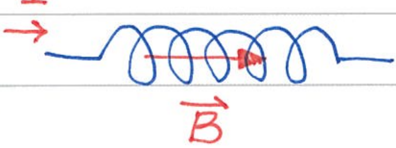
HH0126 Energy in Magnetic Field

An inductor is a circuit element that stores energy in the $\vec{B}$ field. Consider an ideal solenoid carrying current I.



The inductance is defined to be the proportionality constant that relates the current change $\frac{dI}{dt}$ to the EMF $\mathcal{E}$:

$$\boxed{\mathcal{E} = L\frac{dI}{dt}}$$

According to Faraday's law, $\mathcal{E} = -d\Phi_B/dt$. Ignore the sign momentarily,

$$L\frac{dI}{dt} = \frac{d\Phi_B}{dt} \rightarrow LI = \Phi_B \rightarrow \boxed{L = \frac{\Phi_B}{I}}$$

We can now proceed to compute the inductance of the ideal solenoid ☺

$$B = \mu_0 NI/\ell \rightarrow \Phi_B = \overset{\text{N coils!}}{N}BA = (\mu_0 N^2 A/\ell)\, I$$

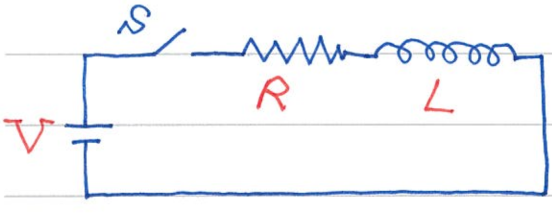
Define $n = N/\ell$ is the number of turns per unit length,

$$\boxed{L = \frac{\Phi_B}{I} = \mu_0 A \ell n^2}$$

Like capacitance, the inductance only involves geometric factors — the cross-sectional area A and the number of turns per unit length n. And, L is proportional to ℓ as expected ☺

⓪ LR circuit

Let us study a circuit with R and L in series as shown below. We close the switch S to complete the circuit at $t=0$.



$$\boxed{V - IR - L\frac{dI}{dt} = 0}$$

$$\rightarrow L\frac{dI}{dt} = -R\left(I - \frac{V}{R}\right) \rightarrow \frac{dI}{dt} = -\left(\frac{R}{L}\right)\left(I - \frac{V}{R}\right)$$

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Introduce the parameters : $I_\infty = V/R$ and $\tau_L = L/R$

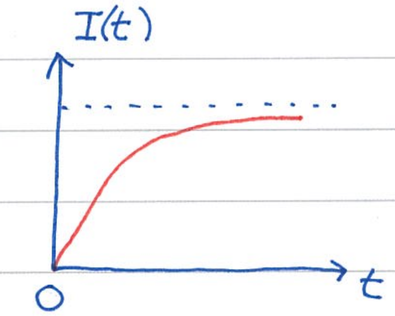
$$\boxed{\frac{dI}{dt} = -\frac{1}{\tau_L}(I - I_\infty)} \rightarrow I - I_\infty = \text{const.}\, e^{-t/\tau_L}$$

Because the switch is turned on at $t=0$, we expect that the initial current is zero, i.e. $I(0)=0$

The const above is thus $-I_\infty$ and the solution becomes

$$\boxed{I(t) = I_\infty - I_\infty e^{-t/\tau_L} = I_\infty (1 - e^{-t/\tau_L})}$$

what is the meaning of I_∞?



① Near $t=0$, $I \approx 0$. The voltage drop is mainly due to the inductor

$$\rightarrow V \approx L\frac{dI}{dt}$$

L dominates!

② In the long-time limit, $t \to \infty$, $dI/dt \approx 0$ and the voltage drop is from the resistor R.

$$\rightarrow V \approx IR$$

R dominates ☺ Note that L is basically invisible for steady I ☺

⑩ Energy stored in $\vec{B}$ field. Let us mutiply the differential equation of the LR circuit by current I. The LHS is

$$\boxed{IV = I^2R + LI\frac{dI}{dt}}$$

just the power provided by the battery. The I^2R term is the Ohmic dissipation power into thermal energy. Thus, we expect the $LI\,dI/dt$ term must present the power into the inductor. Let U_B represent the energy stored in the magnetic field,

$$\boxed{\frac{dU_B}{dt} = LI\frac{dI}{dt}} \quad \text{or } dU_B = LI\,dI$$

At $t=0$, there is no current (and thus no magnetic field) in the circuit. It is natural to set $U_B = 0$ when $I=0$.

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The energy stored in the $\vec{B}$ field can be computed by integrating the differentials,

$$\int_0^{U_B} dU_B = L\int_0^{I} I\,dI \longrightarrow \boxed{U_B = \frac{1}{2}LI^2}$$

Compare the above result with the energy stored in the $\vec{E}$ field of a capacitor, $U_E = \frac{1}{2C}Q^2$

One sees the similarity between U_E and U_B clearly. Because we can express U_E in terms of the electric field directly, we anticipate a similar expression should be available for the $\vec{B}$ field.

