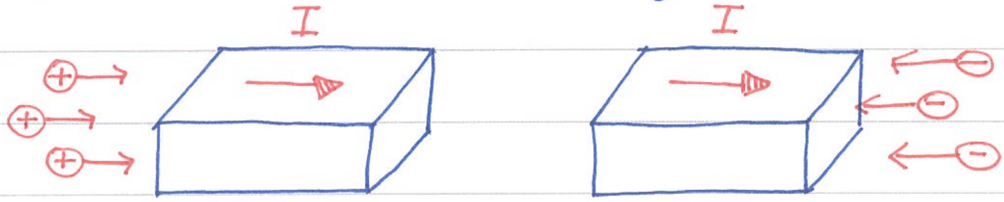
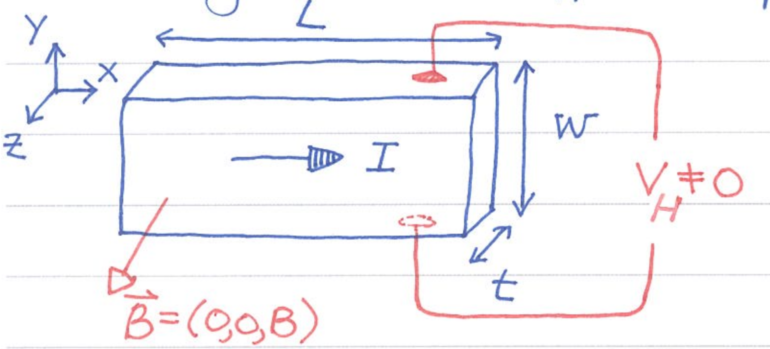
HH0119 Hall Effect

Consider a wire with flowing current. Is it possible to know the charge of the carriers?



Can we differentiate these two cases without microscopic measurement? In 1879, Hall observed that, when an electric current passes through a sample placed in a $\vec{B}$ field, a voltage difference V_H develops in the transverse direction.



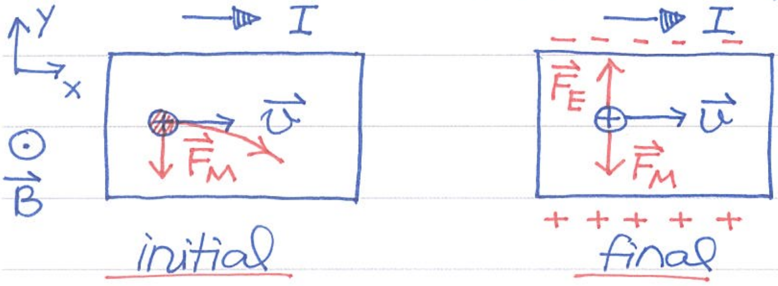
It was found experimentally,

$$\boxed{V_H \propto B I}$$

and its sign depends on the charge of carriers!

∅ Lorentz force revisited.

Suppose the carriers have $q>0$.



Due to the Lorentz force,

$$\vec{F}_M = q\vec{v}\times\vec{B} = (0,-qvB,0)$$

The carriers bend down to the lower edge and charges start to pile up. After a while, the accumulated charges produce an electric field $\vec{E}'$ and the total force is balanced,

$$\vec{F}_E + \vec{F}_M = 0 \rightarrow q\vec{E}' + q\vec{v}\times\vec{B} = 0 \qquad \boxed{\vec{E}' = -\vec{v}\times\vec{B}}$$

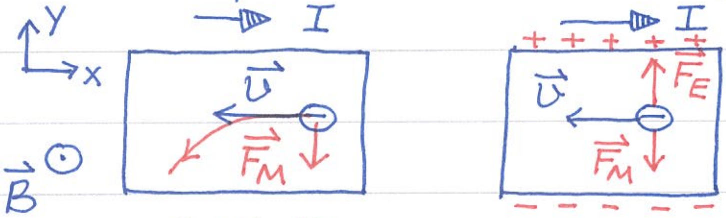
The potential difference between the edges can be computed from $\Delta V = -\int \vec{E}\cdot d\vec{r}$.

