CS 2336
Discrete Mathematics

Lecture 14

Graphs: Euler and Hamilton Paths

Outline

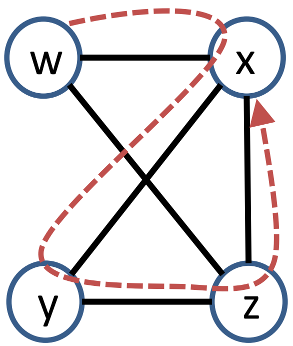
- What is a Path ?
- Euler Paths and Circuits
- Hamilton Paths and Circuits

What is a Path ?

A path is a sequence of edges that begins at a vertex, and travels from vertex to vertex along edges of the graph. The number of edges on the path is called the length of the path.

- Ex : Consider the graph on the right.

  w → x → y → z → x corresponds to a path of length 4



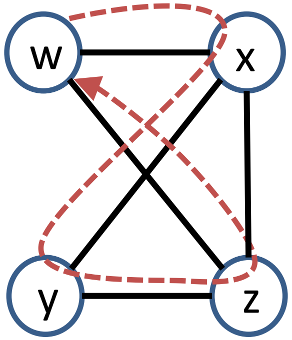
What is a Path ?

If a path begins and ends at the same vertex, the path is also called a circuit.

- Ex : Consider the graph on the right.

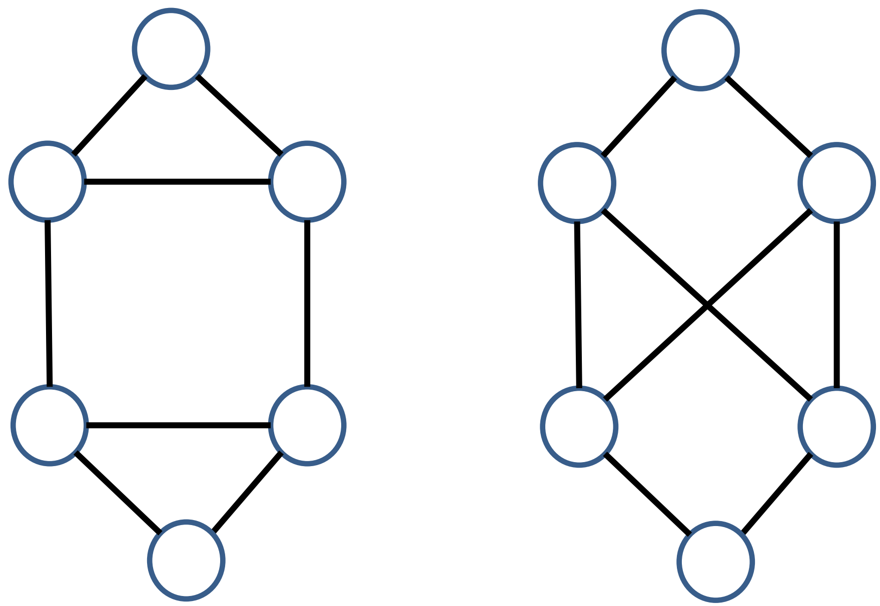
  w → x → y → z → w

  gives to a circuit of length 4



Paths and Isomorphism

Q: How to show that the following graphs are not isomorphic ?

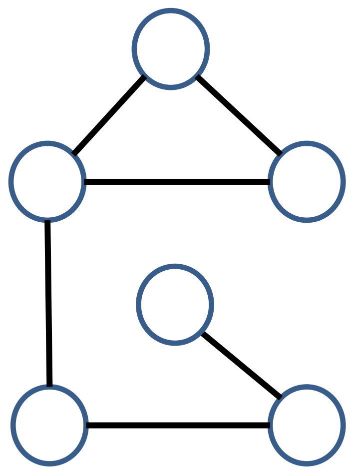


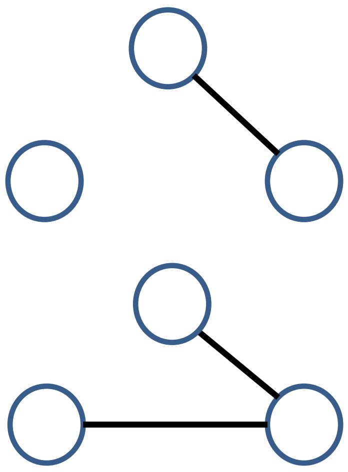
A: One contains a circuit of length 3 (a triangle), while the other does not

Paths and Connected Components

An undirected graph is connected if there is a path between any pair of vertices. Otherwise, it is disconnected.

