HH0127 Maxwell Equations

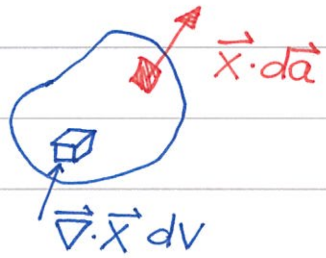
Classical electromagnetic phenomena are described by the set of FOUR Maxwell equations:

$\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$	$\vec{\nabla}\cdot\vec{B} = 0$
$\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$	$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t}$

The differential form is good for analytic calculations. But, it is insightful to write these equations in equivalent integral form so that the geometric features of the field lines are intuitive and manifest.

⓪ Gauss' theorem and Stokes' theorem. In previous notes, we use "tiny cube" and "tiny square loop" to derive the integral form of Maxwell equations. The "tiny cube" method is the simplified version of Gauss' theorem:

$$\boxed{\oint \vec{X}\cdot d\vec{a} = \int \vec{\nabla}\cdot\vec{X}\, dV}$$



It is quite remarkable that the volume integral of $\vec{\nabla}\cdot\vec{X}$ (scalar field) equals the surface integral of $\vec{X}$ (vector field). Let us derive the integral form for the divergence equations.

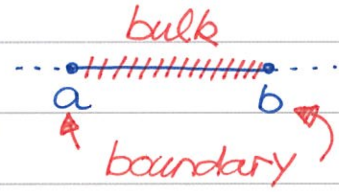
$$\oint \vec{E}\cdot d\vec{a} = \int \vec{\nabla}\cdot\vec{E}\, dV = \int \frac{\rho}{\epsilon_0} dV = Q/\epsilon_0$$

$$\oint \vec{B}\cdot d\vec{a} = \int \vec{\nabla}\cdot\vec{B}\, dV = 0$$

The derivations become simple and clear. The Gauss' theorem may look scary at first glance. But, it is quite similar to the fundamental theorem of calculus. Let us try to compare them to spot the similarity.

The fundamental theorem of calculus states

$$\underbrace{F(b)-F(a)}_{\text{boundary}} = \underbrace{\int_a^b \frac{dF}{dx}\cdot dx}_{\text{bulk.}}$$



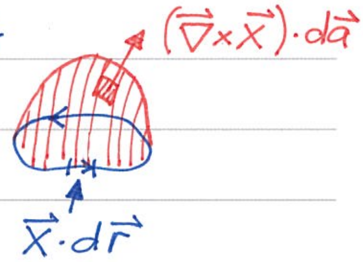
Loosely speaking, we can interpret the theorem as

$$\boxed{\text{boundary sum of } F \quad = \quad \text{bulk sum of its derivative } \frac{dF}{dx}}$$

One can use the same spirit to interpret the Gauss' theorem ☺

What about the "tiny square loop" method? Is there a rigorous theorem behind the curtain? YES ☺

$$\boxed{\oint \vec{X}\cdot d\vec{r} = \int (\vec{\nabla}\times\vec{X})\cdot d\vec{a}}$$



This is the famous Stokes' theorem. Note that the theorem bears the same signature as the fundamental theorem of calculus. Let us derive the integral form for the curl equations.

$$\oint \vec{E}\cdot d\vec{r} = \int (\vec{\nabla}\times\vec{E})\cdot d\vec{a} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a} = \underbrace{-\frac{d\Phi_B}{dt}}.$$

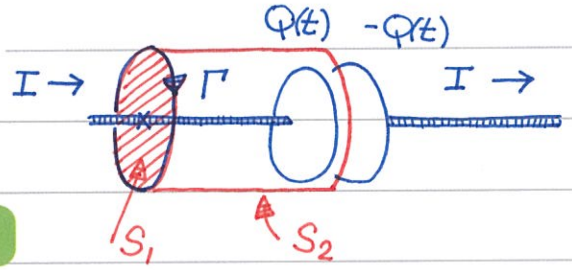
$$\oint \vec{B}\cdot d\vec{r} = \int (\vec{\nabla}\times\vec{B})\cdot d\vec{a} = \int \left(\mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{a}$$

$$\rightarrow \quad \underbrace{\oint \vec{B}\cdot d\vec{r} = \mu_0 I + \frac{1}{c^2}\frac{d\Phi_E}{dt}}$$

