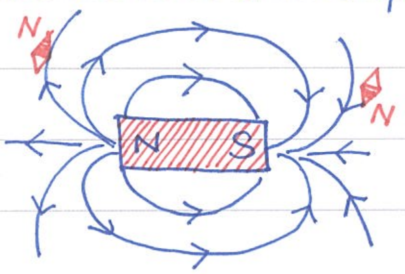
豪豬筆記

HH0117 Can Magnetic Forces Do Work?

In electrostatics, electric charge Q and electric field $\vec{E}$ show interesting relation.

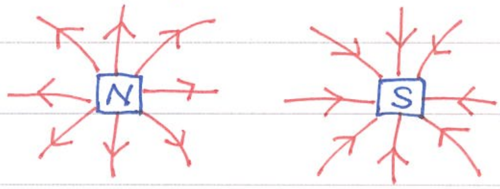
$$\text{charge } Q \;\underset{\vec{F}=q\vec{E}}{\overset{\vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0}{\rightleftarrows}}\; \text{electric field } \vec{E}$$

What about magnetic field? Let us try to understand the notion in a qualitative way first. Imagine we place



many tiny magnet near the magnet. The direction of $\vec{B}$ field is defined as the north pole of the tiny magnet. With enough sensing magnets, one can map out the field lines as shown here.

However, we never find "magnetic charges" (monopoles) in nature. That is to say, N & S poles always show up in pairs. Or, in terms of the field language,

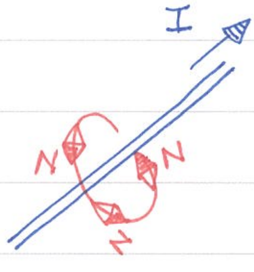


Monopoles does not exist!

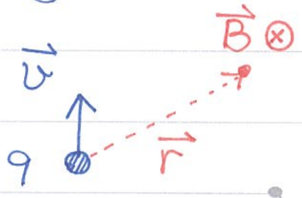
$$\boxed{\vec{\nabla}\cdot\vec{B}=0 \quad \Rightarrow \quad \oint \vec{B}\cdot d\vec{a}=0}$$

no net flux through a closed surface

It is quite interesting that a current-carrying wire also produces magnetic field. In fact, a moving point charge builds up a magnetic field,



$$\boxed{\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\,\frac{q\vec{v}\times\hat{r}}{r^2}}$$

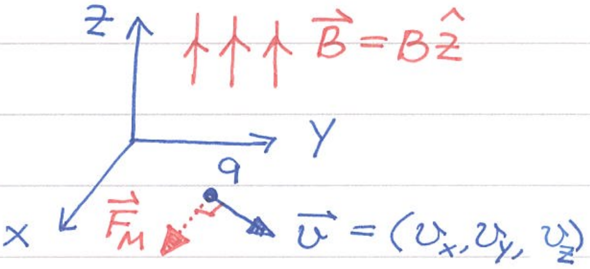


direction by the right-hand rule.

This is the magnetic analog of Coulomb's law for point charge.

⊘ Lorentz force on a moving charge. When a point charge moves through the $\vec{B}$ field, the magnetic force is

$$\boxed{\vec{F}_M = q\vec{v}\times\vec{B}}$$



Compared with its electric version, $\vec{F}_E = q\vec{E}$, the magnetic force is more complicated.

Let us try to solve the trajectory in a uniform magnetic field $\vec{B} = (0, 0, B)$. The Lorentz force on the particle is

$$\vec{F}_M = q\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = q\,(v_y B, -v_x B, 0)$$

no force along the $\vec{B}$ direction.

Write down the EOM for the point charge:

$$\boxed{\begin{aligned} m\frac{dv_x}{dt} &= qBv_y \\ m\frac{dv_y}{dt} &= -qBv_x \\ m\frac{dv_z}{dt} &= 0 \end{aligned}}$$

It is straightforward to see that $v_z(t) = \text{const.}$ → Velocity along the $\vec{B}$ direction remains constant.

