times\vec{E}=0$, we can construct the vector field $\vec{E}$ from the scalar field V.



Choose A, B infinitesimally close to each other, $\vec{r}_B - \vec{r}_A = d\vec{r}$ and $V(\vec{r}_B) - V(\vec{r}_A) = dV$.

$$\rightarrow \boxed{dV = -\vec{E}\cdot d\vec{r} = -E_x dx - E_y dy - E_z dz}$$

The above relation holds for arbitrary $d\vec{r}$. Choose $d\vec{r} = (dx, 0, 0)$

$$(dV)_{y,z} = -E_x dx \rightarrow E_x = -\left(\frac{dV}{dx}\right)_{y,z} = -\frac{\partial V}{\partial x}$$

holding y, z constant

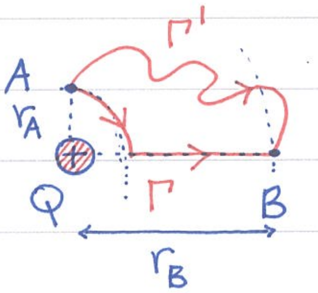
Similarly, $E_y = -\frac{\partial V}{\partial y}$ and $E_z = -\frac{\partial V}{\partial z}$

In vector notation, the $\vec{E}$ field is

$$\boxed{\vec{E} = -\vec{\nabla} V = \left(-\frac{\partial V}{\partial x}, -\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial z}\right)}$$

It shall be a fun exercise to show that $\vec{\nabla}\times\vec{\nabla}V = 0$ as we expected for a conservative field.

Ⓓ Electric potential for different charge distributions. Let us start with a point charge Q at the origin.



$$\Delta V = -\int_{\Gamma'} \vec{E}\cdot d\vec{r} = -\int_{\Gamma} \vec{E}\cdot d\vec{r}$$

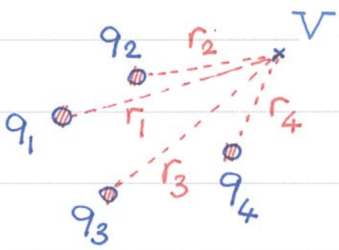
$$= -\int_{r_A}^{r_B} \frac{Q}{4\pi\epsilon_0}\frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0 r_B} - \frac{Q}{4\pi\epsilon_0 r_A}$$

Thus, the electric potential due to a point charge is

$$\boxed{V(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r} + \text{Const}}$$

We often set $V(r\to\infty)=0$ so that Const $=0$.

Because the $\vec{E}$ field satisfies the principle of superposition, the potential V due to many point charges is



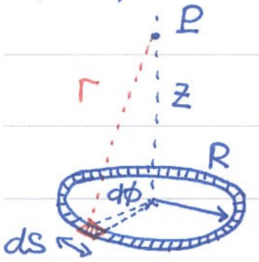
$$V = V_1 + V_2 + \cdots + V_N$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_1} + \frac{q_2}{r_2} + \cdots + \frac{q_N}{r_N}\right)$$

$$\Rightarrow \boxed{V = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i}}$$

r_i is the distance to the i^{th} charge.

Apply it to the charged ring. The linear charge density is λ



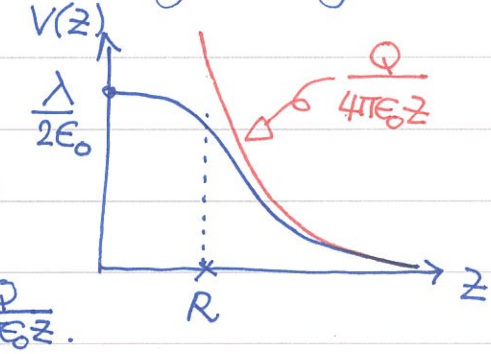
$ds = R\,d\phi$

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\frac{\lambda R\, d\phi}{\sqrt{z^2+R^2}}$$

$$V = \int dV = \frac{\lambda}{4\pi\epsilon_0}\frac{R}{\sqrt{z^2+R^2}}\underbrace{\int_0^{2\pi} d\phi}_{\text{just } 2\pi.}$$

The electric potential for a charged ring is

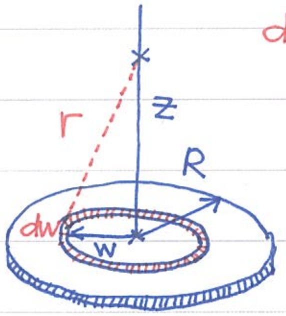
$$\boxed{V = \frac{2\pi R\lambda}{4\pi\epsilon_0}\frac{1}{\sqrt{z^2+R^2}}}$$



In the long-distance limit, $z \gg R$, the potential goes back to the point-charge form $V \approx \frac{Q}{4\pi\epsilon_0 z}$.

