CS 2336
Discrete Mathematics

Lecture 13

Graphs: Introduction

Outline

- What is a Graph?
- Terminology
- Some Special Simple Graphs
- Subgraphs and Complements
- Graph Isomorphism

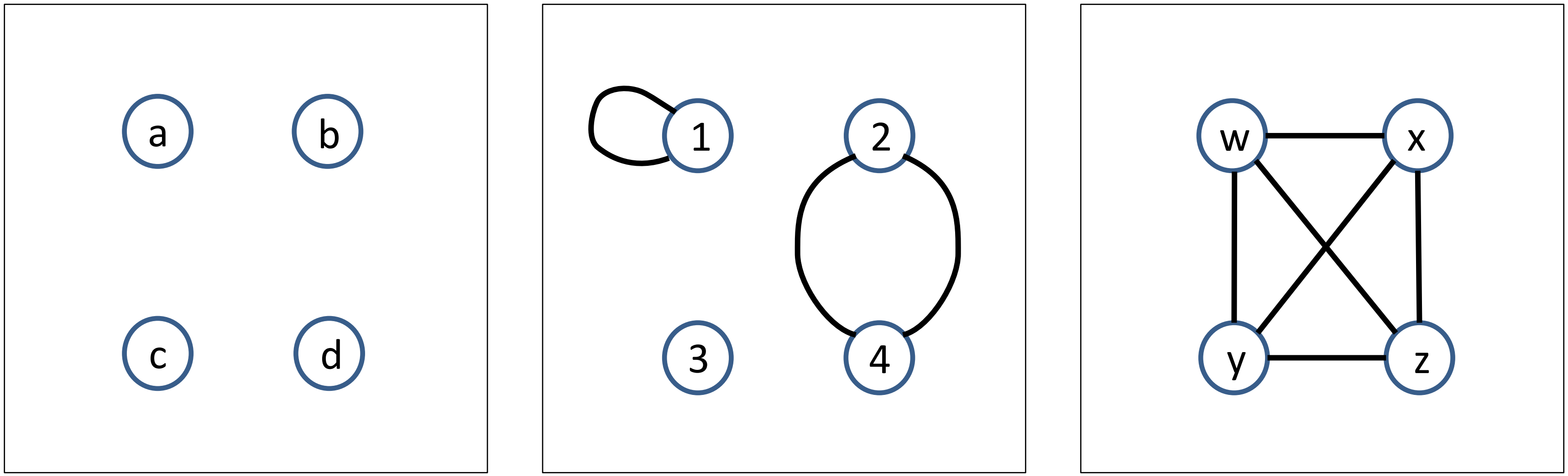
What is a Graph ?

A graph consists of a nonempty set V of vertices and a set E of edges, where each edge in E connects two (may be the same) vertices in V.

- Let G be a graph associated with a vertex set V and an edge set E

  We usually write G = (V, E) to indicate the above relationship

Examples



- Furthermore, if each edge connects two different vertices, and no two edges connect the same pair of vertices, then the graph is a simple graph
- Which of the above is a simple graph ?

