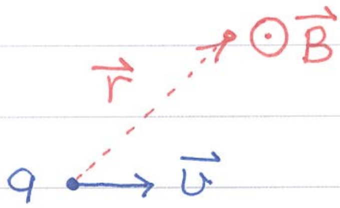
HH0121 Biot-Savart Law

The magnetic field produced by steady currents is described by the so-called magnetostatics. The corresponding Maxwell equations are

magnetostatics		electrostatics
$\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$	$\longleftrightarrow$	$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$
$\vec{\nabla}\cdot\vec{B} = 0$		$\vec{\nabla}\times\vec{E} = 0$

One observes the peculiar symmetry between $\vec{E}$ & $\vec{B}$ fields. In electrostatics, the $\vec{E}$ field generated by a point charge is described by Coulomb law. In magnetostatics, the $\vec{B}$ field from a moving point charge is captured by Biot-Savart law.



$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\frac{q\vec{v}\times\hat{r}}{r^2}}$$

the same inverse-square form

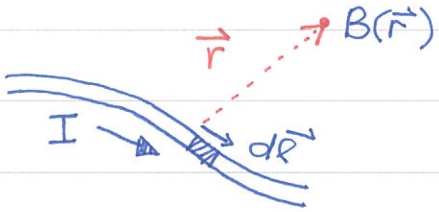
The magnetic constant μ_0 is related to the electric constant ϵ_0 $\longrightarrow$ $\epsilon_0\mu_0 c^2 = 1$

Thus, we can re-express the $\vec{B}$ field as

$$\vec{B} = \frac{1}{4\pi\epsilon_0 c^2}\,\vec{v}\times\frac{q}{r^2}\hat{r} = \frac{\vec{v}}{c^2}\times\frac{q}{4\pi\epsilon_0 r^2}\hat{r} \longrightarrow \boxed{\vec{B} = \frac{\vec{v}}{c^2}\times\vec{E}}$$

The $\vec{B}$ field generated by a moving point charge is simply related to the $\vec{E}$ field from the same point charge. This beautiful relation arises from the Lorentz transformation of $\vec{E}$ & $\vec{B}$ fields in special relativity.

⊘ Biot-Savart law in current form. For practical usage, we often encounter current I, rather than a moving charge.



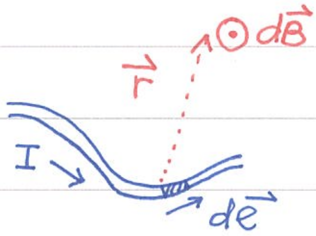
The Biot-Savart law takes the integral form.

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi}\int\frac{I\,d\vec{\ell}\times\hat{r}}{r^2}}$$

We will derive the integral form now

Recall that the current in a wire is $I = \lambda v$ where λ is the linear density.

The $\vec{B}$ field generated by a tiny segment is



$$dB = \frac{\mu_0}{4\pi}\frac{dq\,\vec{v}\times\hat{r}}{r^2}$$

need some "massage".

Note that $\vec{v} \parallel d\vec{\ell}$ in the tiny segment. $\rightarrow$ $\vec{v}\,d\ell = v\,d\vec{\ell}$

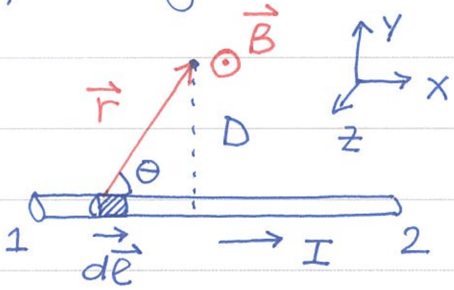
$$dq\,\vec{v} = \frac{dq}{d\ell}\,d\ell\,\vec{v} = \lambda\vec{v}\,d\ell = \lambda v\,d\vec{\ell} \rightarrow \boxed{dq\,\vec{v} = I\,d\vec{\ell}}$$

Integrate all contribution from tiny segments to obtain the $\vec{B}$ field,

$$\vec{B} = \int d\vec{B} = \frac{\mu_0}{4\pi}\int\frac{dq\,\vec{v}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\int\frac{I\,d\vec{\ell}\times\hat{r}}{r^2}$$

Biot-Savart law !!

