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Photons

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In 1905, inspired by Planck's pioneering work on blackbody radiation, Einstein proposed that light exists as discrete quanta, now referred as "photon". The energy of a single photon is related to its frequency,

$$E = \hbar\omega, \tag{1}$$

where $\hbar \equiv h/2\pi$. In 1917, Einstein extended his idea of photons and proposed that a single photon also carries momentum,

$$\boldsymbol{p} = \hbar\boldsymbol{k}, \tag{2}$$

where $\boldsymbol{k}$ is the wave number of light. The above relations reveal the dual nature of photons. On the left-hand side, energy E and momentum $\boldsymbol{p}$ are often associated with the particle perspective. On the right-hand side, (angular) frequency ω and wave number $\boldsymbol{k}$ are associated with waves. Intuitions built upon daily experiences make "particle" and "wave" incompatible concepts and we are keen to know whether light is particle or wave in classical sense. Well, welcome to the quantum world! In quantum wonderland, photons can exhibit both particle and wave nature, depending on how the experiment is set up. We need to put these classical notions aside and cultivate intuitions about quantum phenomena based on experimental facts.

• Compton effect

In 1923, Compton discovered that the wavelength of the x ray increases slightly when scattered from a target. The schematic diagram for the experimental setup can be found in Figure 1. It is rather remarkable that the shift in the wavelength is captured by the simple formula,

$$\Delta\lambda = \frac{h}{m_e c}(1 - \cos\phi), \tag{3}$$

where m_e is the electron mass. Just four years later, Compton received the 1927 Nobel prize "for his discovery of the effect named after him". The

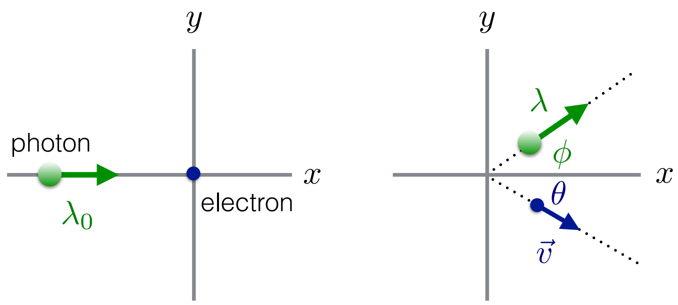

Figure 1: The Compton effect can be explained by collision between photon and electron. A photon of wavelength λ_0 collides with an electron at rest. After the scattering, the photon recoils at an angle ϕ with a larger wavelength λ while the electron flies away with speed v at an angle θ.

Compton effect can be explained easily by the collision between photon and electron. Conservation of energy for the photon-electron system requires that

$$\frac{hc}{\lambda_0} + m_e c^2 = \frac{hc}{\lambda} + \frac{m_e c^2}{\sqrt{1-(v/c)^2}}. \tag{4}$$

The relativistic treatment is necessary because the scattered electron may carry high speed comparable to the speed of light. Meanwhile, the momentum of the photon-electron system is also conserved,

$$\frac{h}{\lambda_0} = \frac{h}{\lambda}\cos\phi + \frac{m_e v}{\sqrt{1-(v/c)^2}}\cos\theta, \tag{5}$$

$$0 = \frac{h}{\lambda}\sin\phi - \frac{m_e v}{\sqrt{1-(v/c)^2}}\sin\theta. \tag{6}$$

In the photon-electron scattering, there are four unknowns, λ, v, ϕ, θ and we have three equations from the conservation laws. One indeed needs a more sophisticated theory to achieve the full understanding. However, we can eliminate two variables v and θ (both associated with the scattered electron) and derive the relation between the remaining variables λ and ϕ. After some algebra, one finds that

$$\lambda - \lambda_0 = \frac{h}{m_e c}(1-\cos\phi). \tag{7}$$

Try to carry out the necessary algebra to eliminate v and θ and derive the Compton shift formula by yourself. You may try eliminating θ first by combining the two equations from the momentum conservation. Then, try to get rid of the square root in the equation from the energy conservation.