Therefore, the Hall voltage is

$$V_H = -\int_0^W E_y\,dy = -vB\int_0^W dy = -vBW < 0$$

Because the current $I \propto v$, we just showed that $V_H \propto BI$ and $V_H < 0$ for positive carriers.

Let us repeat the same analysis for negative carriers.



initial   final

The calculation is the same except the sign of the Hall voltage:

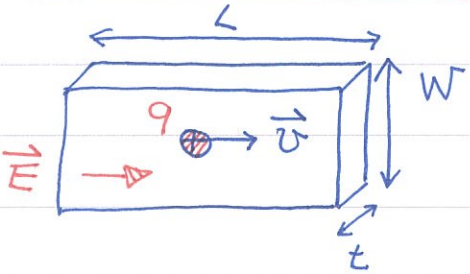
$$V_H = vBW > 0$$

Again, the Hall voltage $V_H \propto BI$ but $V_H > 0$ for carriers with negative charges! Measuring the Hall voltage can tell us whether the carriers have positive or negative charges! Let us try to understand the current flows in wires better ☺

⊘ Ohm's law derived. Assume the dynamics of carriers can be approximated as independent particles. Each carrier is accelerated by the electric field but scattering leads to the retarding force acting on the charge carrier,

$$\boxed{\vec{F}_{scatt} = -\frac{m}{\tau}\vec{v}}$$

The time scale τ is the average time between scattering events.



The EOM for the carrier is

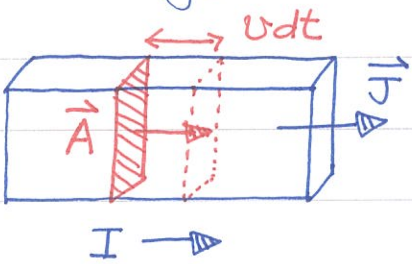
$$\boxed{m\frac{d\vec{v}}{dt} = \underbrace{-\frac{m}{\tau}\vec{v}}_{\text{scattering}} + \underbrace{q\vec{E}}_{\text{external drive}}}$$

In the steady state $\rightarrow$ $d\vec{v}/dt = 0$.

$$-\frac{m}{\tau}\vec{v} + q\vec{E} = 0 \rightarrow \boxed{\vec{v} = \frac{q\tau}{m}\vec{E}}$$

The final velocity $\vec{v}$ is proportional to the $\vec{E}$ field.

The charge current density $\vec{J}$ is related to $I = dQ/dt$.



$$\boxed{I = \vec{J}\cdot\vec{A} = JA}$$ $\rightarrow$ $A = Wt$ is the transverse area.

By counting the passing charges,

$$I = \frac{dQ}{dt} = \frac{Av\cdot dt\cdot\rho}{dt} = Av\rho$$

charge density.

By comparision, we find the important relation between $\vec{J}$ & $\vec{v}$,

$$\boxed{\vec{J} = \rho\vec{v}}$$

The charge current density $\vec{J}$ is the product of charge density ρ and velocity $\vec{v}$ ☺

We are ready to derive the Ohm's law now.

$$I = JA = \rho v A = nqvA$$

ρ: charge density

n: particle # density.

Making use of the previous result, $v = (q\tau/m)E$.

$$I = nqA\cdot\frac{q\tau}{m}E = \left(\frac{nq^2\tau}{m}\right)\frac{A}{L}V$$ $\rightarrow$ $E = \frac{V}{L}$ is used here.

Define the conductivity σ as the following,

$$\boxed{\sigma = \frac{nq^2\tau}{m}} \rightarrow I = \left(\sigma\frac{A}{L}\right)V \propto V.$$

This is the famous Ohm's law ☺. One can find that

$$\boxed{R = \frac{V}{I} = \left(\frac{1}{\sigma}\right)\frac{L}{A}}$$

L/A arises from the geometric setup ☺

material property.