Directed Graph

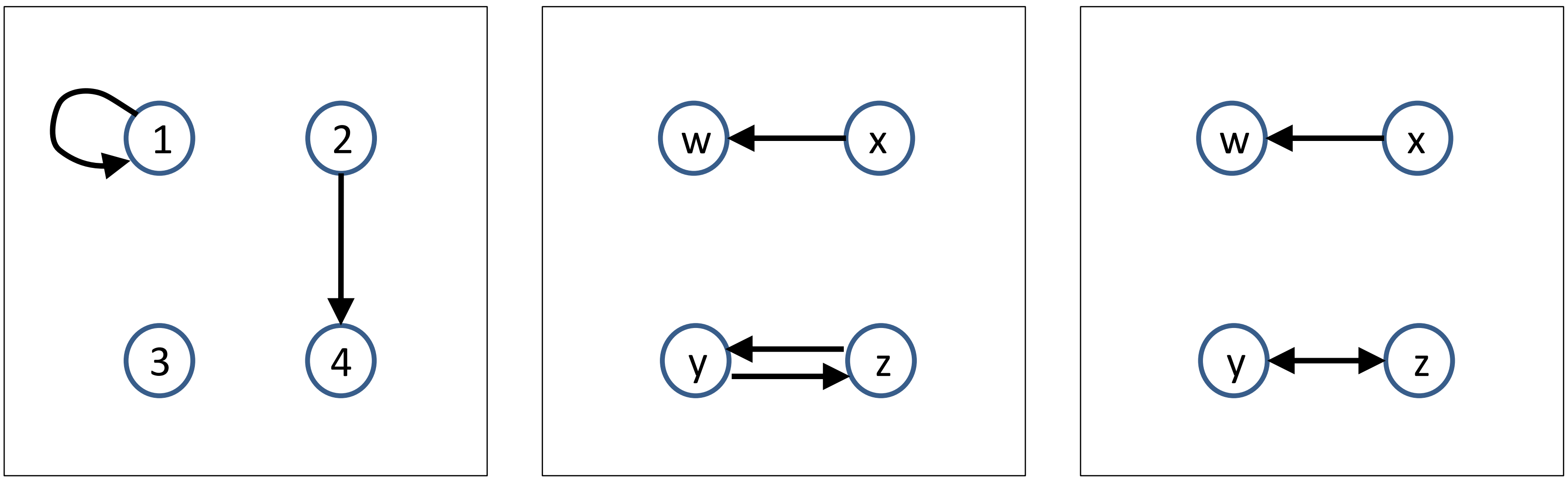
- Sometimes, we may want to specify a direction on each edge

  Example : Vertices may represent cities, and edges may represent roads (can be one-way)

- This gives the directed graph as follows :

> A **directed graph G** consists of a nonempty set V of vertices and a set E of directed edges, where each edge is associated with an ordered pair of vertices. We write G = (V, E) to denote the graph.

Examples



Test Your Understanding

- Suppose we have a simple graph G with n vertices
  What is the maximum number of edges G can contain, if

  (i) G is an undirected graph ?

  (ii) G is a directed graph ?

Terminology (Undirected Graph)

- Let e be an edge that connects vertices u and v

We say (i) e is incident with u and v

(ii) u and v are the endpoints of e ;

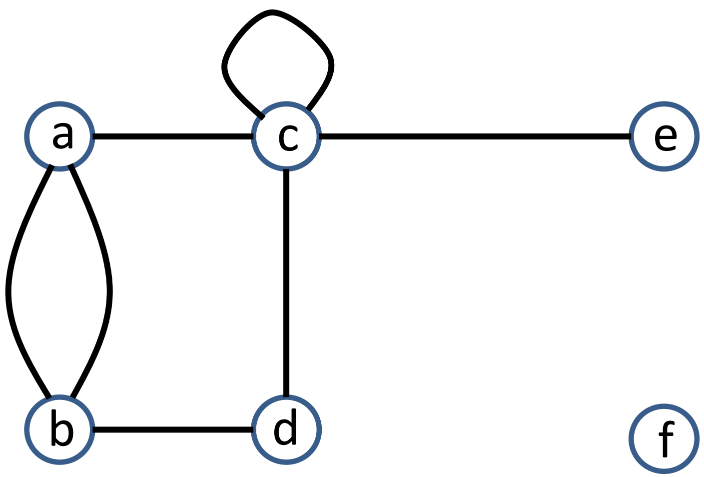
(iii) u and v are adjacent (or neighbors)

(iv) if u = v, the edge e is called a loop

- The degree of a vertex v, denoted by deg(v), is the number of edges incident with v, except that a loop at v contributes twice to the degree of v

Example

- What are the degrees and neighbors of each vertex in the following graph ?



Handshaking Theorem

- Let G = (V, E) be an undirected graph with m edges

Theorem:

$$\sum_{v \in V} \deg(v) = 2m$$

- Proof : Each edge e contributes exactly twice to the sum on the left side (one to each endpoint).

Corollary : An undirected graph has an even number of vertices of odd degree.