• electromagnetic waves in thermal equilibrium

Now let us visit the famous thermal radiation problem originally solved by Planck in 1900. We will try to understand this interesting phenomenon in a slightly different angle suggested by Bose in 1924. According to the Maxwell's equations, the energy of an electromagnetic wave with wave number $\boldsymbol{k}$ (and thus a definite frequency ω) inside a cavity of volume V is

$$E(\omega) = \left\langle \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0} B^2 \right\rangle V. \tag{8}$$

The above form is similar to that for a simple harmonic oscillator. According to statistical mechanics, the average energy of a simple harmonic oscillator in thermal equilibrium is just kT, independent of other detail parameters. This result is known as the equipartition theorem of energy. Not surprisingly, after similar calculations, the average energy $U(\omega)$ of the electromagnetic wave in thermal equilibrium also follows the equipartition theorem,

$$U(\omega) = kT. \tag{9}$$

But, we immediately run into a big trouble. The total energy of the electromagnetic waves in thermal equilibrium is the sum of $U(\omega)$ over all possible wave numbers. Because there are infinite possible wave numbers, the total energy is divergent. This cannot be true because it will directly leads to explosive thermal radiations at finite temperature. So, the marriage between Maxwell (electromagnetism) and Boltzmann (statistical mechanics) doesn't go very well. Who is the one to be blamed? It turns out to be Maxwell in this case, ha!

• Planck distribution

Consider photons with wave number $\boldsymbol{k}$ (and thus a definite frequency ω) in thermal equilibrium. The probability to find n photons is described by the Boltzmann distribution,

$$P_n = C\, e^{-E_n/kT} = C\, e^{-n\hbar\omega/kT}, \tag{10}$$

where C is some constant to be determined. Notice that the Boltzmann distribution only depends on the dimensionless frequency,

$$z = \frac{\hbar\omega}{kT}. \tag{11}$$

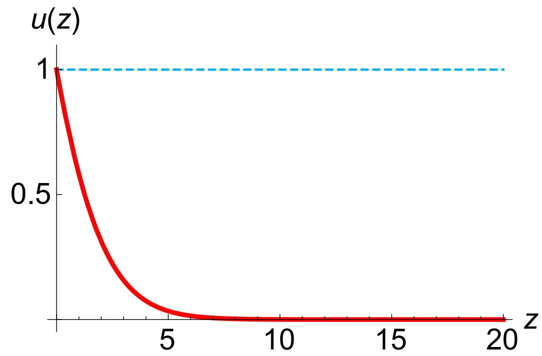

Figure 2: Planck distribution for photons in thermal equilibrium. The Planck distribution is exponentially suppressed in the high-frequency $z \gg 1$ regime in sharp contrast to the dashed line predicted by the classical theory

Making use of the mathematical identity for the geometric series and the unity constraint for the probability distribution,

$$\sum_{n=0}^{\infty} e^{-nz} = \frac{1}{1-e^{-z}}, \quad \rightarrow \quad C = 1 - e^{-z}. \tag{12}$$

The Boltzmann distribution for photons with wave number $\boldsymbol{k}$ in thermal equilibrium takes the simple form,

$$P_n = (1 - e^{-z})\, e^{-nz}, \tag{13}$$

solely depending on the dimensionless frequency $z = \hbar\omega/kT$. Now we are ready to compute the average energy of photons,

$$\begin{aligned} U(\omega) &= \sum_{n=0}^{\infty} n\hbar\omega \times P_n = kT \times \sum_{n=0}^{\infty} nz\, P_n \\ &= kT \times z(1-e^{-z}) \sum_{n=0}^{\infty} n\, e^{-nz}. \end{aligned} \tag{14}$$

Another mathematical identity is needed here, which can be derived by taking derivative on the identity for the geometric series in above,

$$\sum_{n=0}^{\infty} n\, e^{-nz} = \frac{e^{-z}}{(1-e^{-z})^2}. \tag{15}$$

It is convenient to introduce the ratio between the average energy $U(\omega)$ and the thermal energy kT,