As expected, the Stokes' theorem bridges the two forms of Maxwell equations elegantly ☺

(II) Maxwell's correction to Ampere's law. Maxwell found a simple example violating Ampere's law – some correction is needed ☺

Consider an open circuit with a capacitor as shown on the left. The charge $Q = Q(t)$ on the capacitor satisfies,



$$\boxed{\frac{dQ}{dt} = I}$$ charge conservation

Let us choose the Amperian loop Γ with two different surfaces S_1 and S_2. The Ampere's law

$$\oint_\Gamma \vec{B}\cdot d\vec{r} = \begin{cases} \mu_0 I & \text{for } S_1 \\ 0 & \text{for } S_2 \end{cases}$$

runs into BIG trouble.... ← different answers?

Our intuition tells us that $B = \mu_0 I / 2\pi r$ away from the capacitor. Thus, something is missing when the surface S_2 is chosen,

$$\oint \vec{B}\cdot d\vec{r} = \mu_0 I + (\text{something})$$

We would like to figure out the "something".

The electric field inside the capacitor is $E = \sigma/\epsilon_0$ and the corresponding flux is $\Phi_E = \int \vec{E}\cdot d\vec{a} = \sigma/\epsilon_0 \cdot A = Q/\epsilon_0$

$$\frac{d\Phi_E}{dt} = \frac{1}{\epsilon_0}\frac{dQ}{dt} = \mu_0 c^2 \cdot \frac{dQ}{dt} = c^2 \cdot \mu_0 I$$

$$\text{something} = \frac{1}{c^2}\frac{d\Phi_E}{dt}$$

Because we expect that something $= 0$ outside the capacitor and something $= \mu_0 I$ inside, it is reasonable to guess that "something" is $(\frac{1}{c^2})\, d\Phi_E/dt$. Thus, the Ampere-Maxwell law is

$$\boxed{\oint \vec{B}\cdot d\vec{r} = \mu_0 I + \frac{1}{c^2}\frac{d\Phi_E}{dt}} \longrightarrow \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}.$$

Somehow, charge conservation must be embedded in Maxwell equations ☺

(1) Charge Conservation. We start with the identity

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{X}) = 0$$

The identity can be worked out easily in Cartesian coordinates ☺

Apply the identity to Ampere-Maxwell law,

$$\underbrace{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B})}_{=\text{ zero}} = \mu_0\,\vec{\nabla}\cdot\vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{E})$$

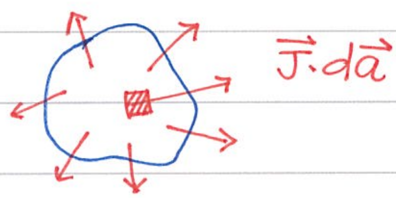
Make use of $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0 = \mu_0 c^2 \rho$,

$$\mu_0\,\vec{\nabla}\cdot\vec{J} + \frac{1}{\cancel{c^2}}\mu_0 \cancel{c^2}\frac{\partial\rho}{\partial t} = 0 \longrightarrow \boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0}$$

The differential equation derived in above implies charge conservation ☺ To see this, it is helpful to write down its integral form.

$$\int \vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t}\cdot dV = 0 \longrightarrow \boxed{\int \vec{J}\cdot d\vec{a} + \frac{dQ_{in}}{dt} = 0}$$

The surface integral $\int \vec{J}\cdot d\vec{a}$ represents the charges flowing out the surface per unit time, i.e.



$$\int \vec{J}\cdot d\vec{a} = \frac{dQ_{out}}{dt} \longrightarrow \frac{dQ_{out}}{dt} + \frac{dQ_{in}}{dt} = 0 \longrightarrow Q_{total} = \text{Const}$$

The total charge $Q_{total} = Q_{out} + Q_{in}$ is constant! We just derive the charge conservation law from Maxwell equations.

