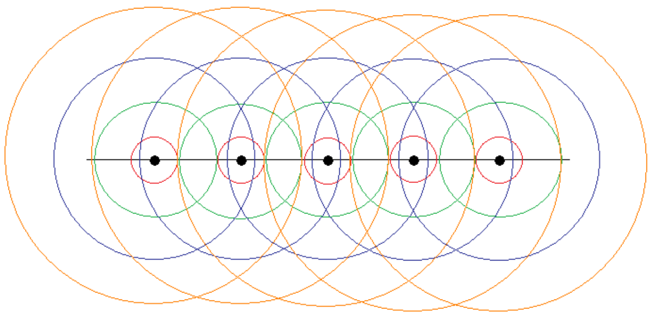
IV Frauhofer diffraction from slits

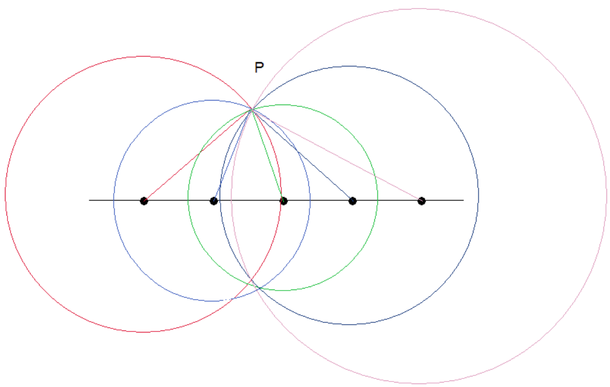
4-1 Huygens-Fresnel principle

∗ Every unobstructed point of a wavefront, at a given instant in time, serves as a source of spherical secondary wavelets (with the same frequency as a source of spherical wave).



4-2 Superposition principle

∗ The amplitude of the optical field at any point beyond is the superposition of the wavelets (considering their amplitudes and relative phases).



4-3 a wave in the complex notation

4-3-1 $$f(x,t) = E_o \cos(\kappa x - \omega t + \delta)$$

where E_o is the amplitude of the wave

κ is the wave number of the wave

ω is the angular frequency of the wave

δ is the phase constant of the wave ; $0 \leq \delta \leq 2\pi$

$$f(x,t_o) = f(x+\lambda, t_o)$$
$$E_o \cos(\kappa x - \omega t + \delta) = E_o \cos(\kappa(x+\lambda) - \omega t + \delta)$$
$$\kappa\lambda = 2\pi$$
$$\kappa = \frac{2\pi}{\lambda}$$

$$f(x,t_o) = f(x, t_o + T)$$
$$E_o \cos(\kappa x - \omega t + \delta) = E_o \cos(\kappa x - \omega(t+T) + \delta)$$
$$\omega T = 2\pi$$
$$\omega = \frac{2\pi}{T}$$

$$\omega = 2\pi\nu$$

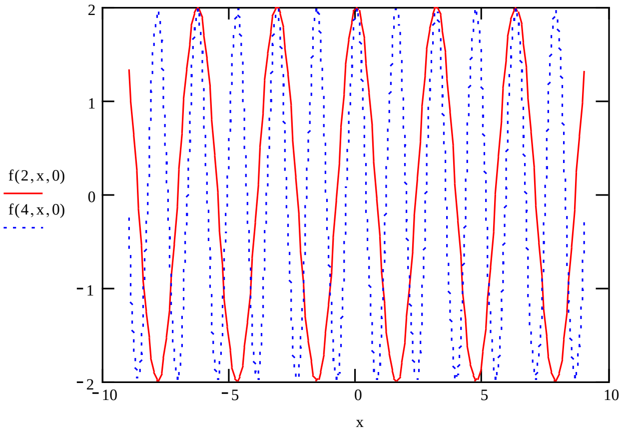
where $\nu = \frac{1}{T}$ is the frequency of the wave

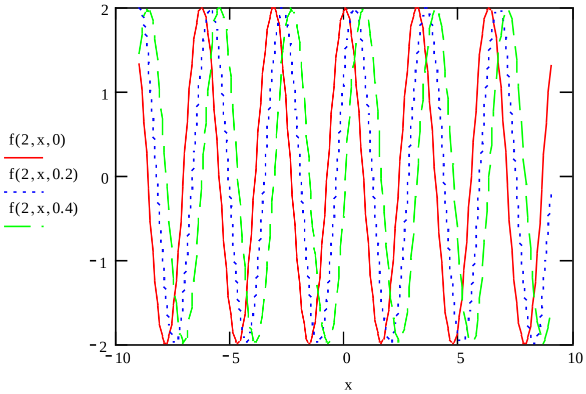
Wave function f(x, t) plotted by Mathcad

Assume that the wave function is in a form of f(a, x, t)=2*cos(a*x-4t)

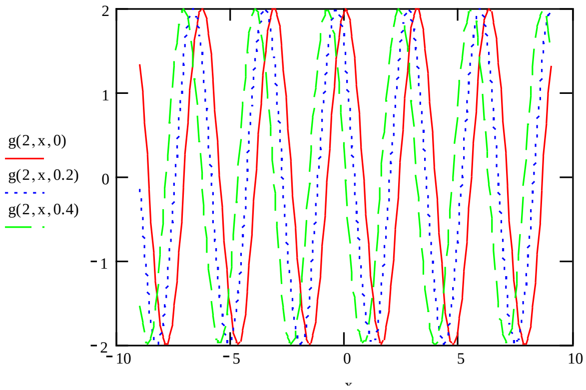
$$x := -9, -8.9 .. 9$$

$$f(a, x, t) := 2\cdot\cos(a\cdot x - 4\cdot t)$$





$$g(a, x, t) := 2\cdot\cos(a\cdot x + 4\cdot t)$$



4-3-2 complex notation

$$e^{ix} = \cos x + i \sin x$$
$$f(x,t) = E_o \cos(\kappa x - \omega t + \delta)$$
$$\tilde{f}(x,t) = E_o[\cos(\kappa x - \omega t + \delta) + i \sin(\kappa x - \omega t + \delta)]$$

$$\tilde{f}(x,t) = E_o e^{i(\kappa x - \omega t + \delta)}$$

define a complex wave function

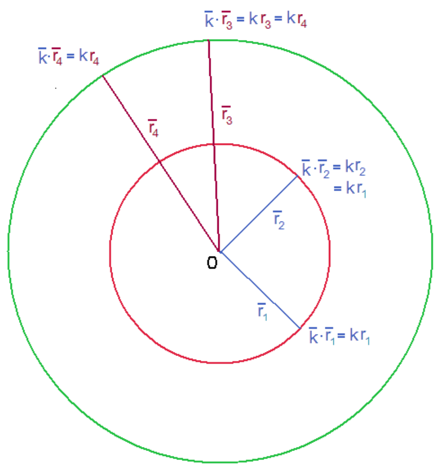
$$\tilde{f}(x,t) = \tilde{E}_o e^{i(\kappa x - \omega t)} \text{ where} \tilde{E}_o = E_o e^{i\delta}$$

The real wave function can be obtained by taking the real part of $\tilde{f}(x,t)$

$$f(x,t) = \mathrm{Re}\left[\tilde{f}(x,t)\right]$$
$$f(x,t) = \mathrm{Re}\left[\tilde{E}_o e^{i(\kappa x - \omega t)}\right]$$
$$f(x,t) = \mathrm{Re}\left[\tilde{E}_o e^{i(\kappa x - \omega t)}\right]$$
$$f(x,t) = \mathrm{Re}\left[E_o e^{i(\kappa x - \omega t + \delta)}\right] = E_o \cos(\kappa x - \omega t + \delta)$$

4-3-3 spherical wave

$$\tilde{E}(r,t) = \frac{\tilde{\xi}_o}{r} e^{i(\kappa r - \omega t)}$$



Intensity is the energy flux = energy/(m^2sec)