⊘ In the presence of $\vec{B}$ field. The EOM is

$$\boxed{m\frac{d\vec{v}}{dt} = -\frac{m}{\tau}\vec{v} + q(\vec{E} + \vec{v}\times\vec{B})}$$

In the steady state, $d\vec{v}/dt = 0$. The RHS should vanish, $-\frac{m}{\tau}\vec{v} + q(\vec{E} + \vec{v}\times\vec{B}) = 0$.

$$v_x = \frac{q\tau}{m}E_x + \frac{q\tau}{m}B\,v_y$$

$$v_y = \frac{q\tau}{m}E_y - \frac{q\tau}{m}B\,v_x$$

$$v_z = \frac{q\tau}{m}E_z$$

Here we choose the $\vec{B}$ field $\vec{B} = (0,0,B)$ along the z-axis. For simplicity, I would set $E_z = 0$ (thus, $v_z = 0$ as well) and focus on v_x and v_y.

Recall that $\sigma = nq^2\tau/m$ and $\omega_c = qB/m$. The current density $\vec{J} = \rho\vec{v} = nq\vec{v}$ and the above equations can be rewritten for J_x and J_y as

$$J_x - (\omega_c\tau)J_y = \sigma E_x$$

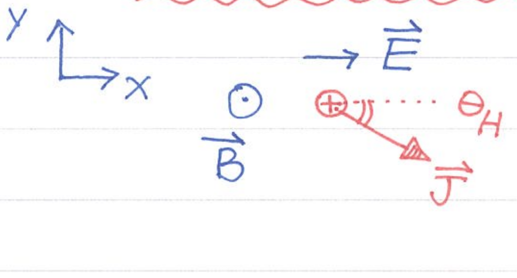
$$J_y + (\omega_c\tau)J_x = \sigma E_y$$

$$J_x = \frac{\sigma}{1+(\omega_c\tau)^2}\left[E_x + (\omega_c\tau)E_y\right]$$

$$J_y = \frac{\sigma}{1+(\omega_c\tau)^2}\left[E_y - (\omega_c\tau)E_x\right]$$

The above results need some digestion ...

(I) Set $E_x \neq 0$ and $E_y = 0$ → The current density $\vec{J}$ is NOT along the $\vec{E}$ field but with a Hall angle θ_H



$$\boxed{\tan\theta_H = \frac{|J_y|}{|J_x|} = \omega_c\tau}$$

(II) Suppose now we control the current density $J_x \neq 0$ and $J_y = 0$. It gives rise to an interesting relation between E_x and E_y.

$$J_y = 0 \quad \rightarrow \quad \boxed{E_y = (\omega_c \tau) E_x}$$

Substitute back to the equation for J_x. It shall be easy to see that ← the same result.

$$\boxed{J_x = \frac{\sigma}{1+(\omega_c\tau)^2}\left[E_x + (\omega_c\tau)^2 E_x\right] = \sigma E_x \,!}$$

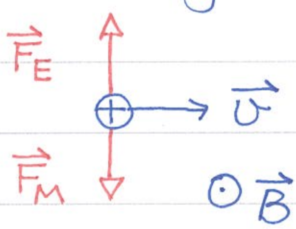
From these relations, we can compute the Hall voltage again.

$$V_H = -E_y W = -(\omega_c\tau)\frac{J_x}{\sigma} W = -\frac{qB}{\cancel{m}}\tau \frac{I/\cancel{Wt}}{nq^2\cancel{\tau}/\cancel{m}} \cancel{W}$$

$$\rightarrow \quad \boxed{V_H = -\left(\frac{1}{nq}\right)\frac{IB}{t}}$$

In the literature, the Hall coefficient $R_H = \frac{1}{nq}$.

Note that the Hall voltage is independent of τ, i.e. the scattering leading to dissipative resistance. Why? Because



the steady state arises from the balance between $\vec{F}_M$ and $\vec{F}_E$

$$\vec{F}_M + \vec{F}_E = 0 \quad \rightarrow \quad \vec{E}' = -\vec{v}\times\vec{B}$$

The electric field $\vec{E}'$ and the magnetic field $\vec{B}$ are simply related by the velocity $\vec{v}$ (proportional to $\vec{J}$) — no scattering details are involved here ☺

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