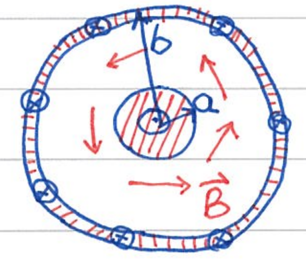
$$U_B = \frac{1}{2}LI^2 = \frac{1}{2}\mu_0 A\ell\, n^2 I^2 = \frac{1}{2\mu_0}(\mu_0 n I)^2 \cdot A\ell$$

Note that $\mu_0 n I = B$ is just the field inside the solenoid and $A\ell$ is the volume where $\vec{B} \neq 0$. Thus, the energy density in the $\vec{B}$ field u_B can be found,

$$\boxed{u_B \equiv \frac{U_B}{V} = \frac{1}{2\mu_0}B^2} \longleftrightarrow u_E = \frac{1}{2}\epsilon_0 E^2$$

Again, the energy can be expressed in terms of the field without refering to the source.

Use a long coaxial cable as another demonstrating example. The currents in the central and the outer parts are the same but in opposite direction.



$$\oint \vec{B}\cdot d\vec{r} = \mu_0 I \longrightarrow B = \frac{\mu_0 I}{2\pi r}$$

The energy density inside the cable is

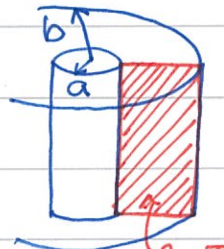
$$\boxed{u_B = \frac{1}{2\mu_0}B^2 = \frac{\mu_0 I^2}{8\pi^2 r^2}}$$

The total energy U_B can be computed,

$$U_B = \int dV\, u_B = \int_a^b \ell \cdot 2\pi \cancel{r}\, dr \cdot \frac{\mu_0 I^2}{8\pi^2 r^{\cancel{2}}}$$

$$= \frac{\mu_0 I^2 \ell}{4\pi} \int_a^b \frac{dr}{r} \longrightarrow \boxed{U_B = \frac{\mu_0 I^2 \ell}{4\pi} \ln \frac{b}{a}}$$

On the other hand, one can try to compute U_B by inductance.



$$\Phi_B = \int B\, dA = \int_a^b \frac{\mu_0 I}{2\pi r} \ell \cdot dr = \frac{\mu_0 \ell I}{2\pi} \int_a^b \frac{dr}{r}$$

$$\longrightarrow \Phi_B = \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{b}{a}\right) \cdot I \longrightarrow \boxed{L = \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{b}{a}\right)}$$

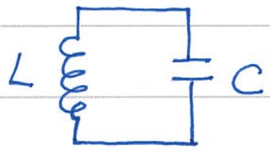
Φ_B through this area ☺

Making use of the formula, $U_B = \frac{1}{2} L I^2$,

$$U_B = \frac{1}{2} \cdot \frac{\mu_0 \ell}{2\pi} \ln\left(\frac{b}{a}\right) I^2 = \frac{\mu_0 \ell I^2}{4\pi} \ln \frac{b}{a} \longleftarrow \text{the same result ☺}$$

Even though we just work out two examples (solenoid and coaxial cable), the expression of the energy stored in the $\vec{B}$ field, $u_B = (1/2\mu_0) B^2$ is true in general.

∅ Electromagnetic oscillation. Consider a LC circuit without any dissipation. The voltage accross the capacitor equals the EMF of the inductor;



$$\boxed{\frac{Q}{C} + L \frac{dI}{dt} = 0} \longrightarrow \frac{Q}{C} + L \frac{d^2 Q}{dt^2} = 0$$

Introduce the EM oscillation frequency of the circuit,

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}}} \longrightarrow \frac{d^2 Q}{dt^2} + \omega_0^2 Q = 0$$

just like the SHO ☺

The solution of the above differential equation is

$$\boxed{Q(t) = Q_M \cos(\omega_0 t + \phi)}$$

Q_M is the maximum charge in the capacitor.

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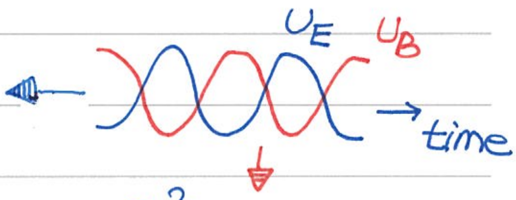
By differentiation, the oscillating current is

$$\boxed{I(t) = \frac{dQ}{dt} = -\omega_0 Q_M \sin(\omega_0 t + \phi)}$$

We are ready to compute U_E and U_B in the LC circuit now:



$$U_E = \frac{1}{2C} Q^2 = \frac{Q_M^2}{2C} \cos^2(\omega_0 t + \phi)$$

$$U_B = \frac{1}{2} L I^2 = \frac{1}{2} L \omega_0^2 Q_M^2 \sin^2(\omega_0 t + \phi) = \frac{Q_M^2}{2C} \sin^2(\omega_0 t + \phi)$$

It is easy to see that the total energy is constant,

$$\boxed{U = U_E + U_B = \frac{Q_M^2}{2C}\left[\cos^2(\omega_0 t + \phi) + \sin^2(\omega_0 t + \phi)\right] = \frac{Q_M^2}{2C}}$$

The energies in the electric and the magnetic fields transform into each other and lead to the electromagnetic oscillation.

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