Intensity is proportional to $\tilde{E}\tilde{E}^*$

If $\tilde{E} = E_R + iE_I = E\cos\theta + iE\sin\theta$

Then $\tilde{\mathrm{E}}\tilde{\mathrm{E}}^* = (\mathrm{E}\cos\theta + i\mathrm{E}\sin\theta)(\mathrm{E}\cos\theta - i\mathrm{E}\sin\theta)$

$$\tilde{\mathrm{E}}\tilde{\mathrm{E}}^* = \mathrm{E}^2(\cos^2\theta + \sin^2\theta) = \mathrm{E}^2$$

➔ energy is proportional to E^2

➔ Total energy is conserved

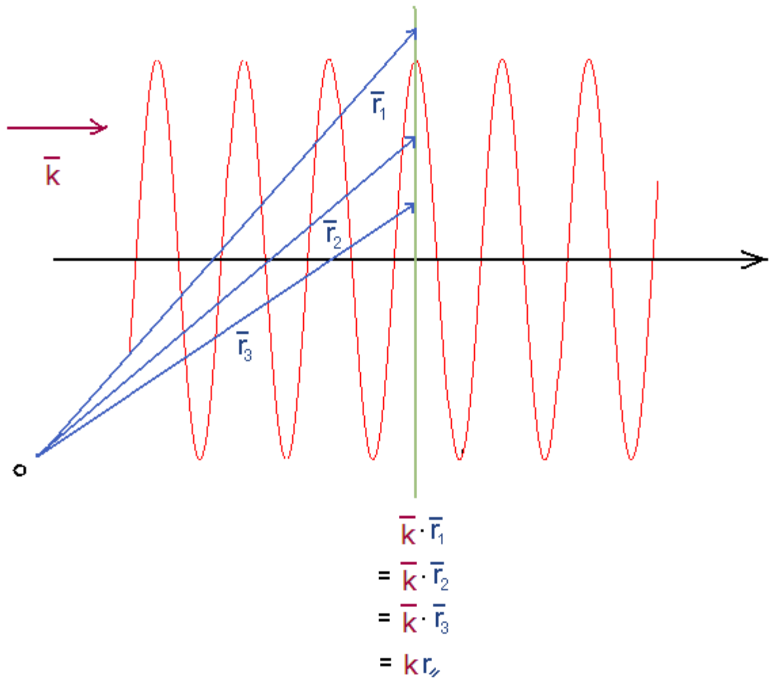
➔ $4\pi r^2 * E^2$ is a constant

➔ E^2 is proportional to $1/r^2$

➔ E is proportional to 1/r for a spherical wave

4-3-4 plane wave

$$\vec{\vec{E}}(\vec{r},t) = \vec{\vec{\xi}}_o e^{i(\vec{\kappa}\cdot\vec{r}-\omega t)}$$

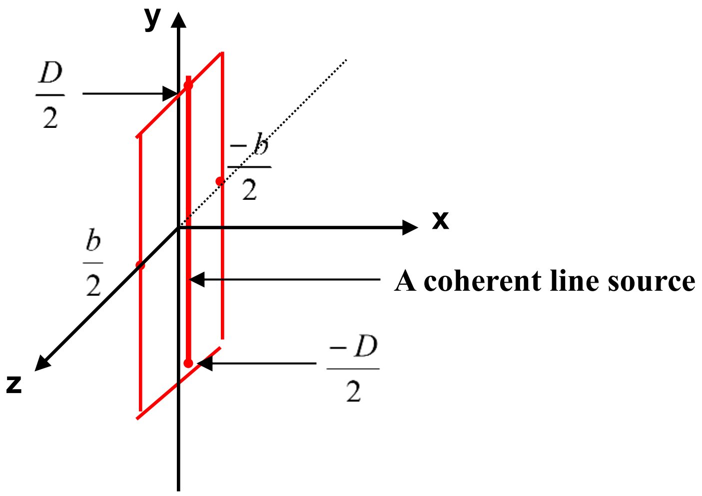


4-4 Frauhofer diffraction (far field diffraction)

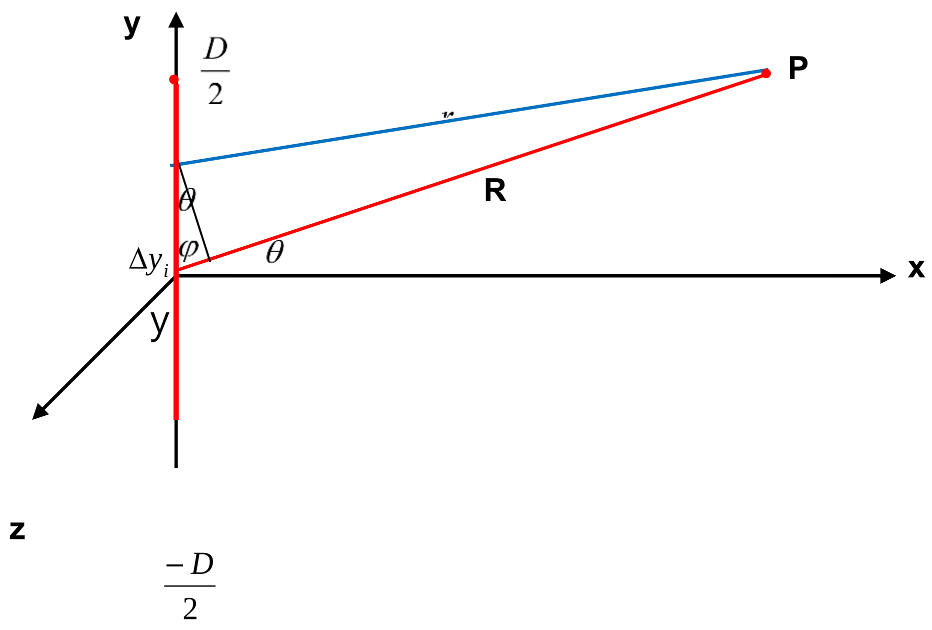
Diffraction patterns from slits

Please find the photo of the diffraction patterns from slits (Fig. 10.20) in "Optics" Eugene Hecht , 2nd edition.

1、diffraction from a single slit



(1) consider diffraction from a coherent line source first.



According to Huygens-Fresnel principle, each point emits a spherical wavelet

$$\tilde{E} = \frac{\tilde{\xi}_o}{r} e^{i(\kappa r - \omega t)}$$

where $\tilde{\xi}_o$ is the source strength at each point.

Suppose that

(a) N is the total number of the source;

(b) D is the width of the coherent line source;

(c) The line is divided into M segments, i.e. i=1、2、3 ....i.....M

The contribution to the electric field at P from the ith segment is

$$\tilde{E}_i = \frac{\tilde{\xi}_o}{r_i} e^{i(\kappa r_i - \omega t)} \frac{N\Delta y_i}{D}$$

Define $\tilde{\xi}_L = \frac{1}{D} \lim_{N\to\propto} (\tilde{\xi}_o N)$

$$\tilde{E}_i = \frac{\tilde{\xi}_L}{r_i} e^{i(\kappa r_i - \omega t)} \Delta y_i$$

The total field at P from all M segments is

$$\tilde{E} = \sum_1^M \tilde{E}_i = \sum_1^M \frac{\tilde{\xi}_L}{r_i} e^{i(\kappa r_i - \omega t)} \Delta y_i$$

For a continue line source

$$\tilde{E} = \tilde{\xi}_L \int_{-\frac{D}{2}}^{\frac{D}{2}} \frac{e^{i(\kappa r - \omega t)}}{r} dy$$