∅ A straight wire. Let us compute the $\vec{B}$ field generated by a segment of straight wire. First of all, we find the $\vec{B}$ field along the z-axis so that $\vec{B} = (0,0,B)$. The Biot-Savart law is greatly simplified,



$$B = \frac{\mu_0}{4\pi}\int_1^2 \frac{I\,dx\cdot\sin\theta}{r^2}$$

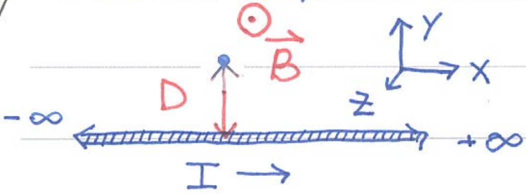
From the geometry, $r = D\csc\theta$ and $x = -D\cot\theta$. Make use of the identity $dx = -D\,d(\cot\theta) = D\csc^2\theta\,d\theta$,

$$B = \frac{\mu_0 I}{4\pi}\int_1^2 \frac{D\cancel{\csc^2\theta}\,d\theta\cdot\sin\theta}{D^2\cancel{\csc^2\theta}} = \frac{\mu_0 I}{4\pi D}\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta$$

The integral is fundamental (and thus simple ☺)

$$\boxed{B = \frac{\mu_0 I}{4\pi D}(\cos\theta_1 - \cos\theta_2)}$$ and $\vec{B} = B\hat{k}$

For an infinite wire, $\theta_1 = 0$ and $\theta_2 = \pi$.



$$B = \frac{\mu_0 I}{4\pi D}(\cos 0 - \cos\pi)$$

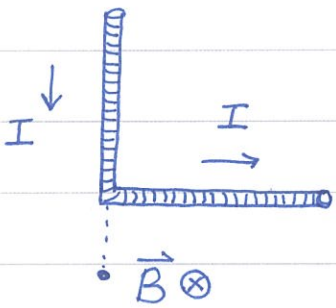
$$\rightarrow \boxed{\vec{B} = \frac{\mu_0 I}{2\pi D}\hat{k}}$$

One can compare it with the $\vec{E}$ field generated by the uniformly charged line,

$$\boxed{\vec{E} = \frac{\lambda}{2\pi\epsilon_0 D}\hat{j}} \leftrightarrow \vec{B} = \frac{I}{2\pi\epsilon_0 c^2 D}\hat{k} = \left(\frac{v}{c^2}\hat{i}\right) \times \left(\frac{\lambda}{2\pi\epsilon_0 D}\hat{j}\right)$$

That is to say, the relation $\vec{B} = (\vec{v}/c^2) \times \vec{E}$ appears again.

The above result can be applied to the semi-infinite wire.



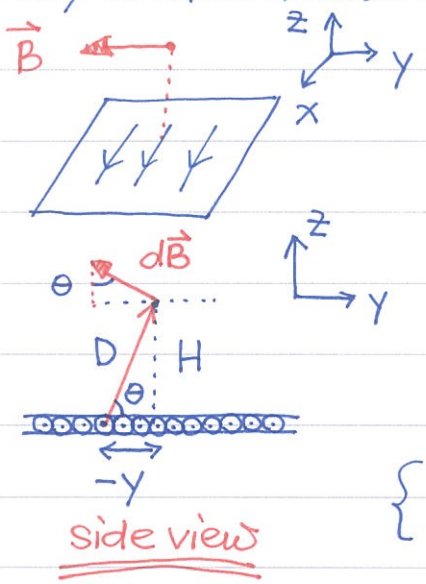
The vertical part does not contribute (why?).

The horizontal part $\rightarrow \theta_1 = \frac{\pi}{2}, \theta_2 = \pi$

$$B = \frac{\mu_0 I}{4\pi D}\left(\cancel{\cos\frac{\pi}{2}}^{\,0} - \cos\pi\right) = \frac{\mu_0 I}{4\pi D}$$

It's value is just half of that for the infinite straight wire 😊

⊘ A current sheet: Now we turn to the $\vec{B}$ field generated by uniform current density on a plane. The surface



current density $j = \frac{I}{W} = \frac{\lambda}{W}v = \sigma v$

where W is width, λ is linear density and σ is surface density. From the symmetry, only the y-component of $d\vec{B}$ survives,

$$B = \int dB \cdot \sin\theta = \frac{\mu_0}{2\pi}\int \frac{dI \cdot \sin\theta}{D}$$

$$\begin{cases} D = H\csc\theta \\ y = -H\cot\theta \end{cases} \rightarrow dy = H\csc^2\theta\, d\theta$$

The tiny current $dI = j \cdot dy = jH\csc^2\theta\, d\theta$.