Terminology (Directed Graph)

- Let e be an edge that connects vertices from u to v

  We say (i) u = initial vertex, v = terminal vertex ;

  (ii) u is adjacent to v;

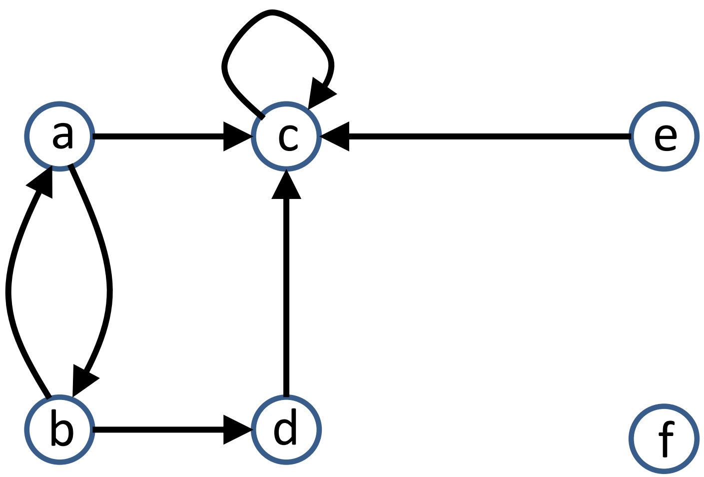
  (iii) v is adjacent from u;

  (iv) if u = v, the edge e is called a loop

- The in-degree of a vertex v, denoted by $\deg^{-}(v)$, is the number of edges with v as terminal vertex
- The out-degree of a vertex u, denoted by $\deg^{+}(u)$, is the number of edges with u as initial vertex

Example

- What are the in- and out-degrees of each vertex in the following graph ?



Handshaking Theorem

- Let G = (V, E) be directed graph with m edges

Theorem:

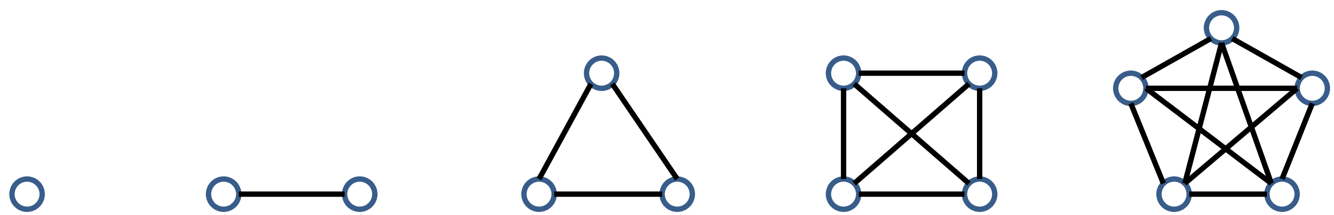
$$\sum_{v \in V} \deg^-(v) = \sum_{u \in V} \deg^+(u) = m$$

- Proof : Each edge e contributes exactly once to the in-degree and once to the out-degree

Some Special Simple Graphs

Definition: A complete graph on n vertices, denoted by K_n, is a simple graph that contains one edge between each pair of distinct vertices

Examples :

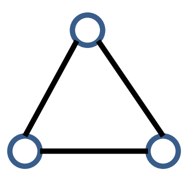


K_1 K_2 K_3 K_4 K_5

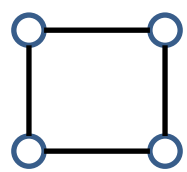
Some Special Simple Graphs

Definition: A cycle C_n, $n \geq 3$, is a graph that consists of n vertices $v_1, v_2, \ldots, v_n$ and n edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, …, $\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$

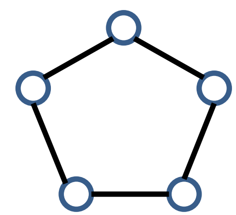
Examples :



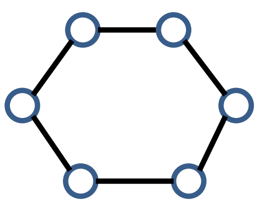
C_3



C_4



C_5

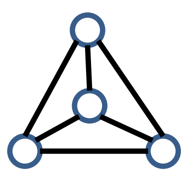


C_6

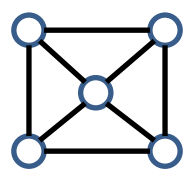
Some Special Simple Graphs

Definition: A **wheel W_n**, $n \geq 3$, is a graph that consists of a cycle C_n with an extra vertex that connects to each vertex in C_n

