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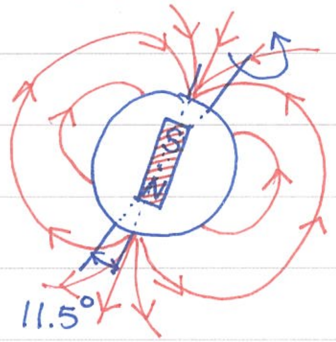
HH0120 How Does the Geomagnetic Field Arise?

The geomagnetic field is well approximated by a magnet dipole with poles close to the geographic poles. Is the origin of the geomagnetic field related to the spinning motion of the Earth?

In 1600, Gilbert provided the first scientific explanation — the Earth is a huge magnet. But the core temperature is high $\rightarrow$ no ferromagnetic order....



Einstein suggested that there may be slight asymmetry between the charges of an electron and a proton. Rotating charged objects lead to $\vec{\mu} \propto \vec{L}$, where $\vec{\mu}$ is the dipole moment and $\vec{L}$ is the angular momentum. Blackett (Nobel Laureate 1948) tried to find the relation between $\vec{\mu}$ and $\vec{L}$ without success.

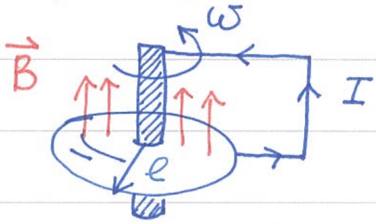
① Sources of the $\vec{B}$ field. A magnet generates the $\vec{B}$ field and the (charge) current also does the magic. If the current is steady, the corresponding Maxwell equations simplify without coupling to the $\vec{E}$ field:

$$\boxed{\begin{aligned} \vec{\nabla}\times B &= \mu_0 \vec{J} \\ \vec{\nabla}\cdot\vec{B} &= 0 \end{aligned}}$$

← Ampere's law in differential form.

It shall be clear that the current density $\vec{J}$ is the source of the $\vec{B}$ field, as ρ is the source of the $\vec{E}$ field. Note that the magnetic dipole moment $\vec{\mu} = -g\mu_B\vec{S}$ (arisen from electron spins) is NOT included in the Maxwell equations, even though it can generate the $\vec{B}$ field. So, $\vec{\mu}$ or $\vec{J}$ cause the geomagnetic field?

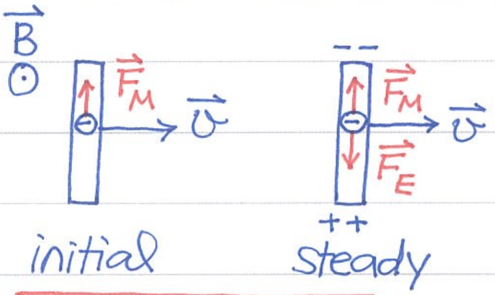
⊘ Faraday's dynamo. It turns out that the origin of geomagnetic field is the current density $\vec{J}$. To understand this phenomenon better, we start with the simple Faraday's dynamo.



A conducting disk rotating at angular velocity ω in the $\vec{B}$ field will generate a motional EMF $\mathcal{E}$ between the center and the edge. The EMF, acting as battery, will drive a current in the loop.

$$\boxed{\mathcal{E} = \frac{1}{2}\omega B \ell^2}$$

Let us consider a simpler geometry first – a moving wire.

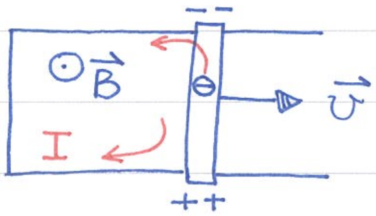


The Lorentz force $\vec{F}_M = q\vec{v}\times\vec{B}$ causes the electrons to move initially. In the steady state, the accumulated charges build up an electric field,

$$\boxed{\vec{E}' = -\vec{v}\times\vec{B}}$$

The voltage difference of a rod (length ℓ) is $\Delta V = vB\ell \neq 0$! ☺

Now, put the moving rod on a U-shape circuit. No magnetic force acts on the charges in the stationary U-shape circuit but the charges near the ends of the moving rod adjust themselves to create an $\vec{E}$ field as discussed before.



Within the rod, electrons move from higher voltage to lower (as in the battery). In the U-shape circuit, electrons move from lower to higher potential (and dissipate energy).

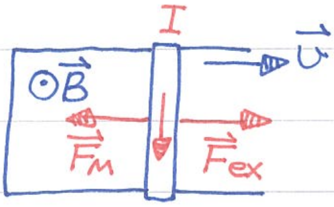
We call this potential difference motional electromotive force (EMF)

$$\boxed{\mathcal{E} = vB\ell}$$

Suppose the resistance of the whole circuit is R,

$$I = \frac{\mathcal{E}}{R} = \frac{vBl}{R} \longrightarrow \boxed{P = I^2 R = \frac{v^2B^2l^2}{R}}$$

This is the power of energy dissipation through Joule heat. Where does the energy come from?