Ⓓ Energy conservation. What about energy conservation? From previous notes HH0112 + HH0126, the energy per unit volume (energy density) of the electromagnetic field is

$$\boxed{u = u_E + u_B = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2}$$

We are curious about its time dependence,

$$\frac{\partial u}{\partial t} = \frac{\epsilon_0}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E}) + \frac{1}{2\mu_0}\frac{\partial}{\partial t}(\vec{B}\cdot\vec{B})$$

Eliminate $\frac{\partial\vec{E}}{\partial t}, \frac{\partial\vec{B}}{\partial t}$ by Maxwell eqs

$$= \epsilon_0\frac{\partial\vec{E}}{\partial t}\cdot\vec{E} + \frac{1}{\mu_0}\frac{\partial\vec{B}}{\partial t}\cdot\vec{B}$$

The changing rate of the energy density is

$$\frac{\partial u}{\partial t} = \epsilon_0 c^2 \left(-\mu_0 \vec{J} + \vec{\nabla}\times\vec{B}\right)\cdot\vec{E} + \frac{1}{\mu_0}\left(-\vec{\nabla}\times\vec{E}\right)\cdot\vec{B}$$

The constants can be simplified by $\epsilon_0 \mu_0 c^2 = 1$

$$\frac{\partial u}{\partial t} + \vec{E}\cdot\vec{J} + \frac{1}{\mu_0}\left[(\vec{\nabla}\times\vec{E})\cdot\vec{B} - \vec{E}\cdot(\vec{\nabla}\times\vec{B})\right] = 0$$

Introduce the Poynting vector $\vec{S}$ for the electromagnetic field,

$$\boxed{\vec{S} \equiv \frac{1}{\mu_0}\vec{E}\times\vec{B}}$$

vector identity $\vec{\nabla}\cdot(\vec{X}\times\vec{Y}) = (\vec{\nabla}\times\vec{X})\cdot\vec{Y} - \vec{X}\cdot(\vec{\nabla}\times\vec{Y})$

Make use of the above vector identity,

$$\boxed{\frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} + \vec{E}\cdot\vec{J} = 0}$$

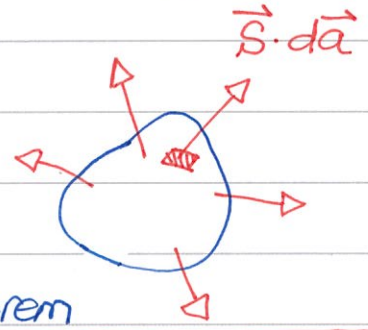
similar to the continuity eq. for charge conservation

The above equation is referred as the Poynting theorem. To reveal its meaning, it is helpful to use the integral form.

① Consider a regime where $\vec{J} = 0$.

$$\int dv \frac{\partial u}{\partial t} + \vec{\nabla}\cdot\vec{S} + \cancel{\vec{E}\cdot\vec{J}}^{\,0} = 0$$

just fields



Convert the volume integral by Gauss' theorem

$$\frac{\partial}{\partial t}\left(\int dv\, u\right) + \int d\vec{a}\cdot\vec{S} = 0 \longrightarrow \boxed{\frac{dU_{in}}{dt} + \int \vec{S}\cdot d\vec{a} = 0}$$

In the chosen regime without $\vec{J}$, the energy of the fields should be conserved,

$$\boxed{\int \vec{S}\cdot d\vec{a} = \text{energy flowing outward per unit time}}$$

Thus, the Poynting vector $\vec{S}$ is the energy current density of the electromagnetic field! It describes how EM energy flows from one point to the other

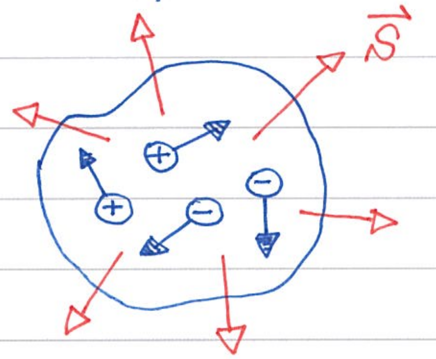
The energy flowing outward causes the energy change outside, $\int \vec{S}\cdot d\vec{a} = \frac{dU_{out}}{dt}$

Thus, the integral form of the Poynting theorem without $\vec{J}$ is

$$\frac{dU_{in}}{dt} + \frac{dU_{out}}{dt} = 0 \longrightarrow \boxed{U_{in} + U_{out} = \text{const}}$$

The total EM energy is conserved. Now we need to include the effect of non-zero (charge) current density $\vec{J}$ ☺

② The general case with $\vec{J} \neq 0$.