$$B = \frac{\mu_0}{2\pi}\int \frac{(j\cancel{H}\cancel{\csc^2}\theta\, d\theta)\cdot \cancel{\sin\theta}}{\cancel{H}\,\cancel{\csc\theta}} = \frac{\mu_0 j}{2\pi}\int_0^{\pi} d\theta$$

$$\rightarrow \boxed{B = \frac{1}{2}\mu_0 j}$$

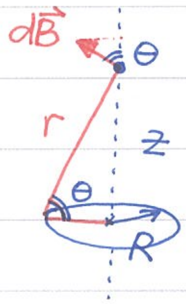
Make use of $\epsilon_0\mu_0 c^2 = 1$ and $j = \sigma v$:

$$B = \frac{1}{2}\frac{1}{\epsilon_0 c^2}\cdot\sigma v = \frac{v}{c^2}\cdot\left(\frac{\sigma}{2\epsilon_0}\right)$$

vector form $\boxed{\vec{B} = \frac{\vec{v}}{c^2}\times\vec{E}}$

For a point charge, a wire, a sheet, the $\vec{E}$ and $\vec{B}$ fields are all related by the simple equation: $\vec{B} = \vec{v}/c^2 \times \vec{E}$!

⑩ A magnet dipole. Consider a circular wire with current I. We know the magnetic dipole moment $\mu = IA = I\pi R^2$. Let us compute the $\vec{B}$ field by Biot-Savart law.



From the symmetry argument, only the z-comp. of $d\vec{B}$ survives.

$$B = \int dB\cdot\cos\theta = \frac{\mu_0 I}{4\pi}\int\frac{d\ell}{r^2}\cos\theta$$

From the geometry, $r = (z^2+R^2)^{\frac{1}{2}}$ & $\cos\theta = R/r$

$$B = \frac{\mu_0 I}{4\pi}\frac{R}{r^3}\int d\ell = \frac{\mu_0 I}{4\pi}\frac{R}{r^3}\cdot 2\pi R = \frac{\mu_0 I R^2}{2r^3}$$

$$\rightarrow \boxed{B = \frac{\mu_0 I R^2}{2(z^2+R^2)^{\frac{3}{2}}}}$$

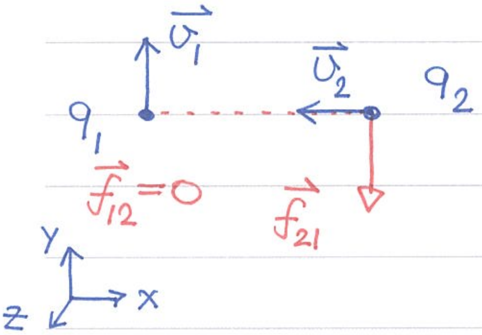
In the long-distance limit $z \gg R$, $B \approx \mu_0 I R^2/2z^3$. Recall the definition of the magnetic dipole moment,

$$\boxed{B(z) \approx \frac{\mu_0 I R^2}{2z^3} = \frac{\mu_0}{2\pi z^3}\cdot\mu \propto \frac{1}{z^3}}$$

dipole field decays as $\frac{1}{z^3}$ ☺

Note that, in the long-distance limit, the magnetic field is proportional to $IR^2 = \mu/\pi$. This is why the current loop can be treated as a magnetic dipole.

(1) Magnetic forces and Newton's 3^{rd} law. Combine the Biot-Savart law with Lorentz force ∘∘∘ We will find the magnetic forces do not follow Newton's 3^{rd} law. Here comes the example discussed in HH0086 last semester. Let us compute $\vec{f}_{12}$ first



$$\vec{f}_{12} = q_1 \vec{v}_1 \times \vec{B}_1 = q_1 \vec{v}_1 \times \left(\frac{\mu_0}{4\pi} \frac{q_2 \vec{v}_2 \times \hat{r}_{12}}{r_{12}^2} \right)$$

$$= \frac{\mu_0}{4\pi} \frac{q_1 q_2}{r_{12}^2} \vec{v}_1 \times (\cancel{\vec{v}_2 \times \hat{r}_{12}}^{\,0}) = 0$$

The Lorentz force on the second particle is

$$\vec{f}_{21} = q_2 \vec{v}_2 \times \vec{B}_2 = q_2 \vec{v}_2 \times \left(\frac{\mu_0}{4\pi} \frac{q_1 \vec{v}_1 \times \hat{r}_{21}}{r_{21}^2} \right) = \frac{\mu_0}{4\pi} \frac{q_1 q_2}{r_{12}^2} \vec{v}_2 \times (\vec{v}_1 \times \hat{r}_{21})$$

Work out the cross products carefully,

$$\boxed{\vec{f}_{21} = -\frac{\mu_0}{4\pi} \frac{q_1 q_2}{r_{12}^2} v_1 v_2 \hat{j} = -\left(\frac{q_1 q_2}{4\pi\epsilon_0 r_{12}^2} \right) \cdot \frac{v_1}{c} \cdot \frac{v_2}{c} \hat{j}}$$

The above form of the magnetic force gives us the hint that the magnetic force can be viewed as relativistic correction of the electric force ☺

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