, where $r = r(y)$

For far field diffraction, $R \gg D$

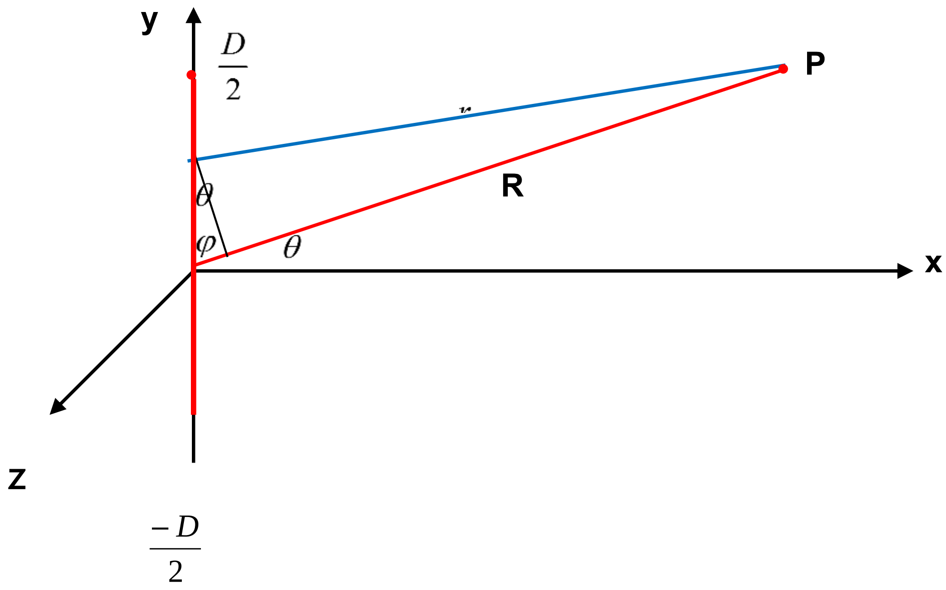
Note that the phase is much more sensitive to the variation in r(y) than the amplitude. Therefore $r_i \cong R$

Then

$$\tilde{E} = \frac{\tilde{\xi}_L}{R} \int_{-\frac{D}{2}}^{\frac{D}{2}} e^{i(\kappa r - \omega t)} dy$$

When r is expanded as a function of y, $r \cong R - y \sin\theta$
, where θ is measured from the x axis in the xy plane

proof : $r \cong R - y\sin\theta$



$$r^2 = R^2 + y^2 - 2Ry\cos\phi$$

$$r^2 = R^2 + y^2 - 2Ry\sin\theta$$

$$\left(\frac{r}{R}\right)^2 = \left(\frac{y}{R}\right)^2 + 1 - \frac{2y}{R}\sin\theta$$

$$\left(\frac{r}{R}\right) = \left[\left(\frac{y}{R}\right) + 1 - \frac{2y}{R}\sin\theta\right]^{\frac{1}{2}}$$

Maclaurin series

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots\ldots$$

$$\frac{r}{R} = 1 - \frac{y}{R}\sin\theta + \frac{1}{2}\left(\frac{y}{R}\right)^2 + \left(-\frac{1}{8}\right)\left(\frac{2y\sin\theta}{R}\right)^2 + \ldots$$

$$r = R - y\cdot\sin\theta + \frac{y^2\cos^2\theta}{2R} + \ldots\ldots$$

$$r \cong R - y\cdot\sin\theta$$

Therefore

$$\tilde{E} = \frac{\tilde{\xi}_L}{R}\int_{-\frac{D}{2}}^{\frac{D}{2}} e^{i(\kappa r-\omega t)}\,dy = \frac{\tilde{\xi}_L}{R}\int_{-\frac{D}{2}}^{\frac{D}{2}} e^{i(\kappa(R-y\sin\theta)-\omega t)}\,dy$$

if $R \gg y$ (Fraunhofer approximation)

$$\tilde{E} = \frac{\tilde{\xi}_L}{R}e^{i(\kappa R-\omega t)}\int_{-\frac{D}{2}}^{\frac{D}{2}} e^{-i(\kappa y\sin\theta)}\,dy$$

$$\tilde{E} = \frac{\tilde{\xi}_L}{R}e^{i(\kappa R-\omega t)}\left.\frac{e^{-i(\kappa y\sin\theta)}}{-i\kappa\sin\theta}\right|_{-\frac{D}{2}}^{\frac{D}{2}}$$

$$\tilde{E} = \frac{\tilde{\xi}_L}{R}e^{i(\kappa R-\omega t)}\frac{-2i\sin(\frac{\kappa D\sin\theta}{2})}{-i\kappa\sin\theta}$$

$$\tilde{E} = \frac{\tilde{\xi}_L D}{R}e^{i(\kappa R-\omega t)}\frac{\sin(\frac{\kappa D\sin\theta}{2})}{\frac{\kappa D\sin\theta}{2}}$$

$$\tilde{E} = \frac{\tilde{\xi}_L D}{R}e^{i(\kappa R-\omega t)}\frac{\sin\gamma}{\gamma}, \text{ where } \gamma = \frac{\kappa D\sin\theta}{2}$$

Intensity is proportional to $\tilde{E}\tilde{E}^*$

$$I = K\,\tilde{E}\tilde{E}^*$$

$$I = K\left[\frac{\tilde{\xi}_L D}{R}e^{i(\kappa R-\omega t)}\frac{\sin\gamma}{\gamma}\right]\left[\frac{\tilde{\xi}_L D}{R}e^{i(\kappa R-\omega t)}\frac{\sin\gamma}{\gamma}\right]^*$$

$$I = K\left(\frac{\xi_L D}{R}\right)^2\left(\frac{\sin\gamma}{\gamma}\right)^2$$

, where $\gamma = \frac{\kappa D\sin\theta}{2}$ and $\left(\frac{\xi_L D}{R}\right)^2 = \tilde{\xi}_L\tilde{\xi}_L{}^*\left(\frac{D}{R}\right)^2$

When $\theta = 0$, $\frac{\sin\gamma}{\gamma} = \frac{\sin(0)}{0} = 1$

$$I(\theta = 0) = K\left(\frac{\xi_L D}{R}\right)^2$$

$$I(\theta) = K\left(\frac{\xi_L D}{R}\right)^2 \left(\frac{\sin\gamma}{\gamma}\right)^2 = I(\theta = 0)\left(\frac{\sin\gamma}{\gamma}\right)^2$$

$$\frac{I(\theta)}{I(\theta = 0)} = \left(\frac{\sin\gamma}{\gamma}\right)^2$$

Discussion

$$\gamma = \frac{\kappa D \sin\theta}{2} = \frac{2\pi}{\lambda}\frac{D \sin\theta}{2} = \frac{\pi D \sin\theta}{\lambda}$$

(a) when $D \gg \lambda$

$$\gamma \to \infty$$

$$\frac{\sin\gamma}{\gamma} \to 0$$

$$I(\theta = 0) = K\left(\frac{\xi_L D}{R}\right)^2$$

$$I(\theta \neq 0) = 0$$

Therefore, the phase of the line source is equivalent to that of a point source located at the center of the line.