Now energy of the fields can be transferred to the charged materials. Following similar derivations, the integral form of the Poynting theorem is

fields + materials.

$$\boxed{\frac{dU_{in}}{dt} + \int \vec{S}\cdot d\vec{a} + \int \vec{E}\cdot\vec{J}\, dV = 0}$$

The $\vec{E}\cdot\vec{J}$ term represents the interaction between the fields and the materials. Compute the power provided by the field per unit volume,

$$p_{EM} = \rho(\vec{E} + \vec{v}\times\vec{B})\cdot\vec{v} = \rho\vec{E}\cdot\vec{v} = \vec{E}\cdot\vec{J} \,!$$

The power for the whole volume is obtained by integration,

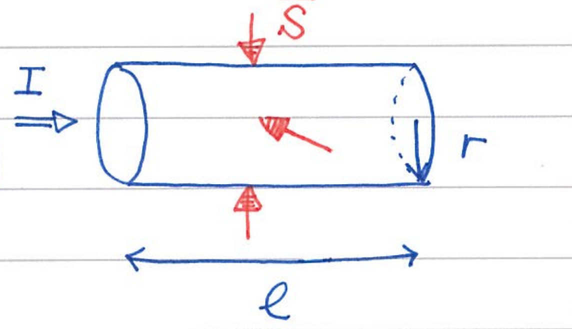
$$\boxed{P_{EM} = \int p_{EM}\, dV = \int \vec{E}\cdot\vec{J}\, dV}$$

power from the fields to the materials.

The integral form of the Poynting theorem has the charming interpretation as the following:

$$\boxed{\begin{pmatrix}\text{EM energy}\\ \text{changing rate}\end{pmatrix} + \begin{pmatrix}\text{energy rate}\\ \text{flowing out}\end{pmatrix} + \begin{pmatrix}\text{energy rate}\\ \text{transferred}\\ \text{to materials}\end{pmatrix} = 0}$$

Let us apply the Poynting theorem to some examples. Consider a wire with steady current I. The $\vec{E}$ and $\vec{B}$ fields are constant,



$$\longrightarrow \boxed{\frac{dU_{in}}{dt} = 0}$$

The rate of EM energy flowing out the surface of the wire is captured by the surface integral,

$$\int \vec{S}\cdot d\vec{a} = -\frac{1}{\mu_0} EB \cdot 2\pi r \ell = -\frac{1}{\cancel{\mu_0}}\left(\frac{J}{\sigma}\right)\left(\frac{\cancel{\mu_0} I}{\cancel{2\pi r}}\right)\cancel{2\pi r}\ell$$

$$\longrightarrow \boxed{\int \vec{S}\cdot d\vec{a} = -I^2\left(\frac{\ell}{\sigma A}\right) = -I^2 R}$$

What does the minus sign mean?

Because $\vec{J} \neq 0$ inside the wire, the electromagnetic field transfers energy to the charged materials,

$$P_{EM} = \int \vec{E}\cdot\vec{J}\, dV = EJ\cdot A\ell = \left(\frac{I}{\sigma A}\right)\left(\frac{I}{\cancel{A}}\right)\cancel{A}\ell$$

$$\longrightarrow \boxed{P_{EM} = I^2\left(\frac{\ell}{\sigma A}\right) = I^2 R}$$

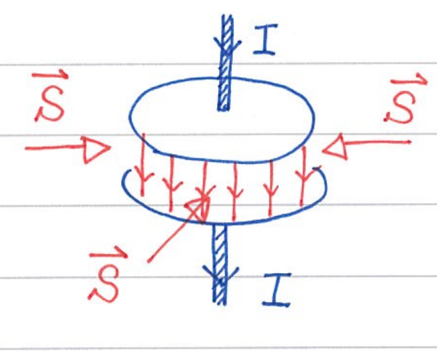
Energy transfers to the charged materials!