- Ex :





Connected

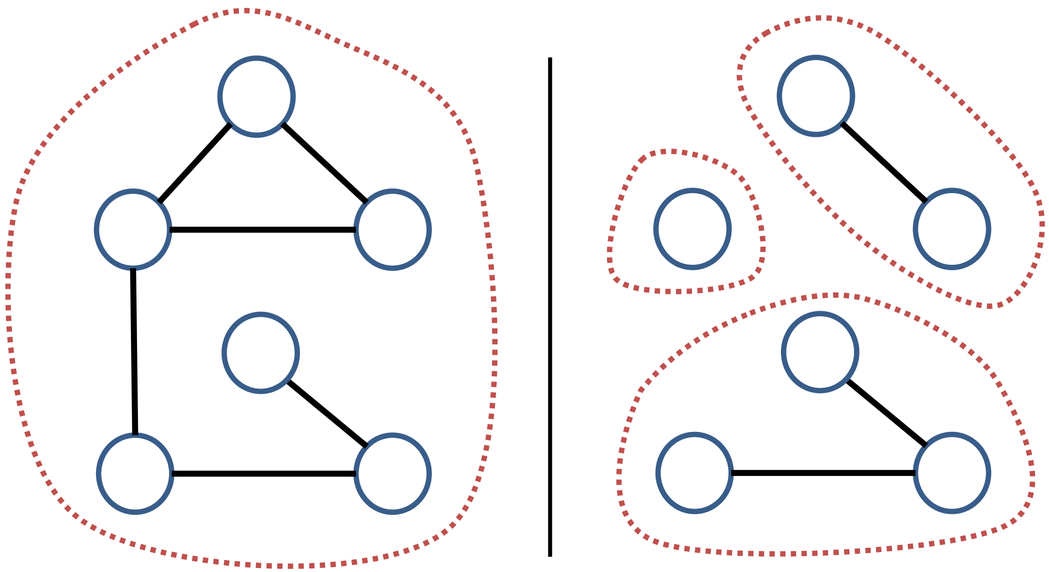
Disconnected

Paths and Connected Components

A connected subgraph is a connected component if it is not contained in any other connected subgraphs.

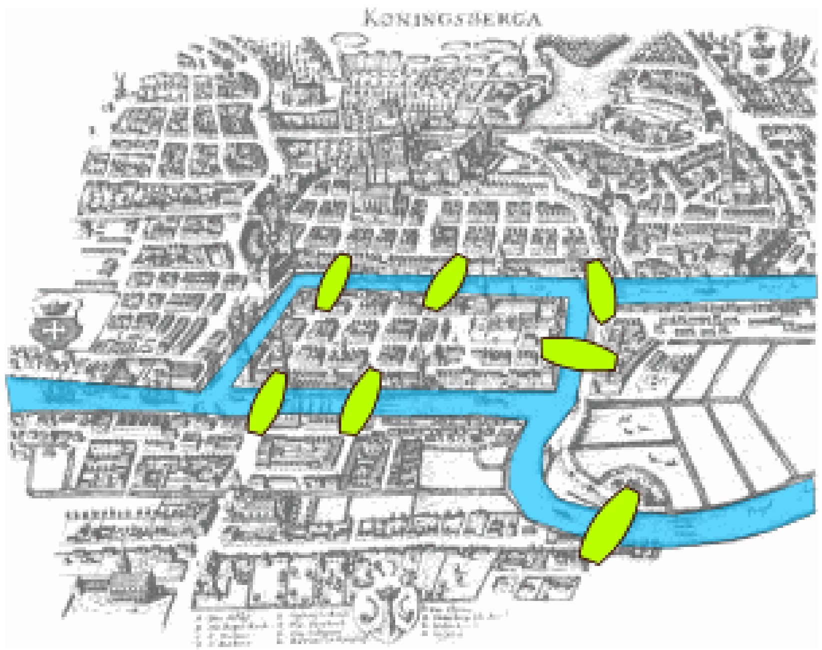
- Ex :