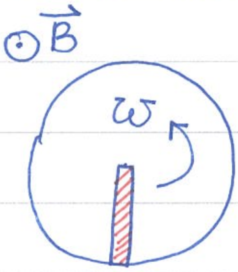


To keep the rod moving at const $\vec{v}$, it is necessary to apply an external force,

$$F_{ex} = IlB \longrightarrow \boxed{P_{ex} = F_{ex} v = \frac{v^2B^2l^2}{R}}$$

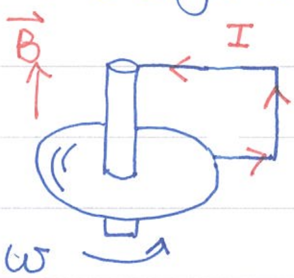
The power provided by the external force P_{ex} is identically the same as the dissipating power. ☺

Now we can come back to the Faraday's disk. The speed at radius r is $v = r\omega$ and the corresponding $\vec{E}$ field is $E = vB = \omega B r$. The EMF can be computed by integration,



$$\mathcal{E} = \int_0^l E \cdot dr = \omega B \int_0^l r\,dr = \frac{1}{2}\omega B l^2$$

Thus, a rotating metal disk in the $\vec{B}$ field acts as a battery to drive the current ☺. Suppose we connect the center with the edge into a circuit of resistance R. The current is



$$\boxed{I = \frac{\mathcal{E}}{R} = \frac{\omega B l^2}{2R}}$$

Let us compute the external τ_{ex} required to keep the dynamo running ☺

$$\tau_{ex} = \int r\,dF_{ex} = \int_0^l r \cdot IB\,dr = IB\int_0^l r\,dr.$$

$$\longrightarrow \boxed{\tau_{ex} = \frac{1}{2}IBl^2 = \frac{\omega B^2 l^4}{4R}}$$ ← not free to run the dynamo

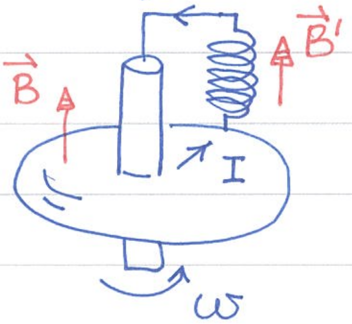
Because the dynamo is in steady state, we expect $P(\text{input}) = P(\text{output})$. The input power is from the external torque,

$$P_{in} = \tau_{ex} \cdot \omega = \frac{\omega B^2 \ell^4}{4R} \cdot \omega = \frac{\omega^2 B^2 \ell^4}{4R}$$

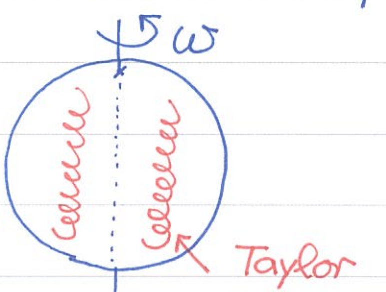
The output power is through Joule heat dissipation,

$$P_{out} = I^2 R = \left(\frac{\omega B \ell^2}{2R}\right)^2 \cdot R = \frac{\omega^2 B^2 \ell^4}{4R} = P_{in} \,!$$

⑩ Dynamo theory for geomagnetic field. So far, we have ignored the $\vec{B}$ field generated by the induced current. Let us try to include the feedback into the dynamo. We can



setup a circuit as shown here. The rotating disk in $\vec{B}$ generates the current I. The current I produces the $\vec{B}'$ field which causes feedback to the current. The steady state depends on how the circuit is set up. Let us apply the idea to the Earth. The



outer core consists of conducting fluid spinning with the Earth. Even a tiny magnetic field will cause some current. In rotating fluids, these currents appear as spiraling Taylor columns → the $\vec{B}$ field is further enhanced. The enhanced $\vec{B}$ field generates more currents and the steady state needs to be solved self-consistently ☺ Because these Taylor columns tend to align with the rotation axis, it explains why the geomagnetic poles are close to the geographic ones.

For the dynamo scenario to work, three criteria must be met:

1. An electrically conducting fluid medium.
2. Kinetic energy provided by the Earth.
3. Convective motion within the fluid.

It may seem a bit strange why convection is necessary. If we assume the generalized Ohm's law is true,

$$\boxed{\vec{J} = \sigma\left(\vec{E} + \vec{v}\times\vec{B}\right)} \longrightarrow \text{substitute into } \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E}$$

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla}\times\left(\vec{v}\times\vec{B} - \frac{1}{\sigma}\vec{J}\right) \longleftarrow \text{using } \begin{array}{l}\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \\ \vec{\nabla}\cdot\vec{B} = 0\end{array} \longrightarrow \vec{\nabla}\times\vec{J} = -\nabla^2\vec{B}$$

The above equation becomes the so-called induction equation,

$$\boxed{\frac{\partial \vec{B}}{\partial t} = \underset{\text{convection}}{\vec{\nabla}\times\left(\vec{v}\times\vec{B}\right)} + \underset{\text{diffusion}}{\frac{1}{\mu_0\sigma}\nabla^2\vec{B}}}$$

Without convection ($\vec{v} = 0$), it is the ordinary diffusion equation and the $\vec{B}$ field decays exponentially. It is quite interesting that convection and diffusion together give a qualitative explanation for the geomagnetic field ☺

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