It is convenient to introduce the cyclotron frequency $\boxed{\omega_c = \frac{qB}{m}}$

The EOM in the x-y plane takes a simpler form ☺

$$\boxed{\begin{aligned} \frac{dv_x}{dt} &= \omega_c v_y \\ \frac{dv_y}{dt} &= -\omega_c v_x \end{aligned}} \quad \rightarrow \quad v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt} = \omega_c(\cancel{v_x v_y - v_x v_y}) \; 0$$

i.e. $\boxed{\frac{d}{dt}(v_x^2 + v_y^2) = 0}$

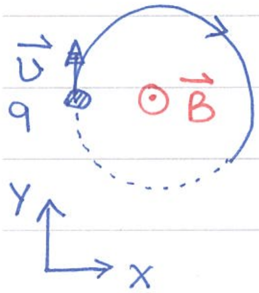
Because $v_x^2 + v_y^2 = \text{const}$,

we can express them as $v_x = v_\perp \sin\theta$, $v_y = v_\perp \cos\theta$

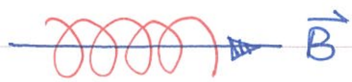
Substitute into the simplified EOM:

$$\frac{dv_x}{dt} = \omega_c v_y \longrightarrow v_\perp \cancel{\cos\theta}\,\frac{d\theta}{dt} = \omega_c v_\perp \cancel{\cos\theta}$$

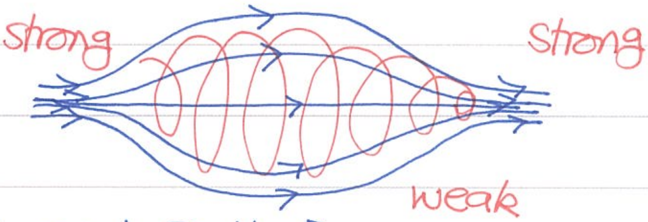
$$\longrightarrow \theta(t) = \omega_c t + \theta_0 \longrightarrow \boxed{\begin{aligned} v_x &= v_\perp \sin(\omega_c t + \theta_0) \\ v_y &= v_\perp \cos(\omega_c t + \theta_0) \end{aligned}}$$

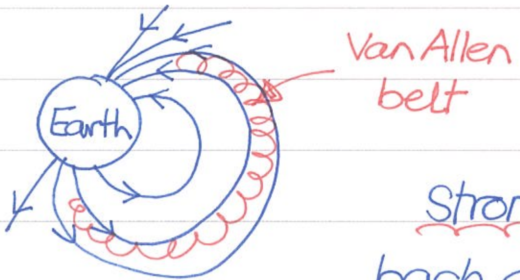


This is nothing but the familiar circular motion in 2D ☺ Combined with the v_z solution, the trajectory is the spiral.



When the $\vec{B}$ field is not uniform, the trajectory becomes more complicated. An interesting case is the so-called "magnetic bottle". The charge particle can be trapped between the strong-field regions, spiraling back and forth!





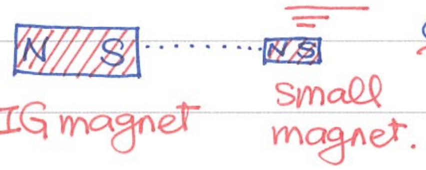
A similar situation occurs in Earth's magnetic field. Charged particles are trapped between the strong-field regions (the poles), spiraling back and forth within a few seconds ☺.

⊘ Work by magnetic force. This seems to be a trivial question. It is easy to compute the power done by the magnetic force,

$$P = \vec{F}_M \cdot \vec{v} = q(\vec{v} \times \vec{B}) \cdot \vec{v} = 0\,!$$

Because the power is always ZERO, we conclude that the work done by the magnetic force shall be ZERO as well.

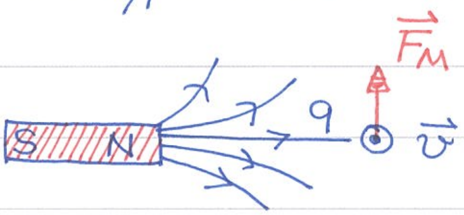
But, this conclusion seems to go against our common sense. The magnets attract each other and do work all the times.

