CS 2336 Discrete Mathematics

Lecture 9

Sets, Functions, and Relations: Part I

Outline

- What is a Set ?
- Set Operations
- Identities
- Cardinality of a Set
  - Finite and Infinite Sets
  - Countable and Uncountable Sets
  - Inclusion-Exclusion Principle (Revisited)

What is a Set ?

- A set is an unordered collection of objects

  ➔ We call the objects: members, items, or elements

- One way to describe a set is to list all its elements inside { }

  Examples :

  1. { Keroro, Giroro, Kururu, Dororo, Tamama }
     is a set containing five elements
  2. { a, e, i, o, u } is the set of vowels

What is a Set ?

- When the items of a set have trends, we may use … to help the description

  Examples :

  1. { 1, 2, …, 9 } is the set of integers from 1 to 9
  2. { …, -2, -1, 0, 1, 2, … } is the set of all integers
  3. { a, b, c, …, z } is the set of English characters

What is a Set ?

- We may also specify the items in the set by stating exactly what their properties are
- This method is called set builder notation

Examples :

1. { x | x is an odd positive integer less than 10 } is exactly the set { 1, 3, 5, 7, 9 }
2. { n | n is a perfect square } is the set of all perfect squares

What is a Set ?

- A set with no items is called an empty set

  ➔ It is denoted by $\{\ \}$ or $\varnothing$

- The order of describing items in a set does not matter : $\{ 1, 3, 5 \} = \{ 3, 5, 1 \}$

- Also, repetition does not matter :

$$\{ 1, 5, 5, 5 \} = \{ 1, 5 \}$$

- Membership Symbol $\in$

$$5 \in \{ 1, 3, 5 \} \qquad 7 \notin \{ 1, 3, 5 \}$$

Test Your Understanding

- How many items in each of the following sets ?

1. $\{ 3, 4, 5, \ldots, 10 \}$
2. $\{ 2, 3, 3, 4, 4, 2, 1 \}$
3. $\{ 2, \{ 2 \}, \{ \{ 1, 2, 3, 4, 5, 6 \} \} \}$
4. $\varnothing$
5. $\{ \varnothing \}$

Russell's Paradox

- As shown in the previous examples, the objects of a set can be sets

  Example : $A = \{ \emptyset, \{ \emptyset \}, \{ \{ \emptyset \} \} \}$

- In that case, we can say something like :

$$\{ \emptyset \} \in A \qquad \{ \emptyset, \{ \emptyset \} \} \notin A$$

  or we may define something like :

$$\{ x \mid x \notin A \}$$

Russell's Paradox

- How about the following set ?

$$\{ x \mid x \notin x \}$$

- Let us call the above set S

  S contains every set x which does not contain itself as an element

- Example :

$$\{ a, b \} \in S \text{ because } \{ a, b \} \notin \{ a, b \}$$

Russell's Paradox

Question : Is S an element of S ?

- Bertrand Russell in 1901 observed that :

Case 1: If the answer is YES

➔ $S \notin S$ (by the property of items in S)

➔ Contradiction with YES

Case 2: If the answer is NO

➔ $S \in S$ (by the property of items in S)

➔ Contradiction with NO

Russell’s Paradox

- There are other similar paradoxes :

  1. Barber’s Paradox
  2. Grelling-Nelson Paradox :

     Some adjective can describe themselves. For instance :

     short, English, polysyllabic, unhyphenated, …

     We call them homological.

     Otherwise, we call them heterological.

     Question : Is “heterological” heterological ?

- Check wikipedia for more information

Subsets

- Given two sets A and B

> We say A is a subset of B, denoted by $A \subseteq B$, if every item in A also appears in B

Ex : A = the set of primes, B = the set of integers

- We say A = B if every item in A is an item in B, and vice versa

  Equivalently, $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

- $A = \{ 1, 2, 3 \}$. Is $\emptyset \subseteq A$? Is $\emptyset \in A$?