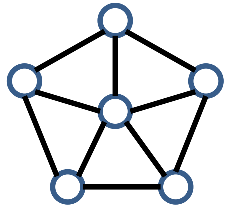
Examples :



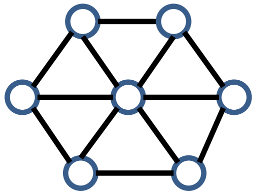
W_3



W_4



W_5

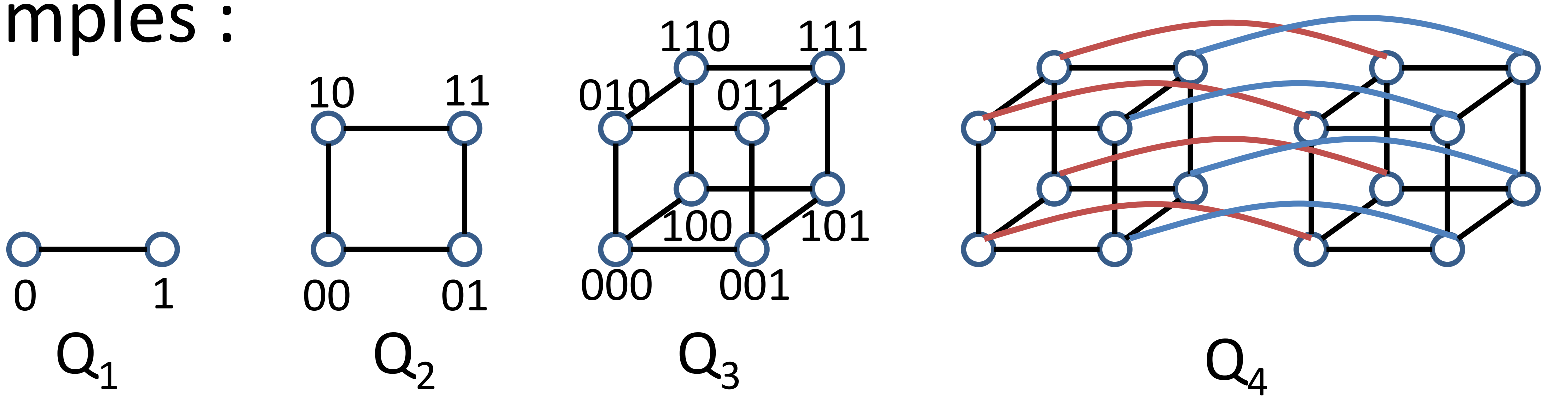


W_6

Some Special Simple Graphs

Definition: An n-cube, denoted by Q_n, is a graph that consists of 2^n vertices, each representing a distinct n-bit string. An edge exists between two vertices ⇔ the corresponding strings differ in exactly one bit position.

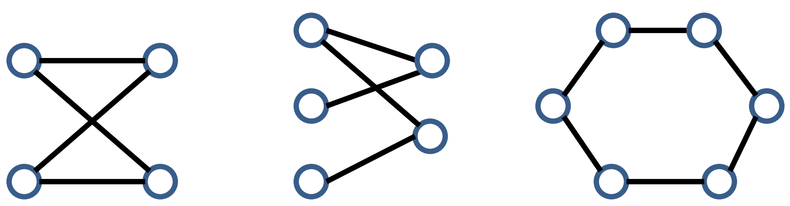
Examples :



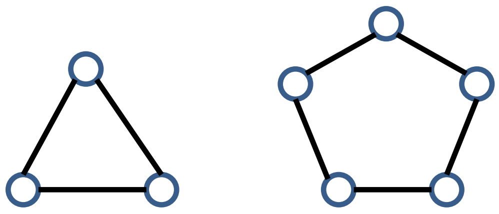
Some Special Simple Graphs

Definition: A bipartite graph is a graph such that the vertices can be partitioned into two sets V and W, so that each edge has exactly one endpoint from V, and one endpoint from W

Examples :