In general, given a static charge distribution, it is easier to compute the scalar field V first. Then, use $\vec{E} = -\vec{\nabla}V$ to obtain the vector field $\vec{E}$.

Let's compute V for a charged disk. The potential from the infinitesimal ring is



$$dq = \sigma\, da = \sigma\, 2\pi w\, dw.$$

$$dV = \frac{dq}{4\pi\epsilon_0}\frac{1}{\sqrt{w^2+z^2}} = \frac{2\pi\sigma}{4\pi\epsilon_0}\frac{w\,dw}{\sqrt{w^2+z^2}}$$

Integrate the variable w to get the potential,

$$V = \frac{\sigma}{2\epsilon_0}\int_0^R \frac{w\,dw}{\sqrt{w^2+z^2}} = \frac{\sigma}{2\epsilon_0}\sqrt{w^2+z^2}\,\Big|_0^R$$

Handle the lower limit carefully to get the potential,

$$\boxed{V = \frac{\sigma}{2\epsilon_0}\left(\sqrt{z^2+R^2} - |z|\right)}$$

① long distance limit, $z \gg R$

$$\sqrt{z^2+R^2} = |z|\left(1+\frac{R^2}{z^2}\right)^{\frac{1}{2}}$$

Making use of binomial expansion,

$$\sqrt{z^2+R^2} \approx |z|\left(1+\frac{1}{2}\frac{R^2}{z^2}\right) \longrightarrow V \approx \frac{\sigma}{2\epsilon_0}\cdot\frac{1}{2}\frac{R^2}{z^2}|z| = \frac{Q}{4\pi\epsilon_0|z|}$$

The potential becomes the poin-charge form when $z \gg R$ ☺

② short distance limit, $z \ll R$,

$$\sqrt{z^2+R^2} = R\left(1+\frac{z^2}{R^2}\right)^{\frac{1}{2}}$$

The potential takes the form,

$$\boxed{V \approx \frac{\sigma}{2\epsilon_0}\left[R + R\cdot\frac{1}{2}\frac{z^2}{R^2} - |z|\right] \approx \frac{\sigma R}{2\epsilon_0} - \frac{\sigma}{2\epsilon_0}|z|}$$

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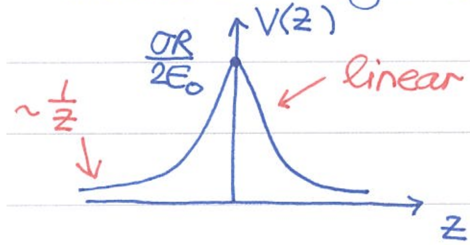
The constant term $\sigma R/2\epsilon_0$ is not important. The linear $|z|$ term gives rise to const $\vec{E}$ field:

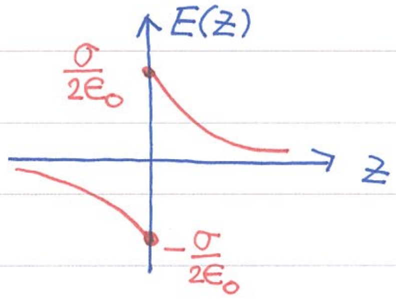
$$\vec{E} = E_z \hat{z}, \quad \boxed{E_z = -\frac{\partial V}{\partial z} = \begin{cases} \sigma/2\epsilon_0 & z>0 \\ -\sigma/2\epsilon_0 & z<0 \end{cases}}$$

The above result agrees with the previous calculations.

Combine both regimes together:

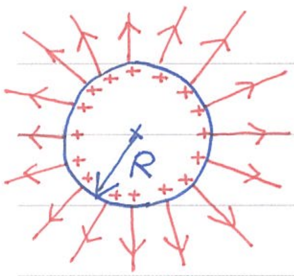


$\rightarrow E_z = -\frac{\partial V}{\partial z}$



In real life, R is always finite and the electric field is <u>NOT</u> really constant ☺

∅ Potential of charged conductors. Consider a spherical conductor with charge Q. The $\vec{E}$ field outside $(r>R)$ is

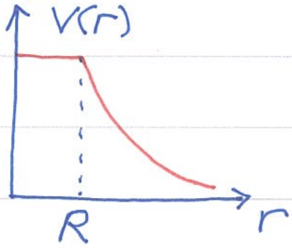


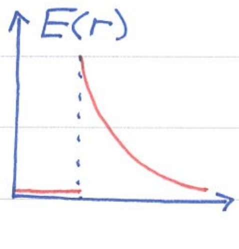
the same as that of a point charge $\rightarrow$