The coherent line source can be envisioned as a single point emitter radiating predominantly in the forward $\theta = 0$ direction ;

In other words, its emission resembles a circular wave in xz plane

(b) when $\lambda \gg D$

$$\gamma \to 0$$

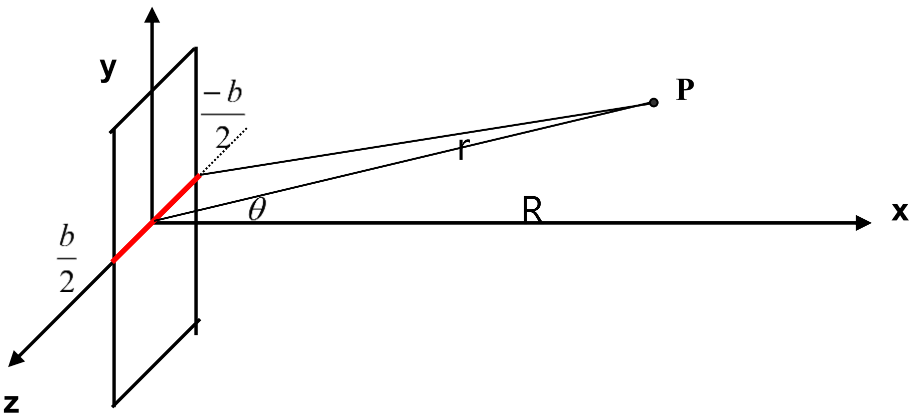
$$\frac{\sin\gamma}{\gamma} \to 1$$

$$I(\theta = 0) = K\left(\frac{\xi_L D}{R}\right)^2$$

$$I(\theta \neq 0) = K\left(\frac{\xi_L D}{R}\right)^2 = I(\theta = 0)$$

the line source resembles a point source emitting spherical waves.

(2) consider diffraction from a single slit



The problem is reduced to that of finding the $\tilde{E}(r,t)$ field in the xz plane from an infinite number of point sources extending across the width of the slit along the z axis.

$$\tilde{E} = \frac{\tilde{\xi}'_L}{R}\int_{-\frac{b}{2}}^{\frac{b}{2}} e^{i(\kappa r - \omega t)}\, dz$$

where $\tilde{\xi}'_L$ is the source strength per unite length.

When r is expanded as a function of z, $r = r(z) \cong R - z\sin\theta$, where θ is measured from the x axis in the xz plane

$$\widetilde{E} = \frac{\tilde{\xi}'_L}{R} \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{i(\kappa r - \omega t)}\, dz = \frac{\tilde{\xi}'_L}{R} \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{i(\kappa(R - z\sin\theta) - \omega t)}\, dz$$

if $R \gg z$ (Fraunhofer approximation)

$$\widetilde{E} = \frac{\tilde{\xi}'_L}{R} e^{i(\kappa R - \omega t)} \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{-i(\kappa z \sin\theta)}\, dz$$

$$\widetilde{E} = \frac{\tilde{\xi}'_L}{R} e^{i(\kappa R - \omega t)} \left. \frac{e^{-i(\kappa z \sin\theta)}}{-i\kappa \sin\theta} \right|_{-\frac{b}{2}}^{\frac{b}{2}}$$

$$\widetilde{E} = \frac{\tilde{\xi}'_L}{R} e^{i(\kappa R - \omega t)} \frac{-2i\sin(\frac{\kappa b \sin\theta}{2})}{-i\kappa \sin\theta}$$

$$\widetilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin(\frac{\kappa b \sin\theta}{2})}{\frac{\kappa b \sin\theta}{2}}$$

$\widetilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta}$, where $\beta = \frac{\kappa b \sin\theta}{2}$

Intensity is proportional to $\widetilde{E}\widetilde{E}^*$

$$I = K' \widetilde{E}\widetilde{E}^*$$

$$I = K' \left[\frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta}\right] \left[\frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta}\right]^*$$

$$I = K' \left(\frac{\xi'_L b}{R}\right)^2 \left(\frac{\sin\beta}{\beta}\right)^2$$

, where $\beta = \frac{\kappa b \sin\theta}{2}$ and $\left(\frac{\xi'_L b}{R}\right)^2 = \tilde{\xi}'_L \tilde{\xi}'^*_L \left(\frac{b}{R}\right)^2$

When $\theta = 0$, $\dfrac{\sin\beta}{\beta} = \dfrac{\sin(0)}{0} = 1$

$$I(\theta = 0) = K'\left(\frac{\xi_L' b}{R}\right)^2$$

$$I(\theta) = K'\left(\frac{\xi_L' b}{R}\right)^2 \left(\frac{\sin\beta}{\beta}\right)^2 = I(\theta = 0)\left(\frac{\sin\beta}{\beta}\right)^2$$

$$\frac{I(\theta)}{I(\theta = 0)} = \left(\frac{\sin\beta}{\beta}\right)^2$$

, where

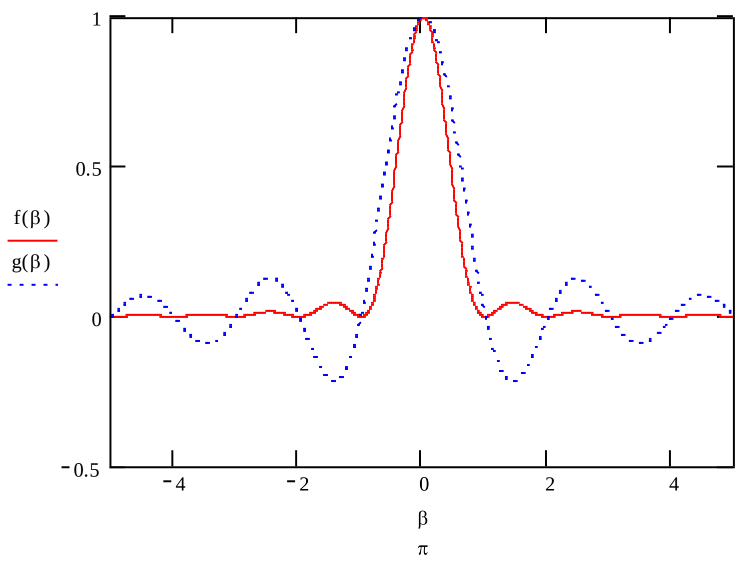
$$\beta = \frac{\kappa b \sin\theta}{2} = \frac{2\pi}{\lambda}\frac{b\sin\theta}{2} = \frac{\pi b \sin\theta}{\lambda}$$

The Fraunhofer diffraction pattern from a single slit

$\beta := -5\pi, -4.99\pi .. 5\pi$

$$f(\beta) := \left(\frac{\sin(\beta)}{\beta}\right)^2$$

$$g(\beta) := \left(\frac{\sin(\beta)}{\beta}\right)$$



The intensity pattern is consistent with the diffraction pattern from a single slit in Fig. 10.20 in Optics" Eugene Hecht , 2nd edition.

