CS 2336
Discrete Mathematics

Lecture 11

Sets, Functions, and Relations: Part III

Outline

- What is a Relation ?
- Types of Binary Relations
- Representing Binary Relations
- Closures

Cartesian Product

- Let A and B be two sets

> The cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs
>
> $$\{ (a, b) \mid a \in A \text{ and } b \in B \}$$

- A = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K } = all ranks

  B = { ♠, ♥, ♦, ♣ } = all suits

  ➔ $A \times B$ = all 52 cards in a deck

  = { (1,♠), (2,♠), (3,♠), ..., (J,♣), (Q,♣), (K,♣) }

Cartesian Product

- Let $A_1, A_2, \ldots, A_k$ be k sets

> The cartesian product of $A_1, A_2, \ldots, A_k$, denoted by $A_1 \times A_2 \times \ldots \times A_k$, is the set of all ordered pairs
>
> $\{ (a_1, a_2, \ldots, a_k) \mid a_j \in A_j \text{ for all } j = 1, 2, \ldots, k \}$

- Let A_j = the set $\Re$ of real numbers, for all j

  ➔ $A_1 \times A_2 \times A_3$ = the 3-d Euclidean space $\Re^3$

Relation

A binary relation from A to B is a subset of the cartesian product $A \times B$

- Example:

  A = { 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K }

  B = { ♠, ♥, ♦, ♣ }

  Spades = { (1,♠), (2,♠), (3,♠), ..., (Q,♠), (K,♠) }

  ➔ Spades is a binary relation

Relation

A k-ary relation is a subset of a cartesian of k sets

- Example:

  $L = \{ (x, y, z) \mid 2x + 3y + z = 0, \text{ and } x, y, z \in \Re \}$

  ➔ the line L is a ternary relation of the space $\Re^3$

Binary Relation

- In the remaining of this lecture, we focus on a special type of relations :

the binary relation from a set A to A

- Such a relation is called a binary relation on A
- Example : A = the set of integers

$$R = \{ (a, b) \mid a - b \geq 10 \}$$

Types of Binary Relations

A binary relation R on A is said to be **reflexive** if $(a, a) \in R$ for every $a \in A$

- Which of the following relations are reflexive ?
  - $R = \{ (a, b) \mid a - b \geq 10,$ a, b are integers $\}$
  - $S = \{ (a, b) \mid a \leq b,$ a, b are integers $\}$
  - $T = \{ (a, b) \mid a < b,$ a, b are integers $\}$
  - $U = \{ (x, y) \mid$ x and y are on the same weekday, x, y are days in April 2013 $\}$

Types of Binary Relations

A binary relation R on A is said to be symmetric if $(a, b) \in R$ implies $(b, a) \in R$

- Which of the following relations are symmetric ?
  - $R = \{ (a, b) \mid a - b \geq 10,\ a, b \text{ are integers} \}$
  - $S = \{ (a, b) \mid a \leq b,\ a, b \text{ are integers} \}$
  - $T = \{ (a, b) \mid a < b,\ a, b \text{ are integers} \}$
  - $U = \{ (x, y) \mid x \text{ and } y \text{ are on the same weekday}, x, y \text{ are days in April 2013} \}$

Types of Binary Relations

A binary relation R on A is said to be antisymmetric if $(a, b) \in R$ implies $(b, a) \notin R$ unless $a = b$

- Which of the following are antisymmetric ?
  - $R = \{ (a, b) \mid a - b \geq 10, \ a, b \text{ are integers} \}$
  - $S = \{ (a, b) \mid a \leq b, \ a, b \text{ are integers} \}$
  - $T = \{ (a, b) \mid a < b, \ a, b \text{ are integers} \}$
  - $U = \{ (x, y) \mid x \text{ and } y \text{ are on the same weekday}, \ x, y \text{ are days in April 2013} \}$

