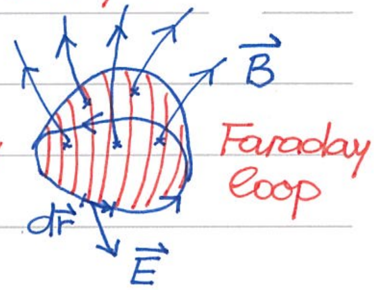
HH0125 Faraday's Law

We first show the equivalence between differential and integral forms of the Maxwell equation:

$$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}} \quad \longleftrightarrow \quad \boxed{\oint \vec{E}\cdot d\vec{r} = -\frac{d\Phi_B}{dt}}$$

Faraday's law

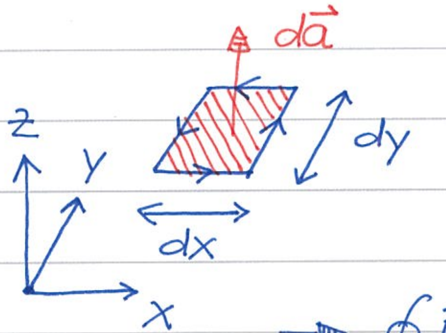
The circulation is calculated around a Faraday loop and the flux is



$$\Phi_B \equiv \int \vec{B}\cdot d\vec{a}$$

the surface enclosed by the Faraday loop

By now, you should be quite familiar with the trick of the "tiny square". The circulation of the $\vec{E}$ field contains 4 parts:



$$\oint_{\square} \vec{E} = \left[-E_x(y+dy) + E_x(y)\right]dx + \left[E_y(x+dx) - E_y(x)\right]dy$$

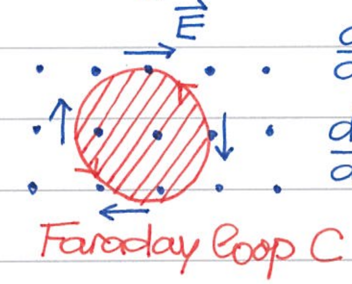
$$\rightarrow \oint \vec{E}\cdot d\vec{r} = \left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right]dx\,dy = (\vec{\nabla}\times\vec{E})\cdot d\vec{a}$$

Make use of the Maxwell equation (differential form),

$$\oint \vec{E}\cdot d\vec{r} = -\frac{\partial \vec{B}}{\partial t}\cdot d\vec{a} = -\frac{d}{dt}(\vec{B}\cdot d\vec{a}) \rightarrow \boxed{\oint \vec{E}\cdot d\vec{r} = -\frac{d\Phi_B}{dt}}$$

The integral form implies that the $\vec{E}$ field is not conservative when the $\vec{B}$ field changes with time!

⊘ Electric field generated by Faraday's law. Consider an increasing $\vec{B}$ field along the z-axis. Apply Faraday's law,



$$\frac{dB}{dt} > 0 \qquad \frac{d\Phi_B}{dt} > 0$$

$$\boxed{\oint \vec{E}\cdot d\vec{r} = -\frac{d\Phi_B}{dt}}$$

$$-E\cdot 2\pi r = -\pi r^2 \frac{dB}{dt}$$

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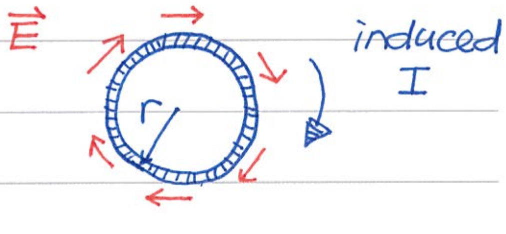
The $\vec{E}$ field generated by the changing $\vec{B}$ field is

$$E = \frac{\pi r^2}{2\pi r}\frac{dB}{dt} \longrightarrow \boxed{E(r) = \frac{1}{2} r \frac{dB}{dt}}$$

Suppose we place a circular wire of resistance R inside the changing $\vec{B}$ field. What's the induced current I in the circular wire?