Discussion

first minimum occurs at $\beta = \pi$

$$\beta = \frac{\pi b \sin\theta}{\lambda}$$

So,

$$\frac{\sin\theta}{\lambda} = \frac{1}{b}$$

second minimum occurs at$\beta = 2\pi$

$$\beta = \frac{\pi b \sin\theta}{\lambda}$$

So,

$$\frac{\sin\theta}{\lambda} = \frac{2}{b}$$

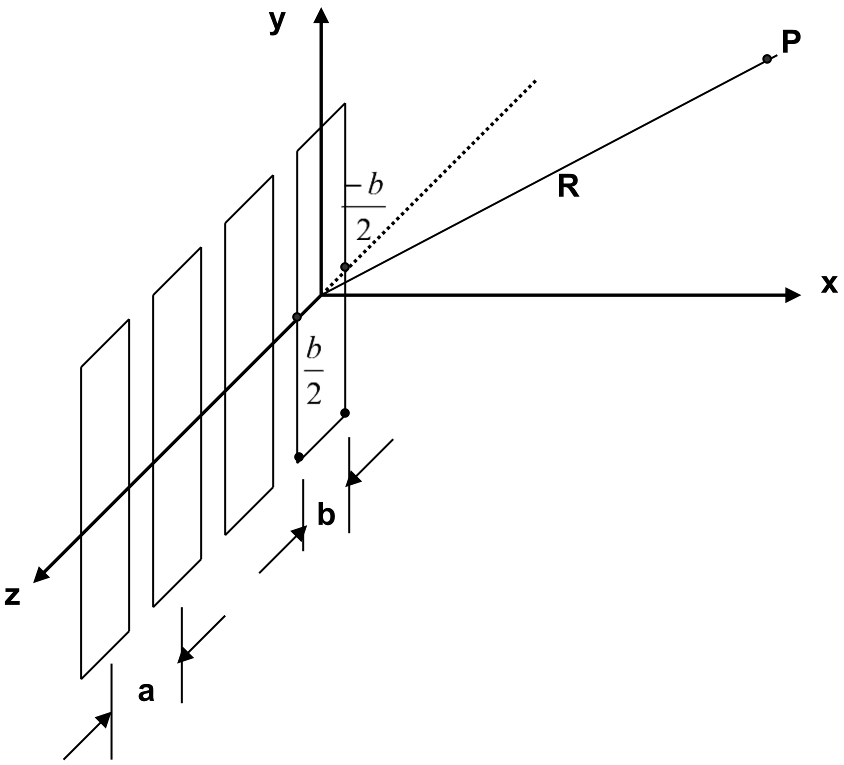
Similarly, the n minimum occurs at

$$\frac{\sin\theta}{\lambda} = \frac{n}{b}$$

Moreover,

the peak width of $I(\theta)$ within first minimum broadens when b shrinks.

(3) diffraction from many slits



The result from a single slit

$$\tilde{E} = \frac{\tilde{\xi}_L'}{R}\int_{-\frac{b}{2}}^{\frac{b}{2}} e^{i(\kappa r-\omega t)}\,dz = \frac{\tilde{\xi}_L'}{R}\int_{-\frac{b}{2}}^{\frac{b}{2}} e^{i(\kappa(R-z\sin\theta)-\omega t)}\,dz$$

if $R \gg z$ (Fraunhofer approximation)

$$\tilde{E} = \frac{\tilde{\xi}_L'}{R}e^{i(\kappa R-\omega t)}\int_{-\frac{b}{2}}^{\frac{b}{2}} e^{-i(\kappa z\sin\theta)}\,dz$$

$$\widetilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin(\frac{\kappa b \sin\theta}{2})}{\frac{\kappa b \sin\theta}{2}}$$

$$\widetilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta}$$

, where $\beta = \frac{\kappa b \sin\theta}{2}$

Based on the superposition principle, the total field strength from N slits is

$$\widetilde{E} = \frac{\tilde{\xi}'_L}{R} \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{i(\kappa r - \omega t)}\, dz + \frac{\tilde{\xi}'_L}{R} \int_{a-\frac{b}{2}}^{a+\frac{b}{2}} e^{i(\kappa r - \omega t)}\, dz + \frac{\tilde{\xi}'_L}{R} \int_{2a-\frac{b}{2}}^{2a+\frac{b}{2}} e^{i(\kappa r - \omega t)}\, dz + \cdots + \frac{\tilde{\xi}'_L}{R} \int_{(N-1)a-\frac{b}{2}}^{(N-1)a+\frac{b}{2}} e^{i(\kappa r - \omega t)}\, dz$$

$$\widetilde{E} = \frac{\tilde{\xi}'_L}{R} \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{i(\kappa(R - z\sin\theta) - \omega t)}\, dz + + \frac{\tilde{\xi}'_L}{R} \int_{a-\frac{b}{2}}^{a+\frac{b}{2}} e^{i(\kappa(R - z\sin\theta) - \omega t)}\, dz + \frac{\tilde{\xi}'_L}{R} \int_{2a-\frac{b}{2}}^{2a+\frac{b}{2}} e^{i(\kappa(R - z\sin\theta) - \omega t)}\, dz + \cdots + \frac{\tilde{\xi}'_L}{R} \int_{(N-1)a-\frac{b}{2}}^{(N-1)a+\frac{b}{2}} e^{i(\kappa(R - z\sin\theta) - \omega t)}\, dz$$

$$\widetilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta} \left[1 + e^{i(-\kappa a \sin\theta)} + e^{i(-2\kappa a \sin\theta)} + \cdots + e^{i(-\kappa(N-1)a\sin\theta)}\right]$$

$$\widetilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta} \left[1 + e^{i(2\alpha')} + e^{i(4\alpha')} + \cdots + e^{i(2(N-1)\alpha')}\right]$$

, where $\alpha' = -\frac{\kappa a \sin\theta}{2}$

$$\tilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta} \sum_{j=0}^{N-1} e^{i(2j\alpha')}$$

$$\tilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta} \sum_{j=0}^{N-1} \left(e^{i2\alpha'}\right)^j$$

$$\tilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta} \sum_{j=0}^{N-1} \left(e^{i2\alpha'}\right)^j$$

Here we use the mathematical equation

$$1 + e^{i\delta} + \left(e^{i\delta}\right)^2 + \left(e^{i\delta}\right)^3 + \cdots + \left(e^{i\delta}\right)^{N-1} = \frac{e^{iN\delta} - 1}{e^{i\delta} - 1}$$

$$\tilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta} \frac{e^{i2\alpha' N} - 1}{e^{i2\alpha'} - 1}$$

$$\tilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta} \left(\frac{e^{i\alpha' N}}{e^{i\alpha'}}\right) \left(\frac{e^{i\alpha' N} - e^{-i\alpha' N}}{e^{i\alpha'} - e^{-i\alpha'}}\right)$$

$$\tilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta} e^{i\alpha'(N-1)} \left(\frac{\left(\cos(\alpha' N) + i\sin(\alpha' N)\right) - \left(\cos(-\alpha' N) + i\sin(-\alpha' N)\right)}{\left(\cos(\alpha') + i\sin(\alpha')\right) - \left(\cos(-\alpha') + i\sin(-\alpha')\right)}\right)$$

$$\tilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta} e^{i\alpha'(N-1)} \left(\frac{2i\sin(\alpha' N)}{2i\sin(\alpha')}\right)$$

$$\tilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta} e^{-i\alpha(N-1)} \left(\frac{\sin(-\alpha N)}{\sin(-\alpha)}\right)$$

Set $\alpha = -\alpha' = \frac{\kappa a \sin\theta}{2}$

$$\tilde{E} = \frac{\tilde{\xi}'_L b}{R} e^{i(\kappa R - \omega t)} \frac{\sin\beta}{\beta} e^{-i\alpha(N-1)} \left(\frac{\sin(N\alpha)}{\sin\alpha}\right)$$

$$\tilde{E} = \frac{\tilde{\xi}'_L b}{R} \frac{\sin\beta}{\beta} \left(\frac{\sin(N\alpha)}{\sin\alpha}\right) e^{i[\kappa R - \omega t - (N-1)\alpha]}$$

Intensity is proportional to $\tilde{E}\tilde{E}^*$

$$I = K\,\tilde{E}\tilde{E}^*$$

$$\mathrm{I} = \mathrm{K}\left[\frac{\tilde{\xi}_L' b}{R}\frac{\sin\beta}{\beta}\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)e^{i[\kappa R-\omega t-(N-1)\alpha]}\right]\left[\frac{\tilde{\xi}_L' b}{R}\frac{\sin\beta}{\beta}\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)e^{i[\kappa R-\omega t-(N-1)\alpha]}\right]^*$$

$$\mathrm{I} = \mathrm{K}\left(\frac{\xi_L' b}{R}\right)^2\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2$$

, where

$\alpha = \frac{\kappa a \sin\theta}{2}$, $\beta = \frac{\kappa b \sin\theta}{2}$ and $\left(\frac{\xi_L' b}{R}\right)^2 = \tilde{\xi}_L' \tilde{\xi}_L'^{\,*}\left(\frac{b}{R}\right)^2$