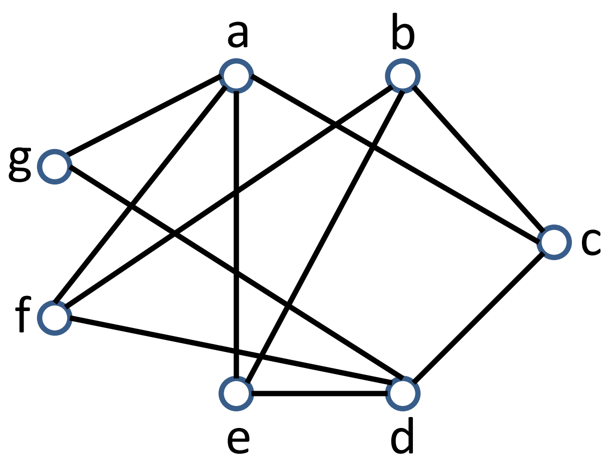
bipartite graphs

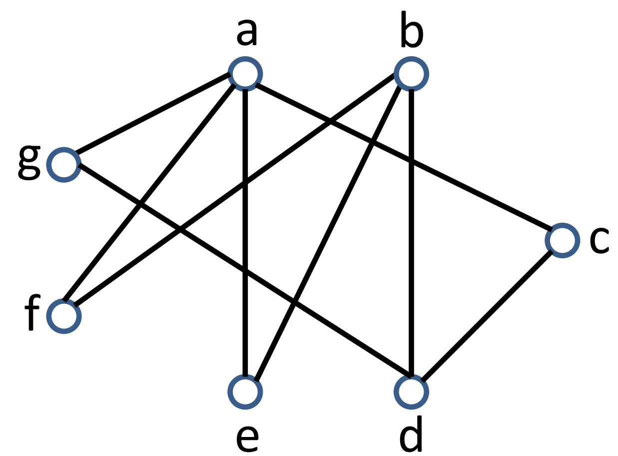


non-bipartite graphs

Some Special Simple Graphs

- Which of the following is a bipartite graph?





Check if a Graph is Bipartite

- The following is a very useful theorem :

Theorem: A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex, so that no two adjacent vertices are assigned the same color

- Proof : If there is a way to color the vertices, the same way shows a possible partition of vertices. Conversely, if there is a way to partition the vertices, the same way gives a possible coloring.

Check if a Graph is Bipartite

- The above implies the following algorithm to check if a connected graph is bipartite :

Step 1 : Pick a vertex u. Color it with white ;

Step 2 : While there are uncolored vertices

(i) for each neighbor of a white vertex,
color it with black ;

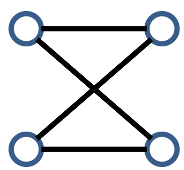
(ii) for each neighbor of a black vertex,
color it with white ;

Step 3 : Report YES if each edge is colored properly.
Else, report NO ;

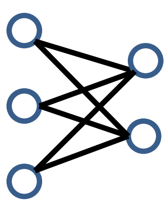
Some Special Simple Graphs

Definition: A complete bipartite graph $K_{m,n}$ is a bipartite graph with vertices partitioned into two subsets V and W of size m and n, respectively, such that there is an edge between each vertex in V and each vertex in W

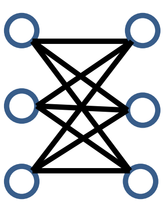
Examples :



