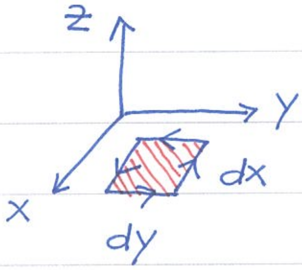
HH0122 Ampere Law

In magnetostatics, one can find the magnetic field by Biot-Savart law (HH0121). Or, one can write the corresponding Maxwell equation in integral form.

$$\vec{\nabla}\times\vec{B}=\mu_0\vec{J} \longrightarrow \boxed{\oint \vec{B}\cdot d\vec{r}=\mu_0 I}$$

Ampere law ☺

Let us try to derive the integral form by choosing a tiny loop.



The circulation consists of 4 parts:

$$\oint \vec{B}\cdot d\vec{r} = \left[B_x(y)-B_x(y+dy)\right]dx + \left[B_y(x+dx)-B_y(x)\right]dy$$

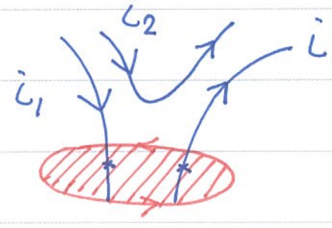
$$\oint \vec{B}\cdot d\vec{r} = \left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)dx\,dy = \left(\vec{\nabla}\times\vec{B}\right)_z dx\,dy$$

Write the above result in vector form,

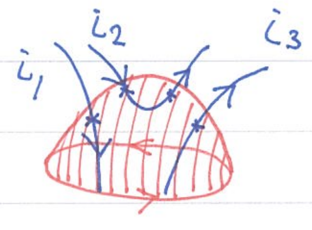
$$\boxed{\oint \vec{B}\cdot d\vec{r} = (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \mu_0\vec{J}\cdot d\vec{a} = \mu_0 I}$$

Ampere law

When using Ampere law, there are different choices of surfaces for the same Amperian loop. Let us consider the following example and see how Ampere law works ☺



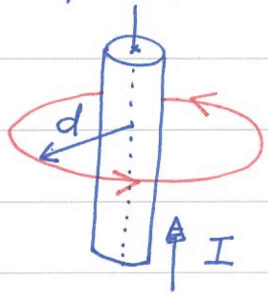
$$\oint \vec{B}\cdot d\vec{r}=\mu_0 I = \mu_0(i_3-i_1)$$



$$\oint \vec{B}\cdot d\vec{r}=\mu_0 I = \mu_0(i_3+\cancel{i_2}-\cancel{i_2}-i_1)$$

As long as the Amperian loop is the same, the enclosed current is always the same despite of diff. chosen surfaces!

① Long straight wire. Consider a long straight wire of radius R, carry current I uniformly. The current density $J = I/A = I/\pi R^2$ is constant. Let us compute the $\vec{B}$ field by Ampere law. For points outside the wire,

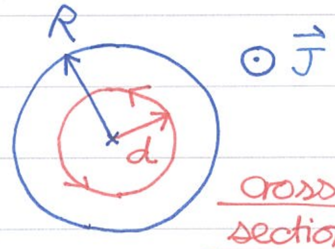


By symmetry argument, the $\vec{B}$ field is along the tangent direction,

$$\oint \vec{B}\cdot d\vec{r} = \mu_0 I \rightarrow B\cdot 2\pi d = \mu_0 I$$

$$\rightarrow B = \frac{\mu_0 I}{2\pi d}$$ the same as derived by Biot-Savart law

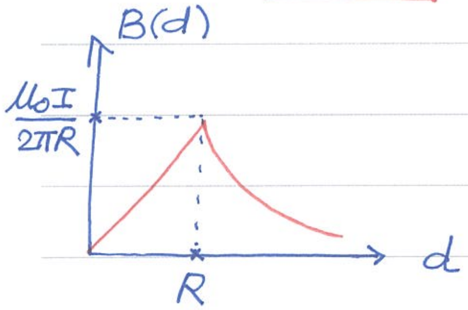
What happens for points inside the wire?



cross section

$$\oint \vec{B}\cdot d\vec{r} = \mu_0 I \rightarrow B\cdot 2\cancel{\pi}\cancel{d} = \mu_0 J\cdot \cancel{\pi} d^2$$

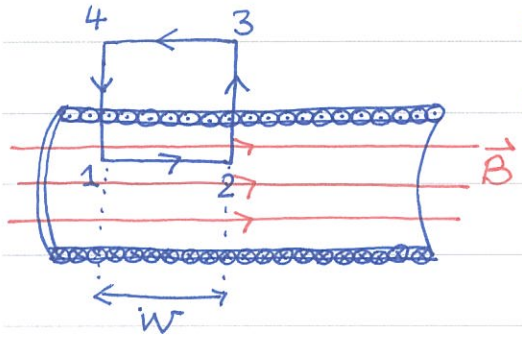
$$\rightarrow B = \frac{1}{2}\mu_0 J d = \frac{\mu_0 I}{2\pi R^2} d \propto d\,!$$



The $\vec{B}$ field at the center of the wire is zero, increasing linearly up to $d=R$, then decreasing as $1/d$ outside the wire.