When $\theta = 0$, $\alpha = \beta = 0$, then

$\frac{\sin\beta}{\beta} = \frac{\sin(N\alpha)}{\sin\alpha} = \frac{\sin(0)}{0} = 1$

$$\mathrm{I}(\theta = 0) = \mathrm{K}\left(\frac{\xi_L' b}{R}\right)^2$$

$$\mathrm{I}(\theta) = \mathrm{I}(\theta = 0)\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2$$

$$\frac{\mathrm{I}(\theta)}{\mathrm{I}(\theta = 0)} = \left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2$$

, where

$$\alpha = \frac{\kappa a \sin\theta}{2} = \frac{\pi a \sin\theta}{\lambda}$$
$$\beta = \frac{\kappa b \sin\theta}{2} = \frac{\pi b \sin\theta}{\lambda}$$

Remarks：

(1)For the multiple slit pattern $a > b$

(2)For the diffraction in a crystal

atomic spacing = a

atomic size = b

$$a \gg b$$

Discussion

$$\frac{I(\theta)}{I(\theta=0)} = \left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2$$

$$\alpha = \frac{\pi a \sin\theta}{\lambda}$$

$$\beta = \frac{\pi b \sin\theta}{\lambda}$$

$\left(\frac{\sin\beta}{\beta}\right)^2$ is a function with its first minimum at $\beta = \pi$

$$\frac{\sin\theta}{\lambda} = \frac{1}{b}$$

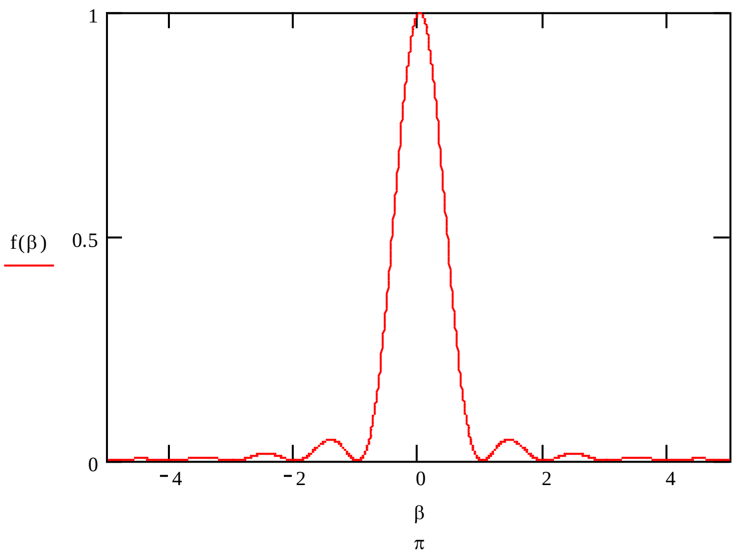
The second minimum occurs at $\beta = 2\pi$

$$\frac{\sin\theta}{\lambda} = \frac{2}{b}$$

The functoin (sin β/β)^2

$\beta := -5\pi, -4.99\pi .. 5\pi$

$$f(\beta) := \left(\frac{\sin(\beta)}{\beta}\right)^2$$



$\left(\frac{\sin\beta}{\beta}\right)^2$ is modulated by $\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2$ that is a periodic function with its periodic maximum at $\alpha = n\pi$

$$\frac{\sin\theta}{\lambda} = \frac{n}{a}$$

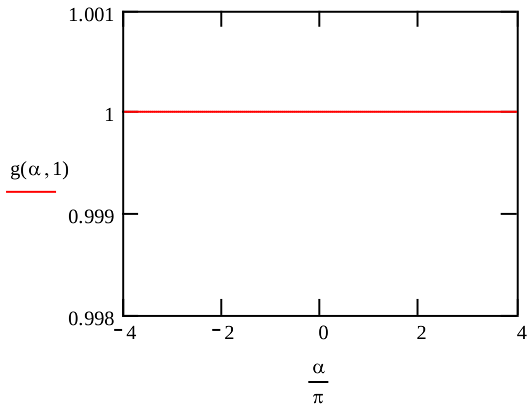
where n is an integer.

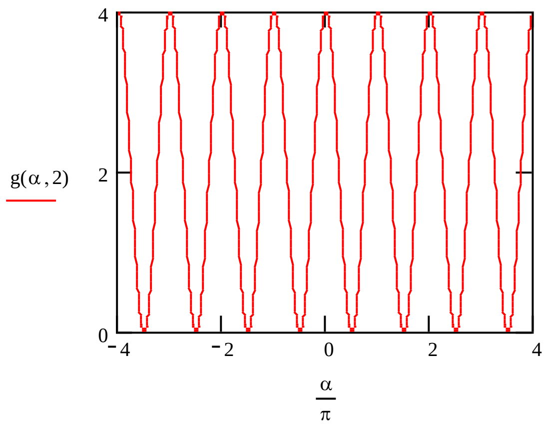
This may explain the diffraction patterns from multiple slits (Fig. 10.20) in "Optics" Eugene Hecht , 2nd edition.

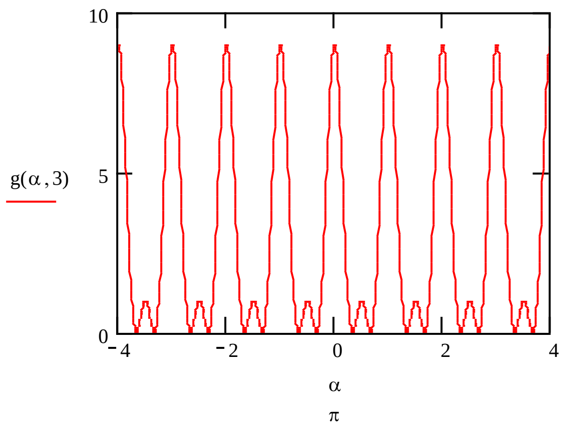
The plots of $\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2$ are shown below.

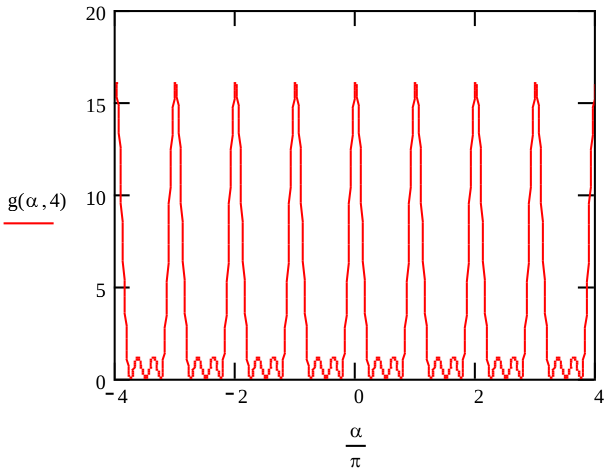
$$\alpha := -4\pi, -3.99\pi \,..\, 4\pi$$

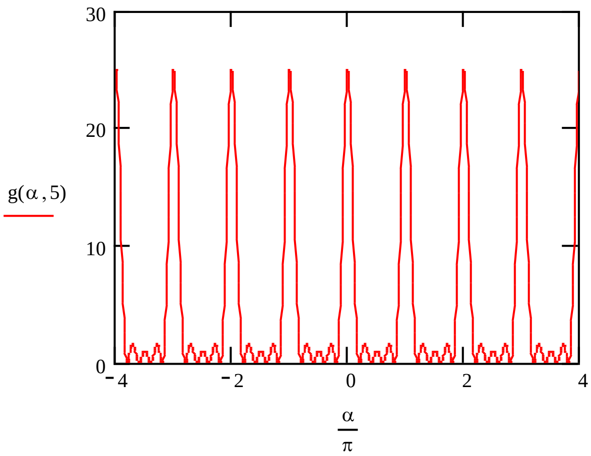
$$g(\alpha, N) := \left(\frac{\sin(N\cdot\alpha)}{\sin(\alpha)}\right)^2$$