1 connected component



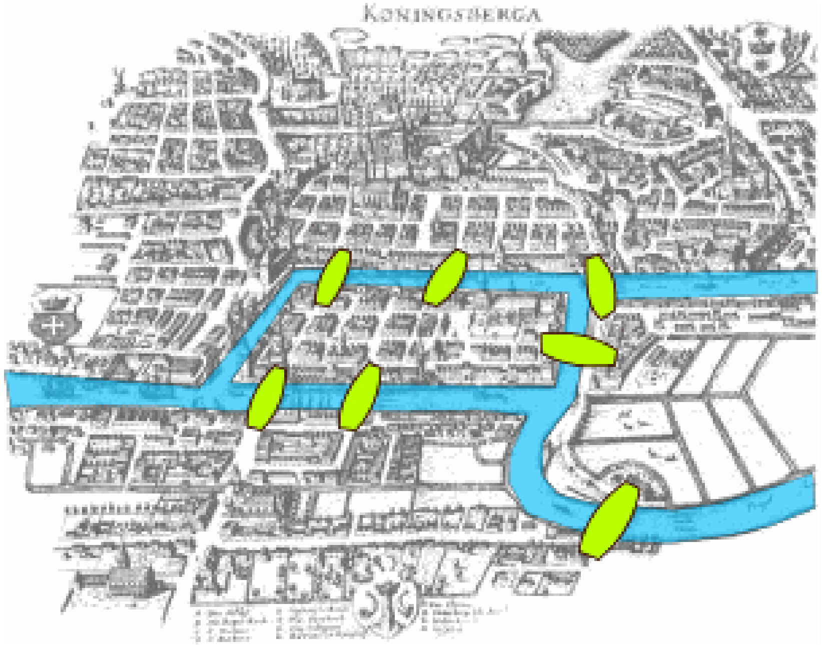
3 connected components

Euler Paths and Circuits



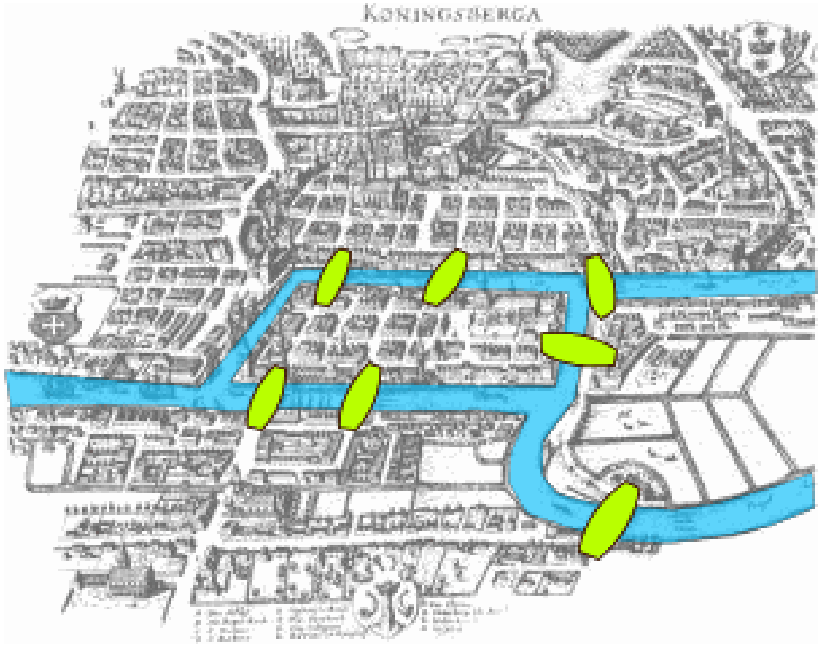
- The above is a map of a Prussian city called Königsberg during the 18th century

Euler Paths and Circuits



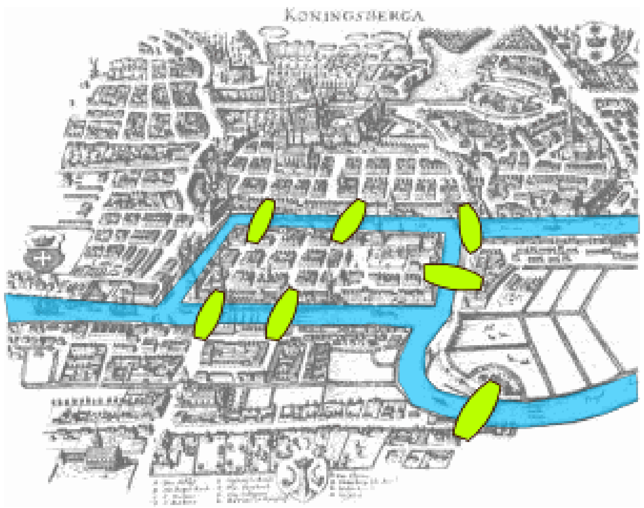
- The Pregel River (blue part) divides the city into 4 parts : Two sides and two large islands

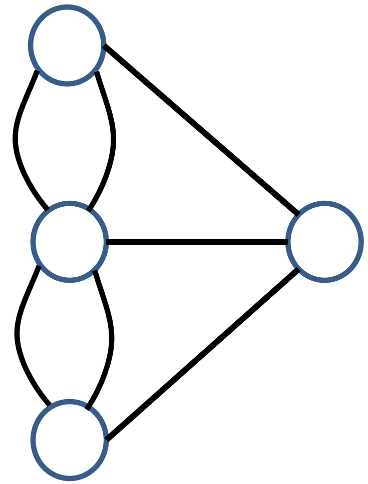
Euler Paths and Circuits



- Seven bridges connect the sides with the islands
- Can we start at some location, travel each bridge exactly once, and go back to the same location ?