Venn Diagram

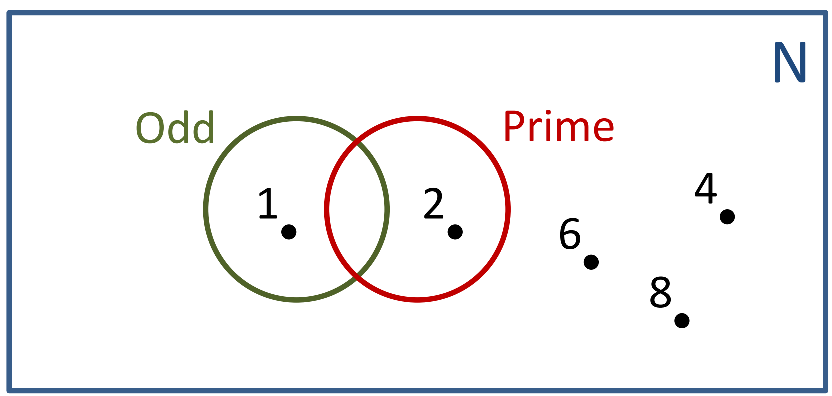
- John Venn in 1881 introduced a graphical way to represent sets and their relations

  rectangle ➔ universal set U

  circle, or figure inside the rectangle ➔ set

  point ➔ item

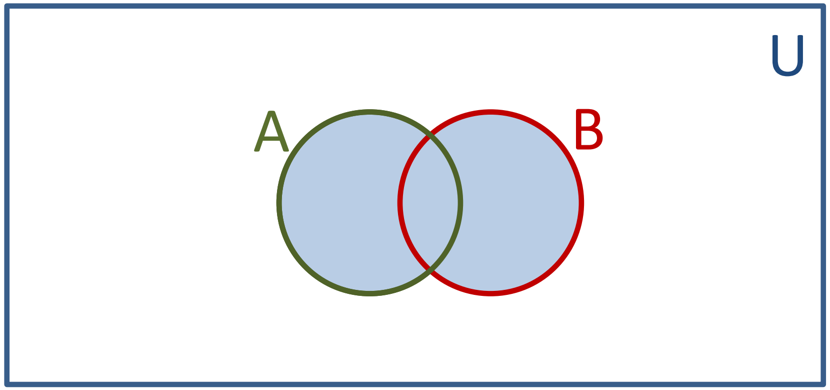
- Example :



Set Operations

- Given two sets A and B

The union of A and B, denoted by $A \cup B$, is the set containing exactly all items in A or in B

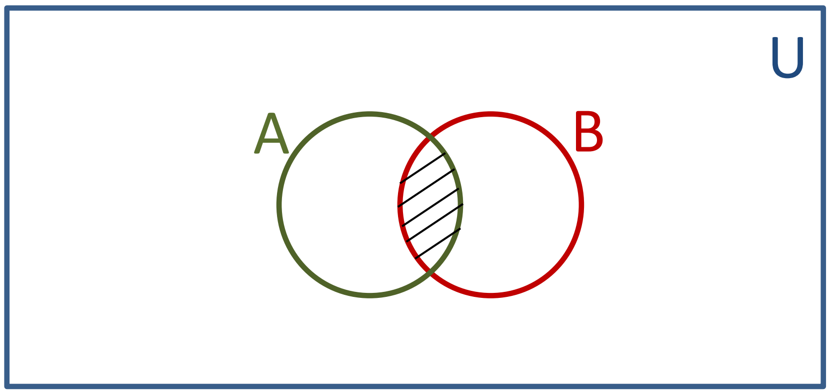


$A \cup B$ is shaded

Set Operations

- Given two sets A and B

The intersection of A and B, denoted by $A \cap B$, is the set containing the common items of A and B

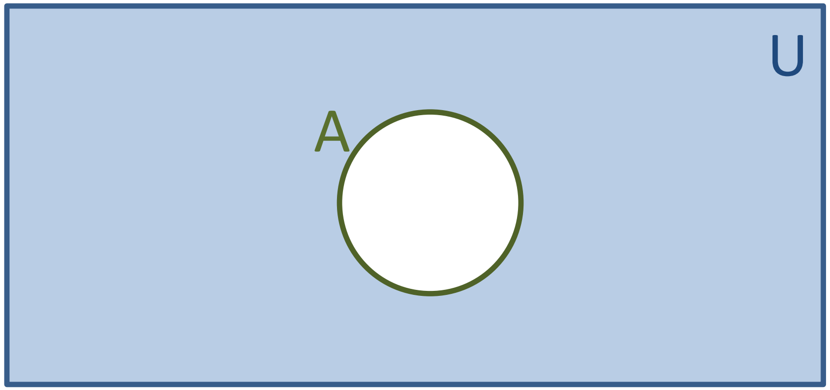


$A \cap B$ is shaded

Set Operations

- Given a set A

The complement of A, denoted by $\overline{A}$, is the set containing exactly those items not in A