$K_{2,2}$



$K_{3,2}$

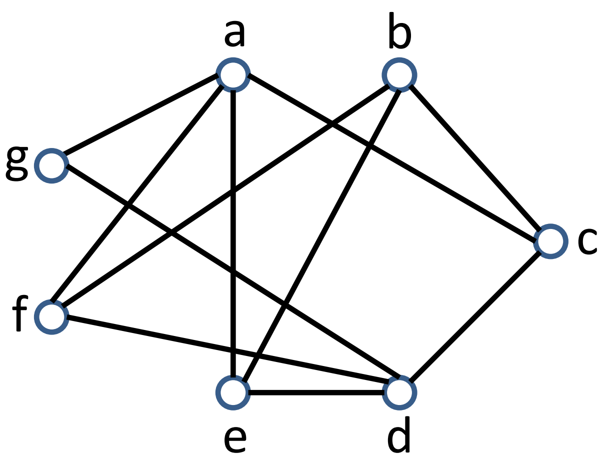


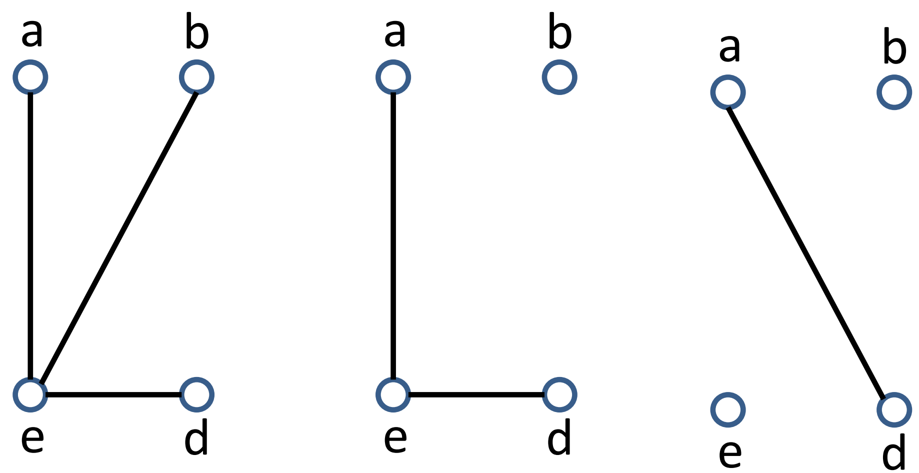
$K_{3,3}$

Subgraphs and Complements

If G = (V, E) is a graph, then G' = (V', E') is called a **subgraph** of G if $V' \subseteq V$ and $E' \subseteq E$.

- Which one is a subgraph of the leftmost graph G ?



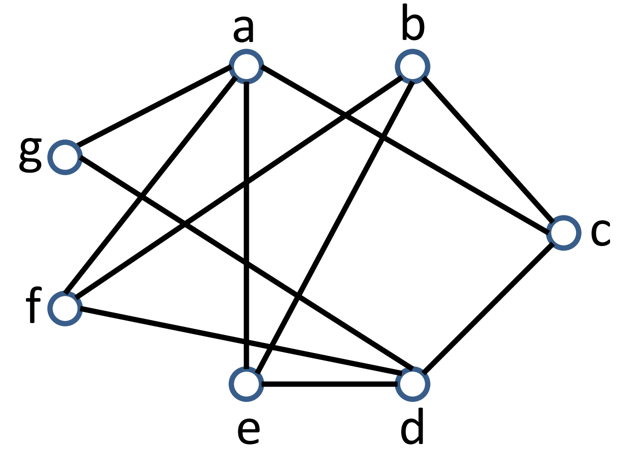


Subgraphs and Complements

If G = (V, E) is a graph, then the subgraph of G induced by U $\subseteq$ V is a graph with the vertex set U and contains exactly those edges from G with both endpoints from U

Ex : Consider the graph on the right side

What is its subgraph induced by the vertex set { a, b, c, g } ?



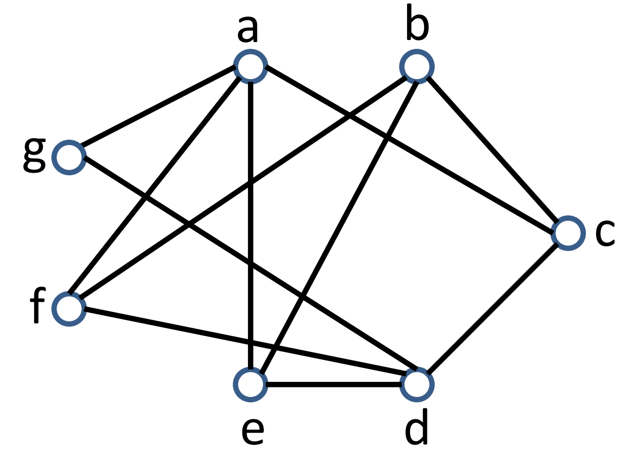
Subgraphs and Complements

If G = (V, E) is a graph, then the complement of G, denoted by $\overline{G}$, is a graph with the same vertex set, such that

an edge e exists in $\overline{G}$ ⇔ e does not exist in G

Ex : Consider the graph on the right side

What is its complement ?