Euler Paths and Circuits





- Euler first represents the four parts and the seven bridges by a graph shown on the right

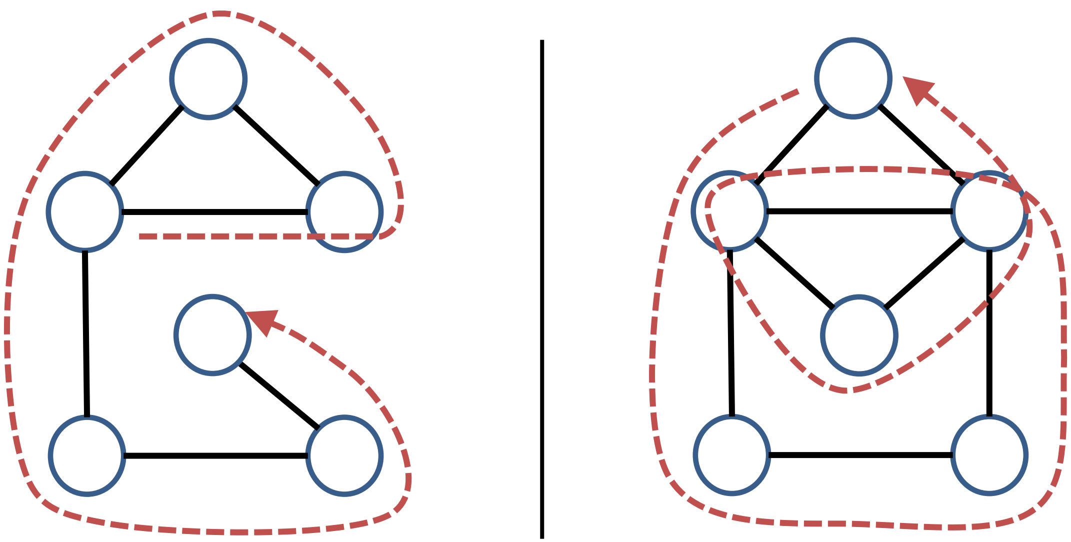
  ➔ The problem will be equivalent to :

  Find a circuit that travels each edge exactly once
- Euler shows that there is NO such circuit

Euler Paths and Circuits

Definition : An Euler path in a graph is a path that contains each edge exactly once. If such a path is also a circuit, it is called an Euler circuit.

- Ex :



Euler path | Euler circuit

Euler Paths and Circuits

Theorem : A connected graph G has an Euler circuit ⇔ each vertex of G has even degree.

- Proof : [The “only if” case]

  If the graph has an Euler circuit, then when we walk along the edges according to this circuit, each vertex must be entered and exited the same number of times.

  Thus, the degree of each vertex must be even.

Euler Paths and Circuits

- Proof : [The “if” case]

  If each vertex has an even degree, we shall use induction (on the number of edges) to show that an Euler circuit exists.

  (Basis) When there is one edge, it must be a self-loop ➔ An Euler circuit exists.

  (Inductive) We start at a vertex x, and obtain a path without using any edge twice, until we end at a vertex without any more unused edge to travel ➔ This vertex must be x (why?)

Euler Paths and Circuits

- Proof : [The “if” case (continued)]

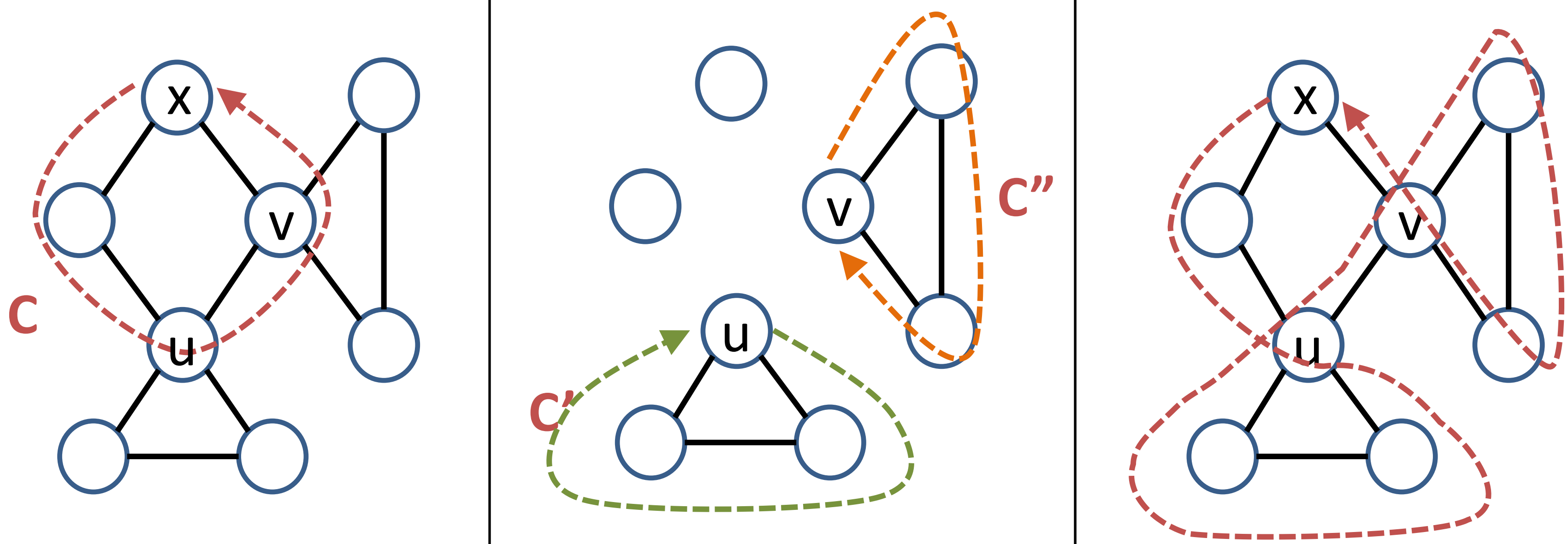
  Let C denote the above circuit.