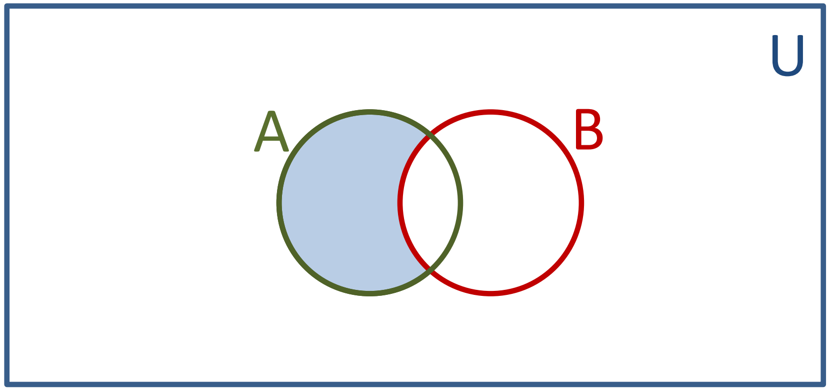


$\overline{A}$ is shaded

Set Operations

- Given two sets A and B

The difference of A and B, denoted by A – B, is the set containing exactly items in A but not in B



A – B is shaded

Set Operations

- Given a set A

> The power set of A, denoted by 2^A, is the set containing exactly all subsets of A

- Example :

$$A = \{ 0, 1 \}$$

$$2^A = \{ \varnothing, \{ 0 \}, \{ 1 \}, \{ 0, 1 \} \}$$

Q: If A has n items, how many items in 2^A ?

Set Identities

- Important Identities
  1. Identity Laws

     $A \cap U \equiv A$ $\quad$ $A \cup \varnothing \equiv A$
  2. Domination Laws (A is gone)

     $A \cap \varnothing \equiv \varnothing$ $\quad$ $A \cup U \equiv U$
  3. Idempotent Laws (okay to apply many times)

     $A \cap A \equiv A$ $\quad$ $A \cup A \equiv A$
  4. Complementation Law

     $\overline{(\overline{A})} \equiv A$

In the above, U is the universe

Set Identities

- Important Identities

5. Commutative Laws

$$A \cap B \equiv B \cap A \qquad A \cup B \equiv B \cup A$$

6. Associative Laws (parentheses may be omitted)

$$A \cap (B \cap C) \equiv (A \cap B) \cap C$$

$$A \cup (B \cup C) \equiv (A \cup B) \cup C$$

7. Distributive Laws (similar to + and × in math expression)

$$A \cap (B \cup C) \equiv (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) \equiv (A \cup B) \cap (A \cup C)$$

Set Identities

- Important Identities

8. De Morgan's Laws

$$\overline{A \cup B} \equiv \overline{A} \cap \overline{B}$$

$$\overline{A \cap B} \equiv \overline{A} \cup \overline{B}$$

9. Absorption Laws (a set may be omitted)

$$A \cap (A \cup B) \equiv A$$

$$A \cup (A \cap B) \equiv A$$

10. Complement Laws

$$A \cap \overline{A} \equiv \varnothing \qquad A \cup \overline{A} \equiv U$$

How to Prove an Identity ?

- Method 1 :

Use a membership table (similar to truth table)

- Example : Prove that $\overline{A \cap B} \equiv \overline{A} \cup \overline{B}$

A	B	$\overline{A}$	$\overline{B}$	$A \cap B$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	1	0	0
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	0	1	1

How to Prove an Identity ?

- Method 2 :

> Show that left side is a subset of right side, and vice versa

- Example : Prove that $\overline{A \cap B} \equiv \overline{A} \cup \overline{B}$

First, for each item x in $\overline{A \cap B}$, $x \notin A \cap B$ (by definition of complement)

Then by definition of intersection, we have

$$\neg ((x \in A) \wedge (x \in B))$$

How to Prove an Identity ?

- Proof (cont) :

  By De Morgan's Law for propositions, we have

  $$(x \notin A) \vee (x \notin B)$$

  Then, by definition of complement, this implies

  $$(x \in \overline{A}) \vee (x \in \overline{B})$$

  Finally, by definition of union, we have

  $$x \in \overline{A} \cup \overline{B}$$

→ The above implies that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

How to Prove an Identity ?

- Proof (cont) :

  It remains to prove: $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

  To do so, we apply a similar approach.

  For each item y in $\overline{A} \cup \overline{B}$, by definition of union, we see that $(y \in \overline{A}) \vee (y \in \overline{B})$

  Then, by definition of complement, we have

  $$\neg (y \in A) \vee \neg(y \in B)$$

  By De Morgan's Law, the above implies

  $$\neg ((y \in A) \wedge (y \in B))$$

How to Prove an Identity ?