$$\boxed{V = \frac{Q}{4\pi\epsilon_0 r}, \quad r>R}$$

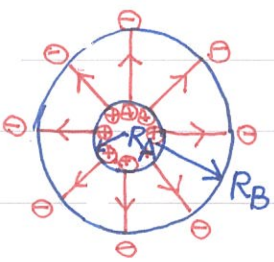
On the other hand, the $\vec{E}$ field inside is zero. $\rightarrow$ no potential difference $\Delta V = 0$

$$\rightarrow \boxed{V = \frac{Q}{4\pi\epsilon_0 R}, \quad r<R}$$





Something interesting happens when we put two spherical conductors together ☺ Suppose the charges on the inner and outer spheres are $+Q$ and $-Q$. By Gauss' law,



$$\boxed{\begin{aligned} &E = 0, \text{ for } r<R_A \text{ or } r>R_B \\ &E = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}, \text{ for } R_A<r<R_B \end{aligned}}$$

$E \neq 0$ only between the two spheres!!

It is straightforward to compute the potential difference between the spheres:

$$\boxed{\Delta V \equiv V_A - V_B = -\int_B^A \vec{E}\cdot d\vec{r} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_A} - \frac{1}{R_B}\right)}$$

Note that $\Delta V \propto Q$ all the times. Introduce the capacitance for the setup:

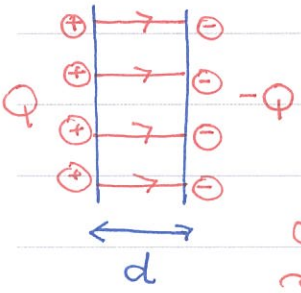
$$\boxed{C \equiv \frac{Q}{\Delta V}} \longrightarrow \boxed{C = 4\pi\epsilon_0 \frac{R_B \cdot R_A}{R_B - R_A}}$$

capacitance for a spherical capacitor.

Because $\vec{E} \neq 0$ only between the two spheres, they can be viewed as a device for charge storage (charge capacitor) without interfering the external environment.

Sometimes, the grounded sphere is taken to infinity, i.e. $R_B \to \infty$. The capacitance for a single sphere then becomes $C = 4\pi\epsilon_0 R_A$. But one should keep in mind that the other sphere is at the infinity.

The simplest capacitor is probably a pair of parallel plates.



The electric field $\vec{E} = \vec{E}_+ + \vec{E}_-$. Outside the plates, the contributions cancel $\longrightarrow \vec{E} = 0$. The $\vec{E}$ field inside the plates is

$$\boxed{E = \frac{\sigma}{2\epsilon_0} + \frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}}$$

$\sigma = \frac{Q}{A}$ surface density.

The potential difference between the plates can be computed easily. $\Delta V = E \cdot d = \frac{Qd}{\epsilon_0 A}$.

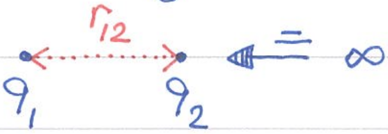
Again, it's easy to see $\Delta V \propto Q$

$$\boxed{C = \frac{Q}{\Delta V} = \epsilon_0 \frac{A}{d}}$$

It is insightful to compare with $C_{sphere} = \epsilon_0 \, 4\pi R_A R_B / (R_B - R_A)$

Capacitance for the plate capacitor. $\longrightarrow C = \epsilon_0 \frac{\text{AREA}}{\text{distance}}$

⑪ Potential energy. Let us start with the simplest case — electrostatic potential energy between two point charges q_1, q_2. Put q_1 at the origin and move q_2 from ∞ to the distance $r_{12} = |\vec{r}_1 - \vec{r}_2|$. The difference



of potential energy can be expressed as

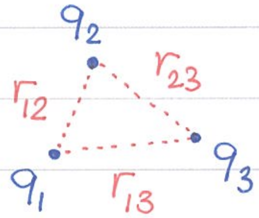
$$\Delta U_E = -\int_{\infty}^{r_{12}} \vec{F} \cdot d\vec{r} = -q_2 \int_{-\infty}^{r_{12}} \vec{E} \cdot d\vec{r} = q_2 \Delta V$$

The energy difference is proportional to potential difference,

$$\boxed{\Delta U_E = U_E(r_{12}) - \cancelto{0}{U_E(\infty)} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{r_{12}}}$$

It is convenient to set $U_E(\infty) = 0$.

The above result can be generalized to N point charges.



Take $N = 3$ as an example.

$$U_E = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_3}{r_{13}}\right)$$

$$= \frac{1}{2}\left(q_1 V_1 + q_2 V_2 + q_3 V_3\right)$$

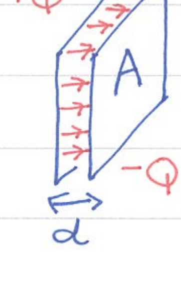
$V_1 = \frac{1}{4\pi\epsilon_0}\left(\frac{q_2}{r_{12}} + \frac{q_3}{r_{13}}\right)$
Similar for V_2, V_3

Thus, the electrostatic potential energy for N point charges is

$$\boxed{U_E = \sum_{\langle ij \rangle} \frac{1}{4\pi\epsilon_0} \frac{q_i q_j}{r_{ij}} = \frac{1}{2} \sum_i q_i V_i}$$

$\langle ij \rangle$ means all distinct pairs.