Types of Binary Relations

A binary relation R on A is said to be transitive if $(a, b), (b, c) \in R$ implies $(a, c) \in R$

- Which of the following are transitive ?
  - $R = \{ (a, b) \mid a - b \geq 10,\ a, b \text{ are integers} \}$
  - $S = \{ (a, b) \mid a \leq b,\ a, b \text{ are integers} \}$
  - $T = \{ (a, b) \mid a < b,\ a, b \text{ are integers} \}$
  - $U = \{ (x, y) \mid$ x and y are on the same weekday, x, y are days in April 2013 $\}$

Representing Binary Relations

- Let A be a finite set
- The binary relation on A can be convenient represented in two different ways :

- Method 1: Matrix Form

A = { 1, 2, 3, 4 }

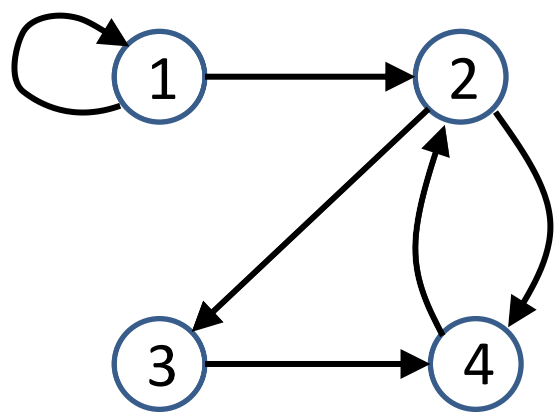
R = { (1,1), (1,2), (2,3),
(2,4), (3,4), (4,2) }

	1	2	3	4
1	✓	✓		
2			✓	✓
3				✓
4		✓		

Representing Binary Relations

- Method 2: Directed Graph
- A = { 1, 2, 3, 4 }

  R = { (1,1), (1,2), (2,3), (2,4), (3,4), (4,2) }



Closures

- Given a binary relation R, we may obtain a new relation R' by adding items into R, such that R' will have certain property

- Example :

  R = { (1,1), (1,2), (2,3), (2,4), (3,4), (4,2) }

  If we add (2,2), (3,3), and (4,4) into R, the resulting relation will be reflexive

Closures

- Let R be a binary relation

The smallest possible relation R' that contains R as a subset, such that R' has a property P, is the closure of R with respect to P

	1	2	3	4
1	✓	✓		
2			✓	✓
3				✓
4		✓		

R

	1	2	3	4
1	✓	✓		
2		✓	✓	✓
3			✓	✓
4		✓		✓

Reflexive closure of R

Closures

- What is the transitive closure of
  R = { (1,1), (1,2), (2,3), (2,4), (3,4), (4,2) } ?

	1	2	3	4
1	✓	✓		
2			✓	✓
3				✓
4		✓		

R

	1	2	3	4
1	✓	✓	✓	✓
2		✓	✓	✓
3		✓	✓	✓
4		✓	✓	✓

Transitive closure of R

Finding Transitive Closure

- Getting the transitive closure seems difficult

  Is there a systematic way to get this ?