$$u(z) \equiv \frac{U}{kT} = \frac{z}{e^z - 1}. \tag{16}$$

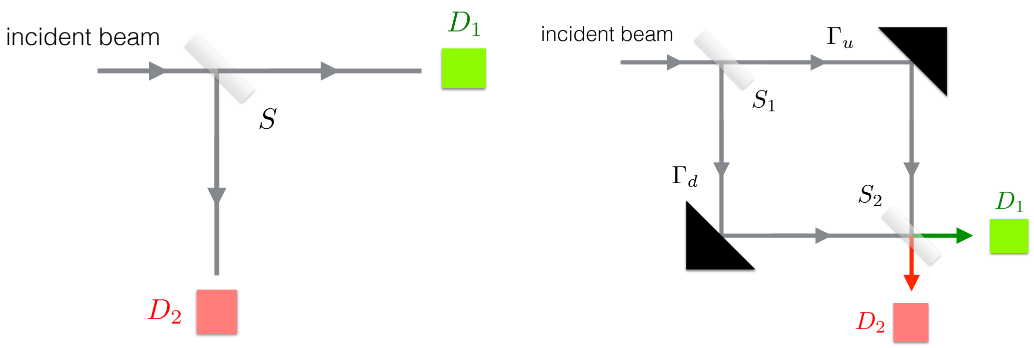

Figure 3: (Left panel) Anti-coincidence apparatus with a single beam splitter S. (Right panel) Interference apparatus for photons with two beam splitters S_1 and S_2.

The classical theory (Maxwell+Boltzmann) will give rise to trivial result $u(z) = 1$ (dashed line in Figure 2). The quantum theory exhibits exponential suppressions in the high-frequency $z \gg 1$ regime and save us from the ultraviolet catastrophe in the classical theory. The physics picture behind the rescue is very simple: high-frequency photons carry higher energies $\hbar\omega \gg kT$, only be found with exponentially small probabilities, and cannot cause any singular behavior.

• wave or particle?

The name of "photon" seems to imply that light should be treated as particle. Is this true? Consider the anti-coincidence apparatus in Figure 3 – an incident light beam passes through a symmetric beam splitter. When detector D_1 signals the presence of the photon, detector D_2 remains silent. On the other hand, when detector D_2 signals, detector D_1 remains silent. Therefore, even though the trains of signals in each detector are random in nature, they show perfect *anti-correlation*, which can be explained naturally in particle perspective.

But, the story hasn't ended yet. Suppose now we use extra mirrors to bring the two light beams back together as shown in Figure 3. The initial setup makes the lengths of the paths Γ_u and Γ_d the same so that no geometric phase shift arises. Note that reflection upon mirror gives rise to a phase shift π and the relative phase shift between transmitted and reflected beams is a symmetric beam splitter is $\pi/2$. For convenience, we assign 0 phase shift for the transmission and $\pi/2$ for the reflection through the beam splitter.

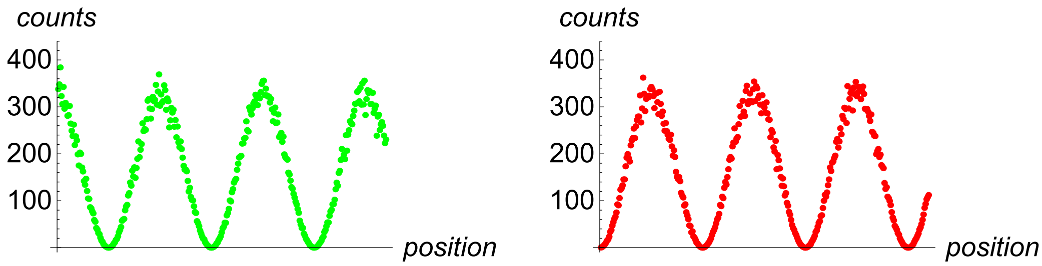

Figure 4: The accumulated counts on the detectors show interference fringes as in the double-slit experiment for light waves.

The signal of D_1 contains two parts: the Γ_u beam includes transmission through beam splitter S_1, reflection by the upper mirror, reflection through beam splitter S_2, while the Γ_d beam includes reflection through beam splitter S_1, reflection by the lower mirror, transmission through beam splitter S_2. The phases accumulated through these optical processes are

$$\phi(\Gamma_u) = 0 + \pi + \pi/2, \qquad \phi(\Gamma_d) = \pi/2 + \pi + 0. \tag{17}$$

The phase difference is zero. If there is interference, we expect it to be constructive. This is indeed true as shown in Figure 4. By moving the position of the upper mirror slightly, the geometric phase shift appears and the interference fringes emerge, as those observed in the double-slit experiment! You can entertain yourself to analyze the interference effect observed in detector D_2. Can you explain the difference observed in detectors D_1 and D_2?

After full digestion of the above experimental results, we are forced to accept an exotic notion: *photons are photons – quantum in nature, neither particles nor waves in any classical sense.*

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