The $\frac{1}{2}$ factor is to remove the double counting of electric interactions. For a given charge distribution, we compute the potential V first. Multiplied by the charge, the potential energy can be computed by summation or integration!



Consider a plate capacitor of area A.

$$\boxed{U_E = \frac{1}{2} Q V_+ + \frac{1}{2}(-Q) V_- = \frac{1}{2} Q \Delta V}$$

U_E comes from charge-charge interactions.

According to the definition, $Q = C\Delta V$.

$$\boxed{U_E = \frac{1}{2} Q \Delta V = \frac{1}{2} C (\Delta V)^2 = \frac{1}{2} \frac{Q^2}{C}}$$

$\Delta V = \frac{\sigma d}{\epsilon_0}$

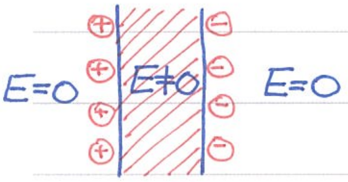
$$U_E = \frac{1}{2} C (\Delta V)^2 = \frac{1}{2} \epsilon_0 \frac{A}{d} \cdot \frac{\sigma^2 d^2}{\epsilon_0^2} = A \cdot \underbrace{\frac{\sigma^2}{2\epsilon_0} d}$$

The potential energy $U_E \propto A$ as expected ☺.

It is quite interesting that the potential energy can also be viewed from the field.

$$\boxed{U_E = \int dV\, u_E \quad \text{and} \quad u_E = \frac{1}{2} \epsilon_0 E^2}$$

energy density u_E from the $\vec{E}$ field!



The energy density outside the plates is 0 ☺

Inside the plates, $E \neq 0$ gives rise to $u_E = \frac{1}{2} \epsilon_0 E^2 = \underbrace{\frac{\sigma^2}{2\epsilon_0}}$

$$\rightarrow U_E = \int dV\, u_E = \underbrace{A \cdot d \cdot \frac{\sigma^2}{2\epsilon_0}}$$

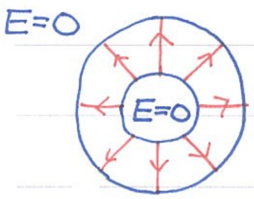
As calculated from the charge-charge interactions.

The electrostatic potential energy can be computed by

① charge-charge interactions
② energy density from the field.

> equivalent for static charge distributions.

Let us check the equivalence for the spherical capacitor:



$$U_E = \frac{1}{2} Q \Delta V = \frac{1}{2} Q \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{R_A} - \frac{1}{R_B} \right)$$

$$= \underbrace{\frac{Q^2}{8\pi\epsilon_0} \left(\frac{1}{R_A} - \frac{1}{R_B} \right)}$$

← charge-charge interactions.

Now let us take the field perspective.

$$u_E = \frac{1}{2} \epsilon_0 E^2 = \frac{1}{2} \epsilon_0 \left(\frac{Q}{4\pi\epsilon_0 r^2} \right)^2 = \underbrace{\frac{Q^2}{32\pi^2\epsilon_0} \frac{1}{r^4}}$$

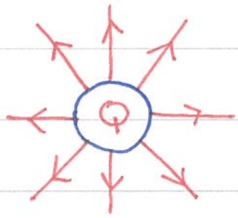
Integrate over the $E \neq 0$ regime:

$$U_E = \int dv\, u_E = \int_{R_A}^{R_B} dr \cdot 4\pi r^2 \frac{Q^2}{32\pi^2\epsilon_0} \frac{1}{r^4}.$$

$$\rightarrow \boxed{U_E = \frac{Q^2}{8\pi\epsilon_0}\int_{R_A}^{R_B}\frac{dr}{r^2} = \frac{Q^2}{8\pi\epsilon_0}\left(\frac{1}{R_A}-\frac{1}{R_B}\right)}$$

The potential energy carried by the field is identical to that from the charge-charge interaction! The equivalence highlights the fundamental importance of the field ☺

The electrostatic potential energy of a spherical conductor can be computed by setting $R_A = R$ and $R_B \rightarrow \infty$



$$\boxed{U_E = \frac{Q^2}{8\pi\epsilon_0 R}} \rightarrow \lim_{R\rightarrow 0} U_E = \text{divergent}$$

According to current theory, an electron is a point without internal structure. So, the electrostatic potential energy of an electron is infinite. Is this really true? Will we run into some serious problem? We don't know the answer yet....

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