According to Ohm's law (HH0119)

$$\boxed{\vec{J} = \sigma \vec{E}} \quad \text{where } R = \left(\frac{1}{\sigma}\right)\frac{L}{A}.$$

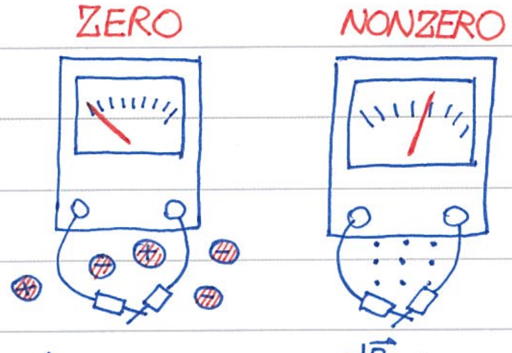
Put in the above geometric relation to compute the current,

$$I = JA = \sigma E \cdot A = \frac{1}{R}(E \cdot L) = \frac{1}{R}(E \cdot 2\pi r)$$

It is convenient to introduce the electromotive force (EMF)

$$\boxed{\mathcal{E} = \oint \vec{E} \cdot d\vec{r}} \longrightarrow \mathcal{E} = E \cdot 2\pi r \longrightarrow \boxed{I = \frac{\mathcal{E}}{R}}$$

It seems that the circular wire with EMF $\mathcal{E}$ is very similar to a normal circuit. But, there are important differences!



In electrostatics, a short-circuited voltmeter always gives ZERO, because $\oint \vec{E} \cdot d\vec{r} = 0$

However, if the magnetic flux changes through the closed circuit, the voltmeter shows NONZERO reading!

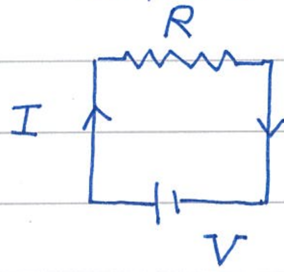
The key difference is the value of EMF. In the presence of changing $\vec{B}$ field, the EMF is not zero,

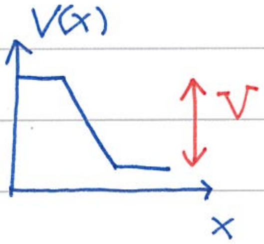
WRONG

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{r} \neq 0 \longrightarrow \cancel{\Delta V = -\int_{\Gamma} \vec{E} \cdot d\vec{r}}$$

potential is ill-defined....

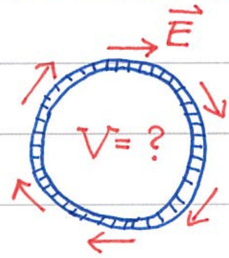
Compare $V = IR$ circuit and $\mathcal{E} = IR$ Faraday loop.

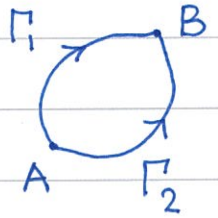




For the ordinary circuit, the electric potential along the circuit is well defined.

On the other hand, for a Faraday loop, the circulation of the $\vec{E}$ field is NOT zero. Thus, the potential is ill-defined.





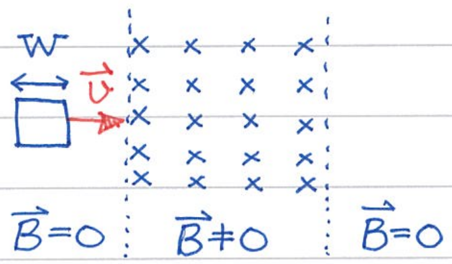
$$\boxed{\int_{\Gamma_1} \vec{E}\cdot d\vec{r} \neq \int_{\Gamma_2} \vec{E}\cdot d\vec{r}}$$

potential NOT defined....

Because we know the energy can be expressed in terms of the field, it is not the end of the day without the electric potential — the $\vec{E}$ field is still there ☺

⑩ Lenz law and magnetic brake. Lenz provided a useful rule to determine the direction of the induced current: the current is always against the flux change ☺

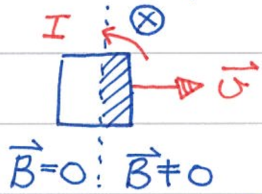
Let us consider a square loop moving acorss a region of constant $\vec{B}$ field. Assume the resistance of the loop is R.