Putting all pieces together, it is easy to see that

$$\frac{dU_{in}}{dt} + \int \vec{S}\cdot d\vec{a} + P_{EM} = 0$$

As predicted by the Poynting theorem ☺

Let us turn to another example, now with a capacitor.



There is no current inside and thus

$$\boxed{P_{EM} = \int \vec{E}\cdot\vec{J}\, dV = 0}$$

Now let's proceed to compute the other 2 terms ☺

豪豬筆記

The rate of flow-out EM energy is

$$\int \vec{S}\cdot d\vec{a} = -\frac{1}{\mu_0} EB\cdot 2\pi r d = -\frac{1}{\cancel{\mu_0}}\frac{Q}{\epsilon_0 A}\frac{\cancel{\mu_0} I}{\cancel{2\pi r}}\cancel{2\pi r} d$$

$$\rightarrow \boxed{\int \vec{S}\cdot d\vec{a} = -\frac{d}{\epsilon_0 A} Q \frac{dQ}{dt} = -\frac{Q}{C}\frac{dQ}{dt}}$$

What about the energy changing rate inside the capacitor?

$$\frac{dU_{in}}{dt} = \frac{d}{dt}\left[\int \frac{1}{2}\epsilon_0 E^2 dV\right] = \frac{d}{dt}\left[\frac{1}{2}\epsilon_0\left(\frac{Q}{\epsilon_0 A}\right)^2 \cdot Ad\right]$$

$$\rightarrow \boxed{\frac{dU_{in}}{dt} = \frac{d}{dt}\left[\frac{1}{2}\frac{d}{\epsilon_0 A} Q^2\right] = \frac{Q}{C}\frac{dQ}{dt}}$$

Energy is kept within the field.

Again, adding all terms together, it is easy to see that

$$\frac{dU_{in}}{dt} + \int \vec{S}\cdot d\vec{a} + P_{EM} = 0$$

The energy never leaves the fields ☺

Obviously, the Poynting theorem is also true here.

⊘ Momentum in the electromagnetic field. In principle, one can follow similar steps (but much harder) to derive the equation for momentum conservation. It turns out that the momentum density (momentum per unit volume) is

$$\boxed{\vec{g} = \frac{1}{c^2}\vec{S} = \epsilon_0 \vec{E}\times\vec{B}}$$

The Poynting vector $\vec{S}$ is the energy current density, i.e. energy change per unit time per unit area. And, surprisingly, $\frac{1}{c^2}\vec{S}$ is also the momentum density!

Let us use "photon" to understand the momentum density in EM waves (Sorry for the "cheating" ☺).

E ● → $\vec{p}$

$$\boxed{E = \hbar\omega, \quad \vec{p} = \hbar\vec{k}}$$

ω angular freq.
$\vec{k}$ wave number.

From the dispersion relation, $\omega = ck$,

$$E = \hbar\omega = \hbar ck \longrightarrow \boxed{E = cp}$$

E, p are linearly related

In consequence, the energy density u and the momentum density g are related in the same way,

$$u = cg$$

Note that $\boxed{X \text{ current density} = X \text{ density} \cdot v}$

We can express the energy density in terms of S,

$$S = uc \longrightarrow \boxed{g = \frac{1}{c} u = \frac{1}{c^2} S}$$

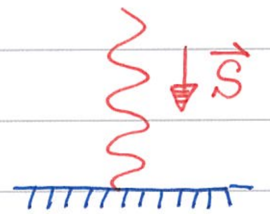
YES!

Because the momentum and the velocity are parallel, $\vec{g}$ and $\vec{S}$ are parallel as well. The vector relation is

$$\boxed{\vec{g} = \frac{1}{c^2}\vec{S}}$$

This relation gives the Poynting vector $\vec{S}$ two distinct physical meanings!