  If we remove C from the graph, the degree of each vertex must still be even (why?). Further, each connected component with edges must share some vertex u with C, and has an Euler circuit C’ (why?)

  ➔ We get an Euler circuit of the original graph, by walking on C until vertex u, then edges on C’, then back to u, and the remaining edges on C

Euler Paths and Circuits

- Example on obtaining an Euler circuit :



Step 1:
Getting a circuit C by starting from a vertex x

Step 2:
Getting C' for each remaining component

Step 3:
Combining C and the C' of each component

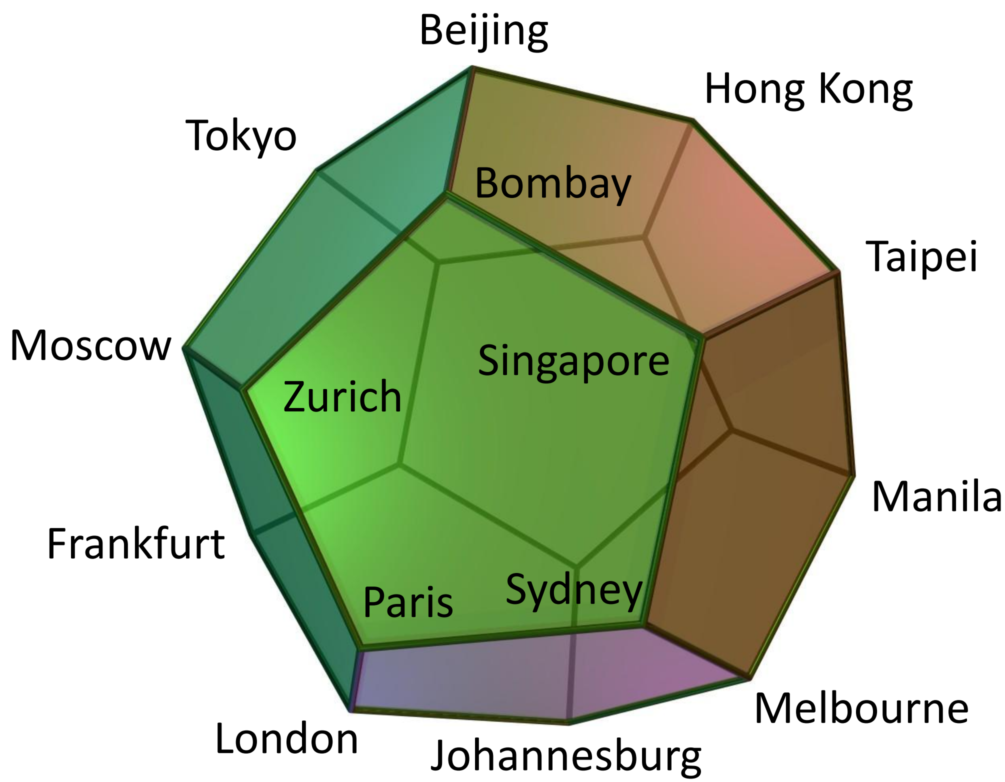
Euler Paths and Circuits

Corollary : A connected graph G has an Euler path, but no Euler circuits ⇔ exactly two vertices of G has odd degree.

- Proof : [The "only if" case] The degree of the starting and ending vertices of the Euler path must be odd, and all the others must be even.