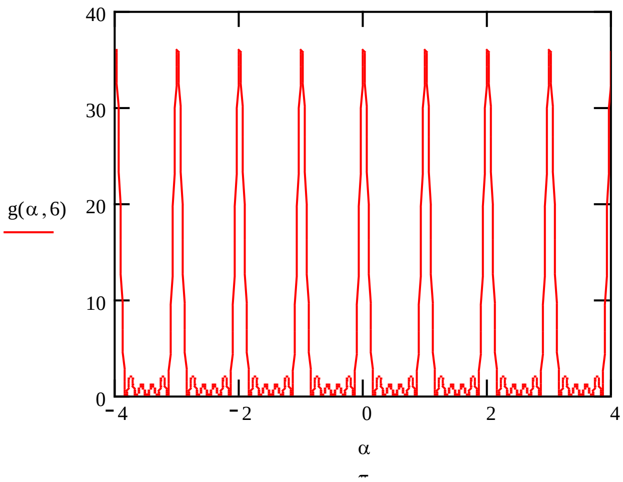


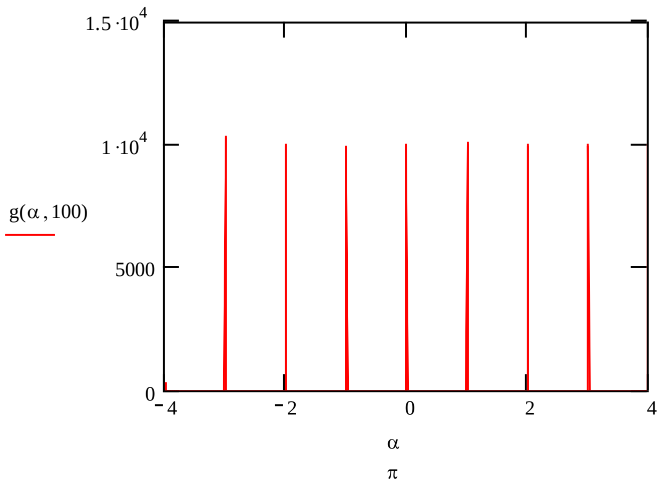





20
15
10
5
0
g(α, 4)
−4
−2
0
2
4
α/π


30
20
10
0
g(α, 5)
−4
−2
0
2
4
α/π


40
30
20
10
0
g(α, 6)
−4
−2
0
2
4
α


1.5 ·10^4
1 ·10^4
5000
0
g(α , 100)
−4
−2
0
2
4
α
π

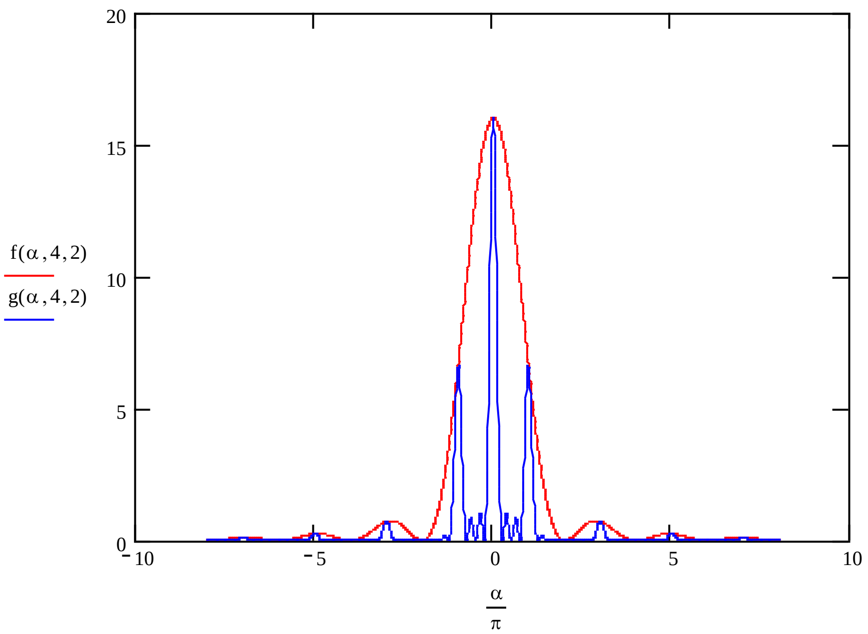
The plots of $\left(\frac{\sin\beta}{\beta}\right)^2$ modulated by $\left(\frac{\sin(N\alpha)}{\sin\alpha}\right)^2$ are shown below.

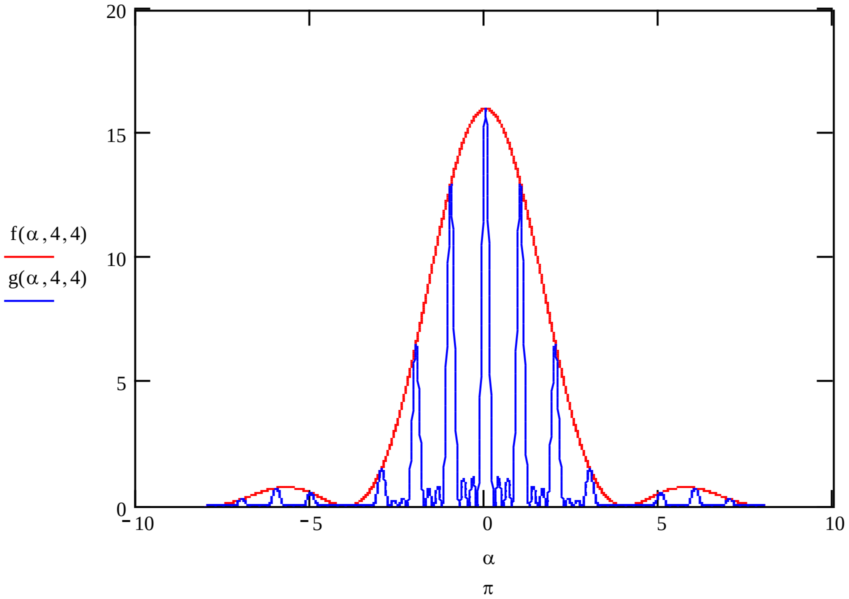
The Fraunhofer diffraction from N slits

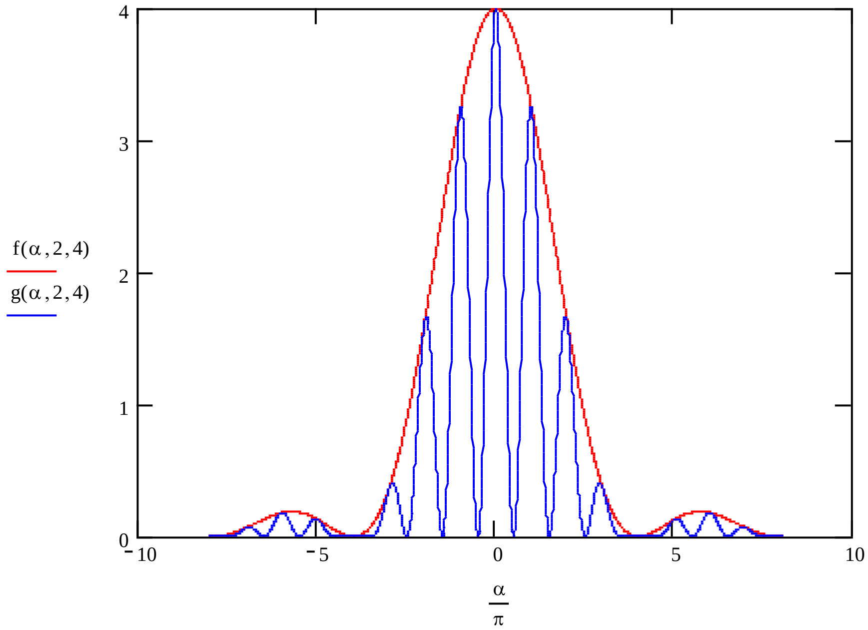
The slit spacing a is M times the slit width b