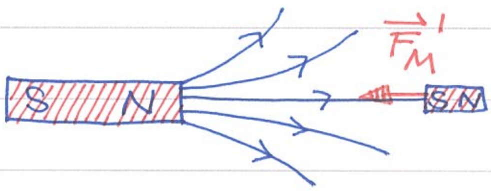


Besides, the magnets do not seem to carry charge. How can they feel the Lorentz force? And, two static magnets still feel the magnetic forces. So, we also have Lorentz force even at $\vec{v}=0$?

$$\vec{F}_M = \cancel{q}^{\,0}(\vec{v}\times\vec{B}) \longrightarrow \text{ZERO.}$$

Is it possible that there are two types of magnetic forces? One type doesn't do any work while the other can do work?



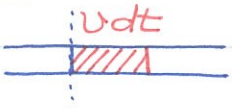


Type I: need charge q and velocity $\vec{v}\neq 0$ $\longrightarrow$ non-zero $\vec{F}_M$.

Type II: need "something" (like a magnet) $\longrightarrow$ $\vec{F}_M'$ can do work!

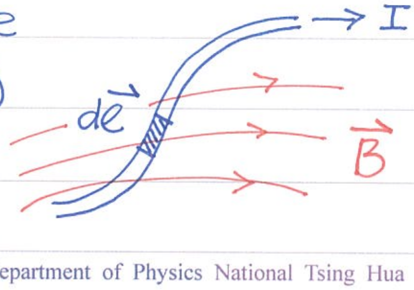
We shall discuss this important issue in the class

∅ Lorentz force on a current-carrying wire. We can generalize the formula for Lorentz force to the wire.



wire $\quad I = \dfrac{dQ}{dt} = \dfrac{\lambda v\, dt}{dt} = \lambda v$!

$dQ = \lambda\cdot(v dt)$. We can compute the Lorentz force on $d\vec{\ell}$ (as shown $\rightarrow$)



$$d\vec{F} = dq\, \vec{v}\times\vec{B} = \lambda d\ell\, \vec{v}\times\vec{B}$$

Notice that $\vec{v} \parallel d\vec{\ell} \rightarrow d\ell\,\vec{v} = v\,d\vec{\ell}$.

$$d\vec{F} = \lambda v\, d\vec{\ell} \times \vec{B} = \underbrace{I\, d\vec{\ell} \times \vec{B}}$$

Integrate to get the force on the wire,

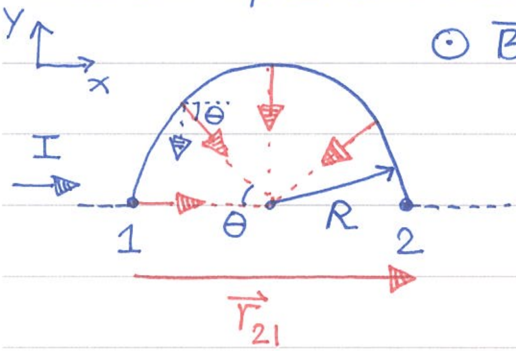
$$\boxed{\vec{F} = \int d\vec{F} = I \int_1^2 d\vec{\ell} \times \vec{B}}$$

Because the integral contains the cross product, it can be challenging in a non-uniform $\vec{B}$ field.

But, if the $\vec{B}$ field is uniform, the integral is trivial,

$$\vec{F} = I \int_1^2 d\vec{\ell} \times \vec{B} = I \left(\int_1^2 d\vec{\ell} \right) \times \vec{B} = \underbrace{I\, \vec{r}_{21} \times \vec{B}}$$

An example will help here ☺ Consider semicircle as shown on the left. The total magnetic force is



$$\vec{F} = I\, \vec{r}_{21} \times \vec{B} = I \cdot 2R \cdot B\, \hat{i} \times \hat{k}$$

$$\rightarrow \boxed{\vec{F} = -2IRB\,\hat{j}}$$

One can perform the angular integral to get the same ☺

$$\vec{F} = F_y\,\hat{j} = -I \int_0^{\pi} d\theta\, R \cdot B \sin\theta\,\hat{j} = -IRB\,\hat{j} \underbrace{\int_0^{\pi} d\theta \sin\theta}_{=2!}$$

Thus, the total force is $\vec{F} = -2IRB\,\hat{j}$ as computed before.

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