① entering stage:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = +B\frac{dA}{dt}$$

$$\longrightarrow \boxed{\mathcal{E} = BWv}$$



$\mathcal{E}>0$, counterclockwise induced current.

The induced current is

$$\boxed{I = \frac{\mathcal{E}}{R} = \frac{W}{R}vB} \longrightarrow P = I^2R = \frac{W^2}{R}v^2B^2$$

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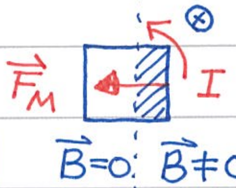
To maintain the constant velocity, an external force $\vec{F}_{ex}$ is necessary to balance the magnetic force $\vec{F}_M$:

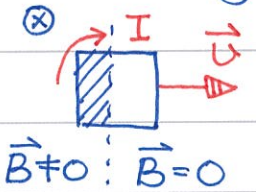


$$F_M = IWB = \underbrace{\frac{W^2}{R}}B^2 v$$ ← pulling back.

Because $\vec{F}_M + \vec{F}_{ex} = 0$, the external force provides a positive power into the square loop.

$$P_{ex} = F_{ex} \cdot v = \underbrace{\frac{W^2}{R}} v^2 B^2$$ the same as the Ohmic dissipation in the loop ☺

② exiting stage : By Lenz law, a clockwise I is induced.

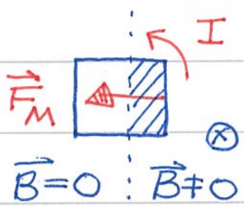


$$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt} = B\frac{dA}{dt} = -BWv < 0}$$

The induced current is the same except the direction is opposite. One can check the input power is again equal to output power $P = I^2R$. The interesting point is the magnetic force $\vec{F}_M$ is opposite to the velocity $\vec{v}$.

→ $\boxed{\vec{F}_M \cdot \vec{v} < 0}$ Because it tends to reduce the speed, the phenomenon is referred as "magnetic brake". Furthermore, $F_M = (W^2B^2/R)\, v \propto v$. It is particularly useful for objects moving at high speed ☺

Ⅱ Transformation of energies. Consider the same moving square at the entering stage but the external force is removed. The velocity $\vec{v} = \vec{v}(t)$ now changes with time. Write down the EOM for the system.



$$m\frac{dv}{dt} = F_M = -\underbrace{\left(\frac{W^2B^2}{R}\right)}v$$ ← like frictional force ☺

Introduce braking time $\underbrace{\tau_b = \frac{mR}{W^2B^2}}$ → $$\boxed{\frac{dv}{dt} = -\frac{1}{\tau_b}v}$$ YES!

The velocity $v(t)$ can be solved from the EOM,

$$\boxed{v(t) = v_0\, e^{-t/\tau_b}}$$

The velocity decreases exponentially! magnetic braking

One can compute the kinetic energy and its decreasing rate,

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m v_0^2\, e^{-2t/\tau_b} \longrightarrow \boxed{\frac{dK}{dt} = -\frac{m v_0^2}{\tau_b}\, e^{-2t/\tau_b}}$$

Recall the definition of braking time $\tau_b = mR/W^2B^2$,

$$\longrightarrow \boxed{\frac{dK}{dt} = -\frac{W^2B^2v_0^2}{R}\, e^{-2t/\tau_b}}$$

But, where does the energy go?

Compute the thermal energy rate from Ohmic dissipation,

$$\frac{dQ}{dt} = I^2 R = \frac{\mathcal{E}^2}{R} = \frac{W^2B^2}{R} v_0^2\, e^{-2t/\tau} \longrightarrow \boxed{\frac{dQ}{dt} + \frac{dK}{dt} = 0}$$

It is quite remarkable that Faraday's law transforms the mechanical energy into thermal energy through Ohmic dissipation of the induced current $I(t)$.

mechanical energy (K) + thermal energy (Q) = const

The energy is conserved, not only for Newtonian mechanics but also for the electromagnetic fields governed by the Maxwell equations!

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