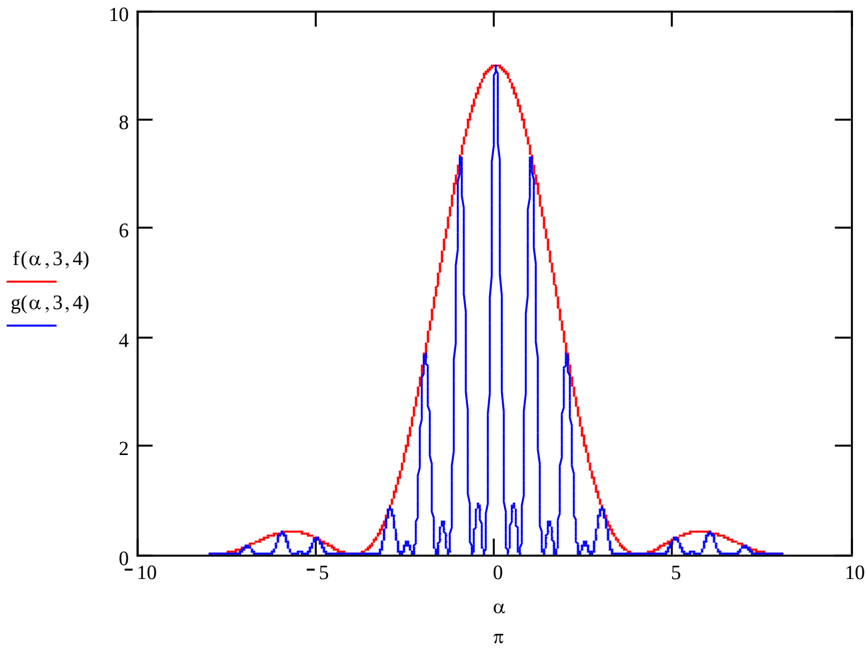
$$\alpha := -8\pi, -7.99\pi .. 8\pi$$

$$f(\alpha, N, M) := \left(\frac{\sin\left(\frac{\alpha}{M}\right)}{\frac{\alpha}{M}}\right)^2 N^2 \qquad g(\alpha, N, M) := \left[\left(\frac{\sin\left(\frac{\alpha}{M}\right)}{\frac{\alpha}{M}}\right)^2\right]\left(\frac{\sin(N\cdot\alpha)}{\sin(\alpha)}\right)^2$$






4
3
2
1
0
f(α, 2, 4)
g(α, 2, 4)
− 10
− 5
0
5
10
α
π


10
8
6
4
2
0
f(α, 3, 4)
g(α, 3, 4)
− 10
− 5
0
5
10
α
π

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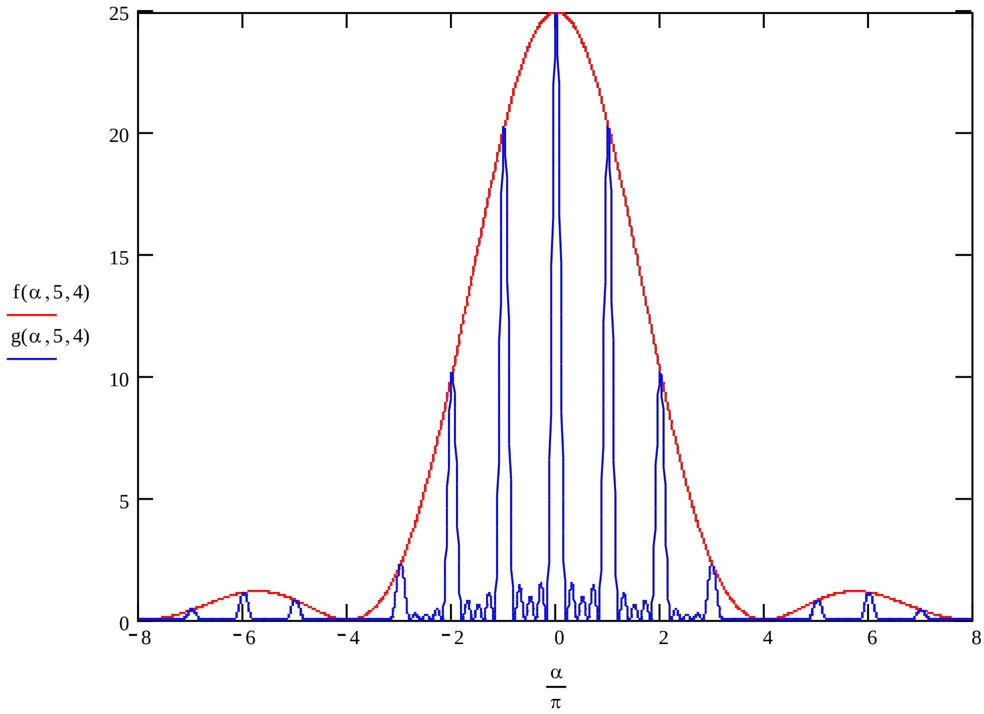

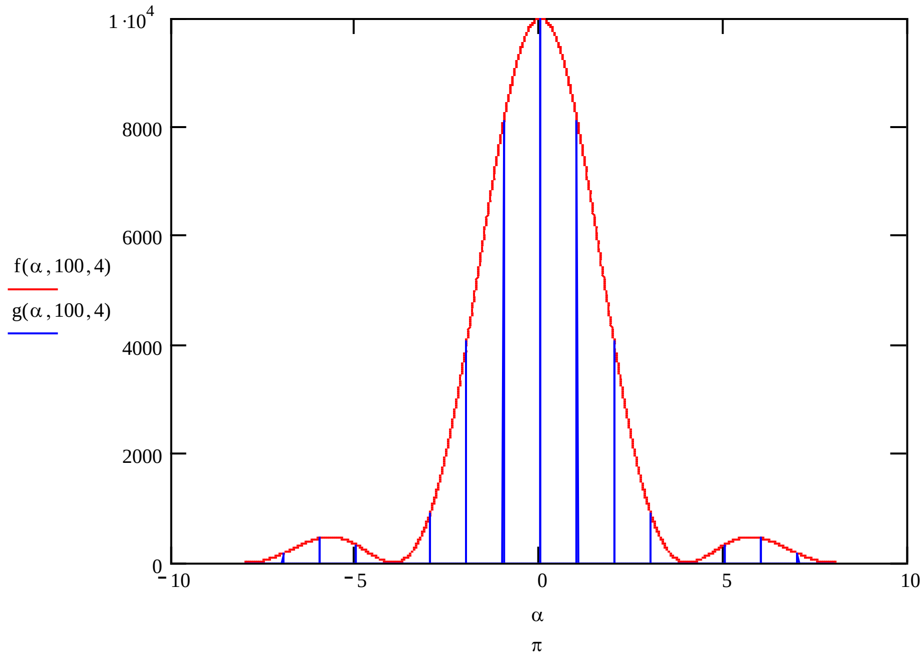

Note that a is the spacing between adjacent slit, which is equivalent to a one-dimensional periodic structure

The periodic maximum occurs at $\frac{\sin\theta}{\lambda} = \frac{n}{a}$.

Remarks:

(1) There exists a one-dimensional periodic structure in the space of $\frac{\sin\theta}{\lambda}$
We will illustrate that the $\frac{\sin\theta}{\lambda}$ space is the momentum κ space in the next chapters.

(2)$\frac{\sin\theta}{\lambda} = \frac{n}{a}$ is similar to Bragg’ s law

$$2d\sin\theta = n\lambda$$
$$\frac{\sin\theta}{\lambda} = \frac{n}{2d}$$