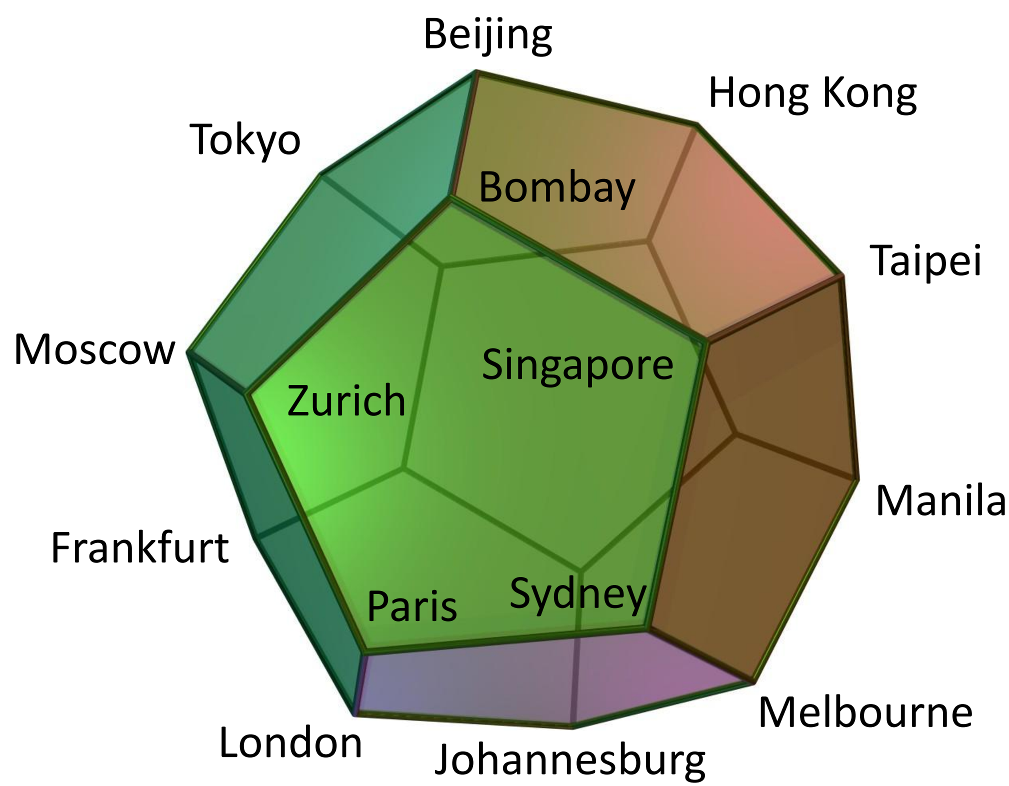
  [The "if" case] Let u and v be the vertices with odd degrees. Adding an edge between u and v will produce an Euler circuit → Removal of this edge thus implies an Euler path in the graph

Hamilton Paths and Circuits



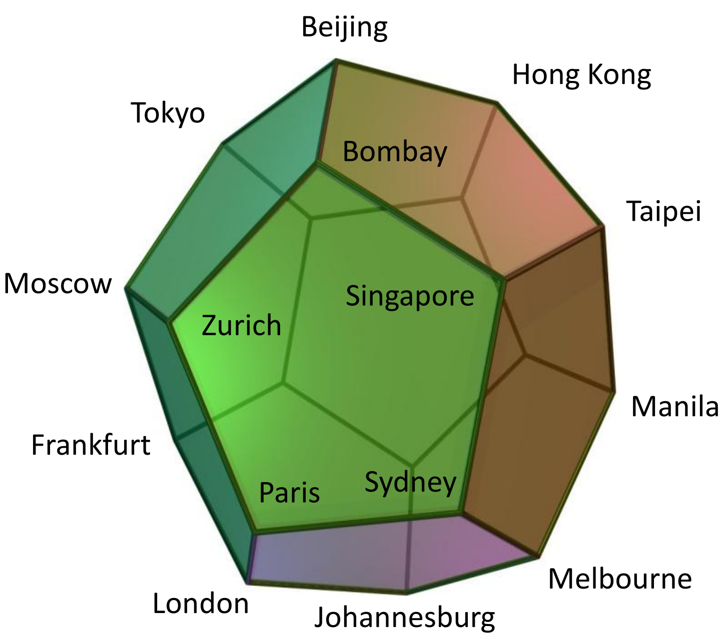
- The above is a regular dodecahedron (12-faced) with each vertex labeled with the name of a city

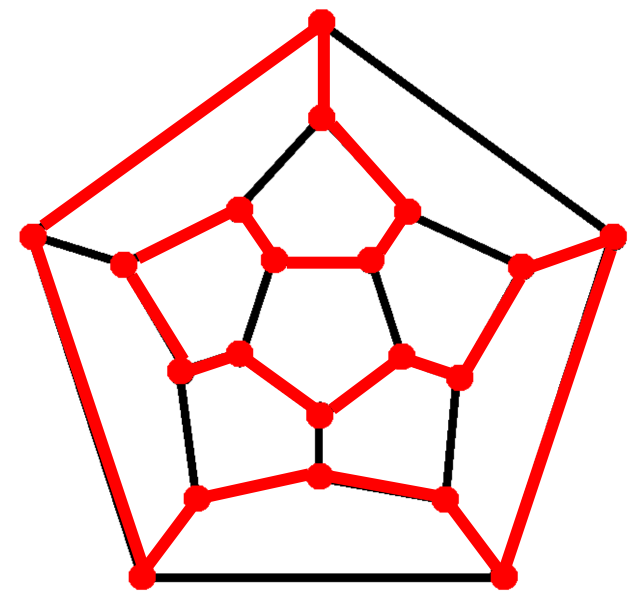
Hamilton Paths and Circuits



- Can we find a circuit (travelling along the edges) so that each city is visited exactly once ?