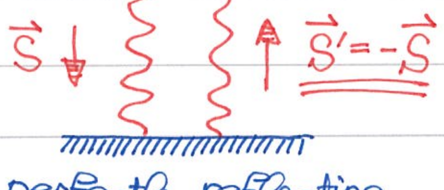
Let us apply the above result to compute the radiation pressure on a perfectly absorbing surface.



perfectly absorbing

$$\boxed{\begin{aligned} P = \frac{1}{A}\frac{dp}{dt} &= \text{momentum current density} \\ &= (\text{momentum density}) \cdot c = gc = \frac{S}{c} \end{aligned}}$$

The above calculation can be applied to a perfectly reflecting surface



perfectly reflecting

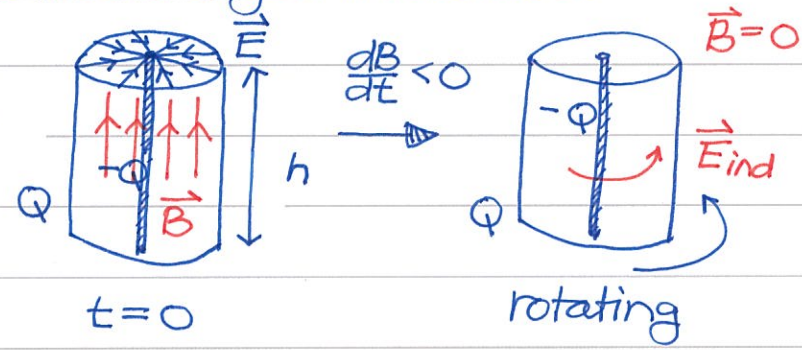
$$\boxed{P = 2gc = \frac{2S}{c}}$$

radiation P doubles!

Although the effects of radiation pressure on ordinary objects are hard to observe, you can find it in the sky — the comet tail caused by radiation pressure from the Sun.

Let us consider an interesting example to see how the angular momentum in the EM field can be transferred to the charged materials.

At $t=0$, the system is at rest. The momentum density inside the cylinder is,



$$\vec{g} = \epsilon_0 \vec{E}\times\vec{B} \longrightarrow \boxed{\vec{\ell} = \vec{r}\times\vec{g} = \epsilon_0\, \vec{r}\times(\vec{E}\times\vec{B})}$$

Because $\vec{r}, \vec{E}, \vec{B}$ are mutually perpendicular,

$$\vec{\ell} = \ell\,\hat{z} \quad \text{and} \quad \ell = \epsilon_0 r E B = \cancel{\epsilon_0}\cancel{r}\cdot\frac{\lambda}{2\pi\cancel{\epsilon_0}\cancel{r}}\cdot B = \frac{QB}{2\pi h}$$

The angular momentum density is a constant. The total angular momentum carried by the EM field is

$$L = \int dV\, \ell = \frac{QB}{2\pi h}\cdot \pi r^2 h \longrightarrow \boxed{L = \frac{1}{2} r^2 Q B}$$

Suppose we decrease the $\vec{B}$ field gradually to zero. Where does the angular momentum of the EM field go?

$$\boxed{\frac{dL}{dt} = \frac{1}{2} r^2 Q \frac{dB}{dt} < 0}$$

Let us calculate the torque on the charged cylinder. According to Faraday's law, the induced electric field is

$$E_{ind}\cdot 2\pi r = -\frac{d\Phi_B}{dt} = -\pi r^2 \frac{dB}{dt} \longrightarrow \boxed{E_{ind} = -\frac{1}{2} r \frac{dB}{dt}}$$

The torque on the cylinder can be computed

$$\boxed{\tau_m = r\cdot F_{ind} = r Q E_{ind} = -\frac{1}{2} r^2 Q \frac{dB}{dt} > 0}$$

Making use of the EOM for rotation

$$\tau_m = \frac{dL_m}{dt} = -\frac{1}{2} r^2 Q \frac{dB}{dt}$$

L_m is angular momentum for cylinder.

Combine both contributions together,

$$\boxed{\frac{dL}{dt} + \frac{dL_m}{dt} = 0 \longrightarrow L + L_m = \text{Const.}}$$

Initially at $t=0$, the angular momentum is stored in the electromagnetic field. As we decrease the $\vec{B}$ field,

$$L(t_i) = \frac{1}{2} r^2 Q B_0 \xrightarrow{\quad} L(t_f) = 0$$

$$L_m(t_i) = 0 \qquad \frac{dB}{dt} < 0 \qquad L_m(t_f) = \frac{1}{2} r^2 Q B_0$$

The angular momentum is transferred from L (the field) to L_m (the material). BUT, the total angular momentum is still conserved ☺

清大東院
2014.04.14