Try to obtain the result by Biot-Savart law — you shall find it rather messy

Ⓓ Long solenoid. Consider an ideal solenoid — the $\vec{B}$ field inside is uniform and vanishes outside the solenoid. Choose the Amperian loop as shown,



$$\oint \vec{B}\cdot d\vec{r} = \mu_0 I\cdot\left(\frac{W}{L}N\right)$$

wire number inside the Amperian loop

The circulation can be separated into 4 parts:

$$\oint \vec{B}\cdot d\vec{r} = \int_1^2 \vec{B}\cdot d\vec{r} + \int_2^3 \vec{B}\cdot d\vec{r} + \int_3^4 \vec{B}\cdot d\vec{r} + \int_4^1 \vec{B}\cdot d\vec{r}$$

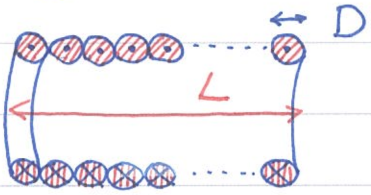
$$= B\cdot W + 0 + 0 + 0 \rightarrow \oint \vec{B}\cdot d\vec{r} = BW$$

Thus, Ampere law tells us:

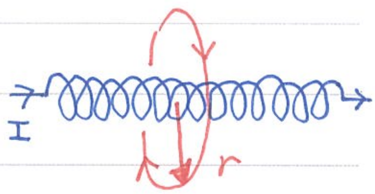
$$B \cdot \cancel{W} = \mu_0 I \cdot \left(\frac{N}{L}\right) \cancel{W} \longrightarrow \boxed{B = \mu_0 I \frac{N}{L}}$$

We can estimate the $\vec{B}$ field outside to see the above approximation is reasonable or not

Suppose the solenoid is tightly packed spirally. The total length $L = ND$ where D is the diameter of the wire. To estimate the $\vec{B}$ field outside, one can choose the following Amperian loop.



$$B_{out} \cdot 2\pi r = \mu_0 I$$

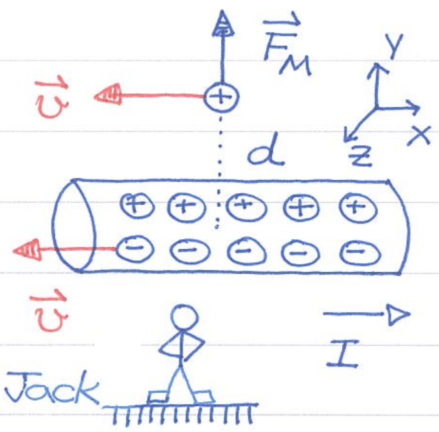


Because the wire is spiraling, there is one wire passing through the surface. That's why we put just $\mu_0 I$ on the right-hand side.

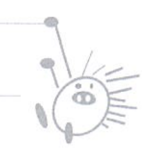
$$\boxed{B_{out} = \frac{\mu_0 I}{2\pi r}} \longrightarrow \frac{B_{out}}{B_{in}} = \frac{\cancel{\mu_0 I}/2\pi r}{\cancel{\mu_0 I} N/L} = \frac{L}{2\pi r N} = \frac{D}{2\pi r}$$

Usually, the diameter of the wire is rather small $D \ll r$. The $\vec{B}$ field outside can be safely ignored

Electromagnetism and relativity. Consider the magnetic force on a moving charge as shown below. The linear charge densities $\lambda_+ = \lambda_- = \lambda_0$ so that the total charge density is zero. $\rightarrow \vec{E} = 0$.



$$\boxed{\begin{aligned} B &= \frac{\mu_0 I}{2\pi d} \longrightarrow \vec{B} = \frac{\mu_0 \lambda_0 v}{2\pi d}\hat{z} \\ \vec{F}_M &= q\vec{v} \times \vec{B} = \frac{\mu_0 q \lambda_0}{2\pi d} v^2 \hat{y} \end{aligned}}$$