Hamilton Paths and Circuits



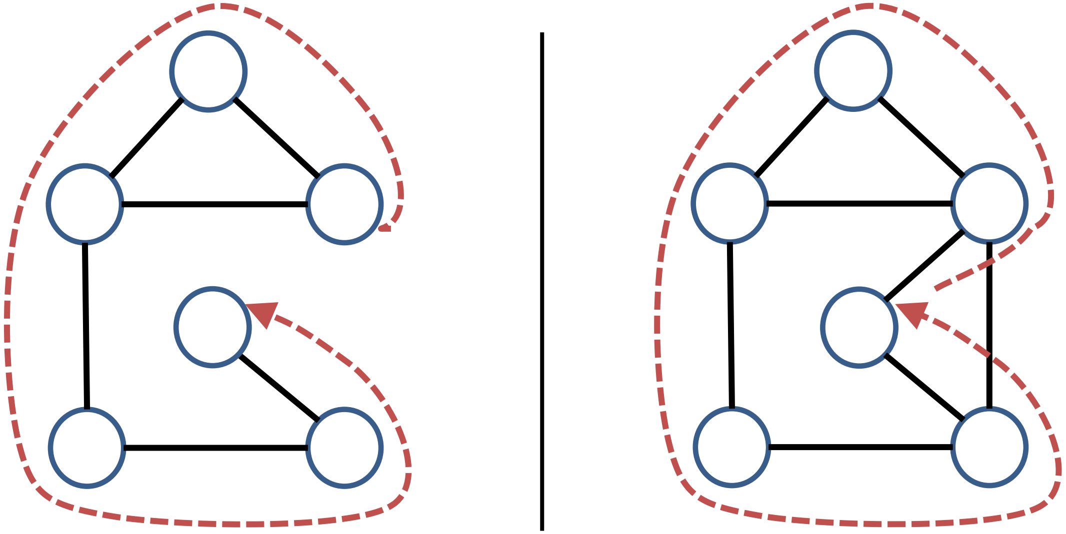


- The right graph is isomorphic to the dodecahedron, and it shows a possible way (in red) to travel

Hamilton Paths and Circuits

Definition : A Hamilton path in a graph is a path that visits each vertex exactly once. If such a path is also a circuit, it is called a Hamilton circuit.

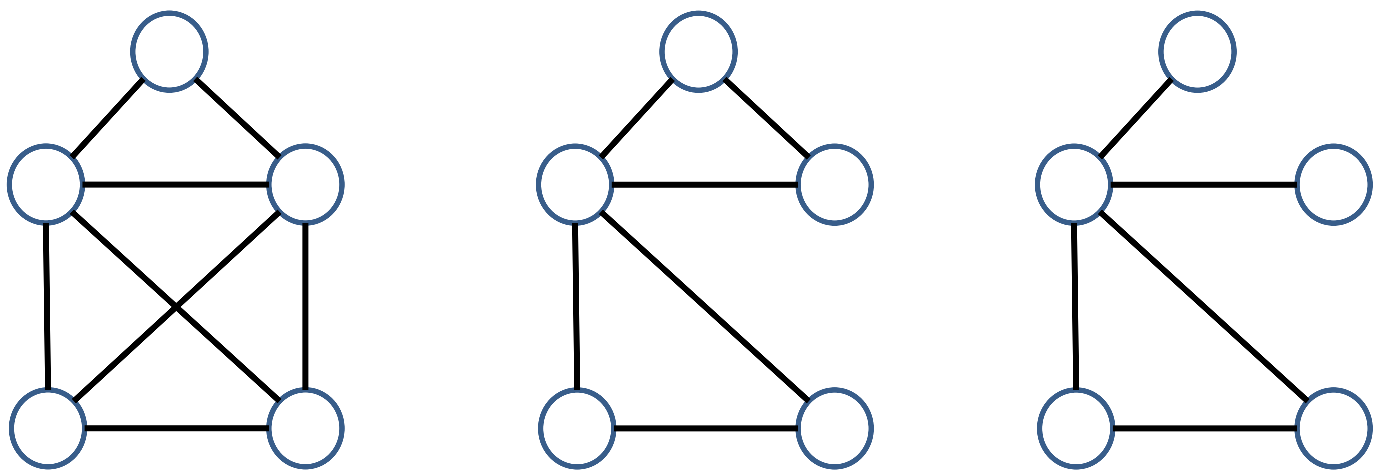
- Ex :



Hamilton path Hamilton circuit

Hamilton Paths and Circuits

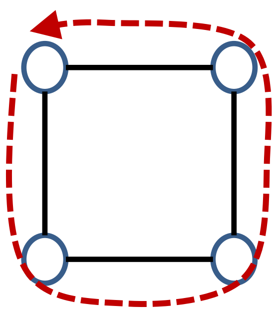
- Which of the following have a Hamilton circuit or, if not, a Hamilton path ?