- Proof (cont) :

By definition of intersection, we get

$$\neg (y \in A \cap B)$$

Finally, by definition of complement, we have

$$y \in \overline{A \cap B}$$

➔ The above statements imply that

$$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$$

➔ Since each side is a subset of the other side, the two sides are equivalent

How to Prove an Identity ?

- Method 3:

Apply known set identities

- Example : Prove that

$$\overline{A \cup (B \cap C)} \equiv (\overline{C} \cup \overline{B}) \cap \overline{A}$$

Proof : $\overline{A \cup (B \cap C)} \equiv \overline{A} \cap (\overline{B} \cup \overline{C})$ [why?]

$\equiv \overline{A} \cap (\overline{C} \cup \overline{B})$ [why?]

$\equiv (\overline{C} \cup \overline{B}) \cap \overline{A}$ [why?]

Cardinality of a Set

- Given a set A

> If there are exactly n distinct items in A, where n is a nonnegative integer, then we say A is finite and n is the cardinality of A. The cardinality of A is denoted |A|

- Example :

$$A = \{ 0, 1, orange, apple \}$$

➔ $|A| = 4$

Cardinality of a Set

- If a set is not finite, then it is an infinite set
- Examples :

  (1) The set of natural numbers

  (2) The set of real numbers
- Although the number of items in an infinite set is infinite, mathematicians still want to define cardinality for infinite sets, and use that to compare if they have the same size …

Cardinality of a Set

- We denote the cardinality of the set of natural numbers as $\aleph_0$
- For comparison, we use the following :

  Given a set A and a set B

> if there is a one-to-one correspondence between the items in A and the items in B, then we say they have the same cardinality, and we write $|A| = |B|$

Cardinality of a Set

- Which of the following sets have cardinality $\aleph_0$?

  That is, with same cardinality as the set of natural numbers

  (1) The English alphabet

  (2) The set of nonnegative integers

  (3) The set of integers

  (4) The set of even numbers

  (5) The set of rational numbers

  (6) The set of real numbers

Countable and Uncountable

Definition :

If a set is finite, or if it has cardinality $\aleph_0$, then we say the set is countable

Else, the set is uncountable

Georg Cantor in 1891 showed the following amazing result :

The set of real numbers is uncountable

The Set of Reals is Uncountable

- Proof :

First, we can see that a subset of a countable set must be countable (It is a bit tricky …)

➔ To obtain the desired result, it is sufficient to show that the set R' of real numbers in (0, 1) is uncountable.

Suppose on the contrary that R' is countable. Thus, there is a one-to-one correspondence between R' and the natural number set N

Let x_k = the number in R' corresponding to k in N

The Set of Reals is Uncountable

- Next, we construct x whose k^{th} digit is equal to "the k^{th} digit of x_k (mod 2) + 1

E.g.	x_1	0.71828182845904523536…
	x_2	0.44269504088896340740…
	x_3	0.14159265358979323846…
	x_4	0.41421356237309504880…
	…	…
	x =	0.2121…

- x is in (0,1) but without any correspondence

➔ contradiction occurs and the proof completes

Principle of Inclusion-Exclusion

- Let A and B be finite sets

  We can easily argue that :

$$| A \cup B | = |A| + |B| - | A \cap B |$$

- We can generalize the above for the case of three finite sets A, B, and C :

$$| A \cup B \cup C | = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

Principle of Inclusion-Exclusion

- In fact, the above can be further generalized for union of a collection of finite sets :

$$
\begin{aligned}
& |\, A_1 \cup A_2 \cup \ldots \cup A_k \,| \\
= \; & |A_1| + |A_2| + \ldots + |A_k| \\
& \quad - |A_1 \cap A_2| - |A_1 \cap A_3| - \ldots - |A_{k-1} \cap A_k| \\
& \quad + |A_1 \cap A_2 \cap A_3| + \ldots + |A_{k-2} \cap A_{k-1} \cap A_k| \\
& \quad + \ldots \\
& \quad + (-1)^{k-1} \, |A_1 \cap A_2 \cap \ldots \cap A_k \,|
\end{aligned}
$$

- This is called the principle of inclusion-exclusion

Principle of Inclusion-Exclusion

- Example 1 :

  How many integers between 1 and 250 are divisible by any of the numbers 2, 3, 5, or 7 ?

- Example 2 :

  How many nonnegative integral solutions does

$$x + y + z = 11$$

  have, when $x \leq 3$, $y \leq 4$, $z \leq 6$?