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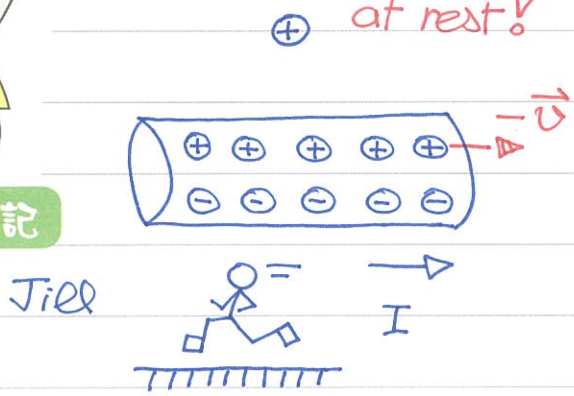
Now switch to Jill's moving frame. The charge is at rest and thus the magnetic force $\vec{F}_M = 0$! What happens to the repulsive force between the point charge and the wire ?



It turns out the charge densities change due to length contraction.

$\lambda_0 = Q_0/L_0$ $\qquad$ $\lambda = Q/L$

(1) Charge conservation gives $Q = Q_0$

(2) Lorentz contraction leads to $L = \sqrt{1 - v^2/c^2}\, L_0$

Thus,
$$\boxed{\lambda = \frac{1}{\sqrt{1 - v^2/c^2}} \lambda_0}$$
charge density is larger for moving charges.

In Jill's frame, the positive charges move $\rightarrow \lambda_+ > \lambda_0$

The negative charges become at rest $\rightarrow \lambda_- < \lambda_0$

$$\boxed{\begin{aligned} \lambda_+ &= \frac{1}{\sqrt{1 - v^2/c^2}} \lambda_0 \\ \lambda_- &= \sqrt{1 - v^2/c^2}\, \lambda_0 \end{aligned}}$$

The total charge density $\lambda = \lambda_+ - \lambda_-$

$$\lambda = \left(\frac{1}{\sqrt{1 - v^2/c^2}} - \sqrt{1 - v^2/c^2} \right) \lambda_0$$

$$= \frac{1}{\sqrt{1 - v^2/c^2}} \left[1 - \left(1 - \frac{v^2}{c^2}\right) \right] \lambda_0 \neq 0 !$$

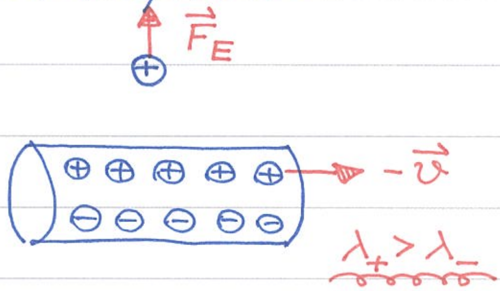
The wire now carries a slight charge density $\lambda > 0$ due to relativistic correction

$$\boxed{\lambda = \frac{1}{\sqrt{1 - v^2/c^2}} \cdot \frac{v^2}{c^2} \lambda_0 \approx \left(\frac{v}{c}\right)^2 \lambda_0}$$
not charge neutral anymore ...

The electric field generated by λ is

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0 d}\,\hat{y}$$

$$= \frac{\lambda_0}{2\pi\epsilon_0 d}\left(\frac{v}{c}\right)^2 \hat{y}$$



The electric force on the charge can be easily computed,

$$\boxed{\vec{F}_E = q\vec{E} = \frac{q\lambda_0}{2\pi\epsilon_0 d}\left(\frac{v}{c}\right)^2 \hat{y}}$$

Making use of the identity, $\epsilon_0\mu_0 c^2 = 1$. The comparison between $\vec{F}_M$ and $\vec{F}_E$ becomes transparent ☺

$$\boxed{\vec{F}_M\,(\text{Jack}) = \frac{\mu_0 q\lambda_0}{2\pi d}\,v^2\hat{y} = \frac{q\lambda_0}{2\pi\epsilon_0 d}\left(\frac{v}{c}\right)^2\hat{y} = \vec{F}_E\,(\text{Jill})}$$

It is remarkable that magnetic force in Jack's frame turns into electric force in Jill's frame. Thus, the $\vec{B}$ field can be viewed as relativistic correction of the $\vec{E}$ field. To make full sense of relativity theory, we must treat both fields together → $E_x, E_y, E_z, B_x, B_y, B_z$ 6-component field. Not a scalar, not a vector, so, WHAT IS IT ?

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