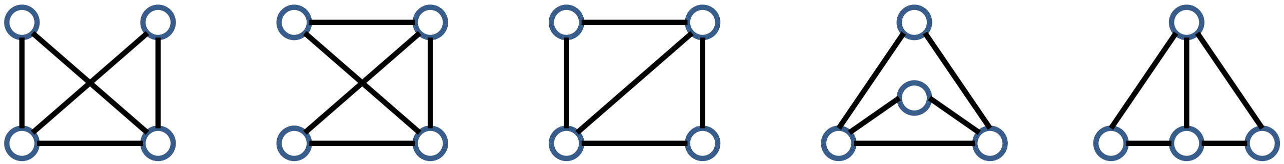
Graph Isomorphism

Graphs $G = (V, E)$ and $H = (U, F)$ are isomorphic if we can set up a bijection $f : V \rightarrow U$ such that

$$x \text{ and } y \text{ are adjacent in } G$$

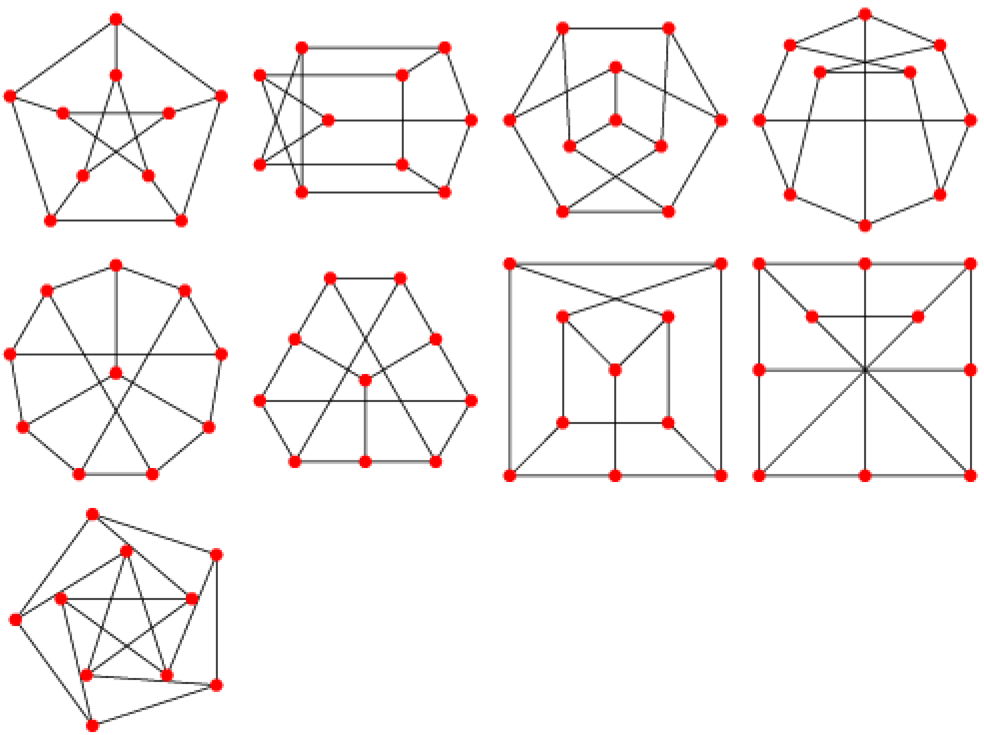
$$\Leftrightarrow f(x) \text{ and } f(y) \text{ are adjacent in } H$$

Ex : The following are isomorphic to each other :



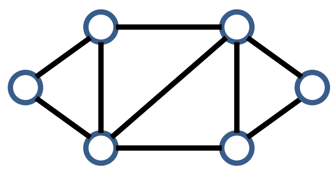
Graph Isomorphism

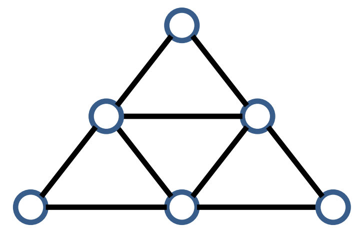
The following graphs are isomorphic to each other.
This graph is known as the Petersen graph :



Graph Isomorphism

How to show the following are not isomorphic ?





How to show the following are not isomorphic ?