- Consider the directed graph representation

  ➔ R = all pairs of vertices where one can reach the other in 1 step

- We can (not easily) show that for vertices x and y,

> x can reach y in the graph
>
> ⇔ (x, y) is in the transitive closure of R

Finding Transitive Closure

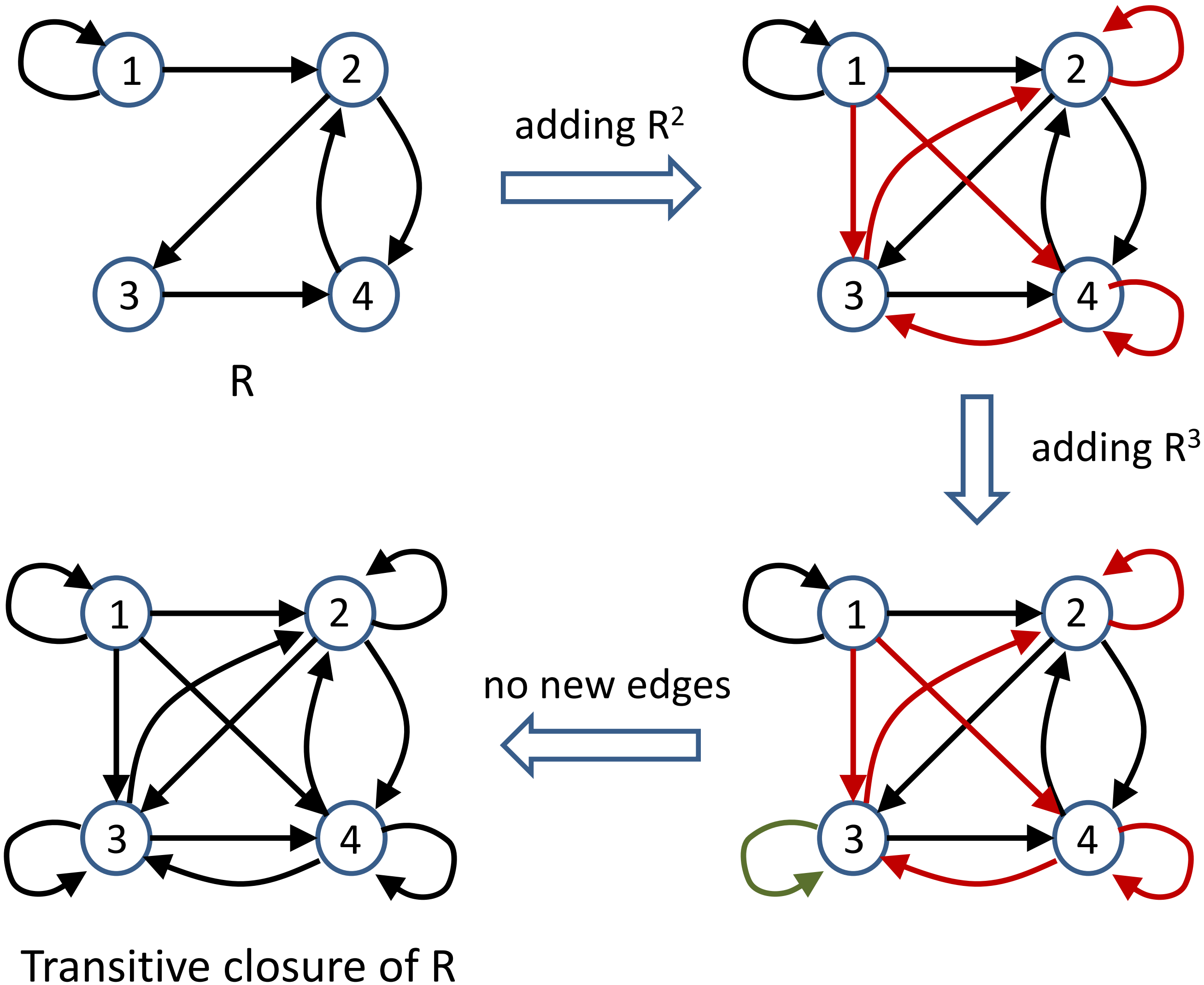
- Let R^k = all pairs of vertices where one can reach the other in exactly k steps

  ➔ $R = R^1$

- We can repeatedly obtain R^2, R^3, and so on, until we cannot add any new edges

  ➔ the resulting graph corresponds to the transitive closure of R

Finding Transitive Closure



Finding Transitive Closure

- Apart from the previous method, there are faster ways to compute the transitive closure

- If the matrix form is given, and |A|= n
  1. Recursive doubling algorithm : $O(n^3 \log n)$ time
  2. Floyd-Warshall algorithm : $O(n^3)$ time

     → Code is super simple : Only 3 for loops !