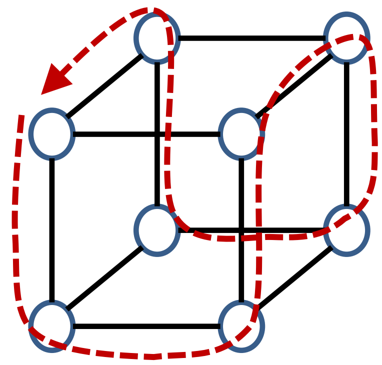
Hamilton Paths and Circuits

- Show that the n-dimensional cube Q_n has a Hamilton circuit, whenever $n \geq 2$

- Ex :



Q_2



Q_3

Hamilton Paths and Circuits

- Unlike Euler circuit or Euler path, there is no efficient way to determine if a graph contains a Hamilton circuit or a Hamilton path

  ➔ The best algorithm so far requires exponential time in the worst case

- However, it is shown that when the degree of the vertices are sufficiently large, the graph will always contain a Hamilton circuit

  ➔ We shall discuss two theorems in this form

Hamilton Paths and Circuits

- Before we give the two theorems, we show an interesting theorem by Bondy and Chvátal (1976)

  ➔ The two theorems will then become corollaries of Bondy-Chvátal theorem

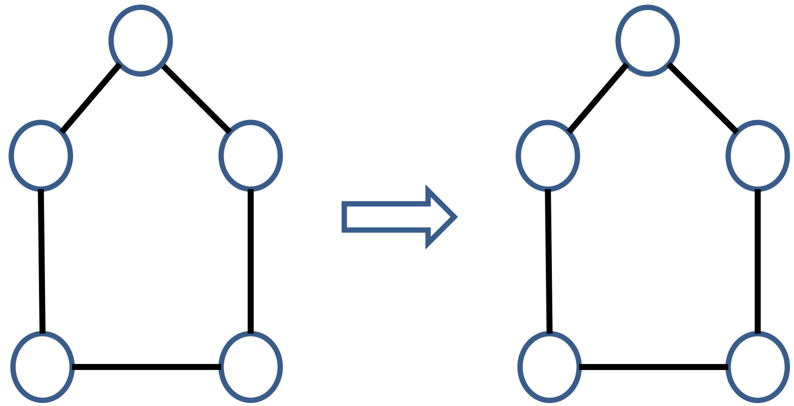
- Let G be a graph with n vertices

Definition : The Hamilton closure of G is a simple graph obtained by recursively adding an edge between two vertices u and v, whenever

$$\deg(u) + \deg(v) \geq n$$

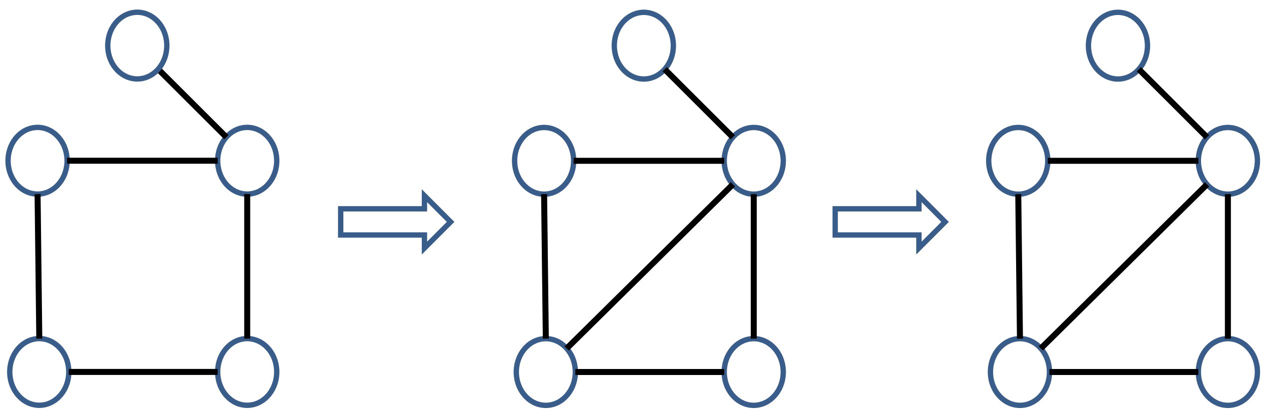
Hamilton Paths and Circuits

- Ex :



G Hamilton closure

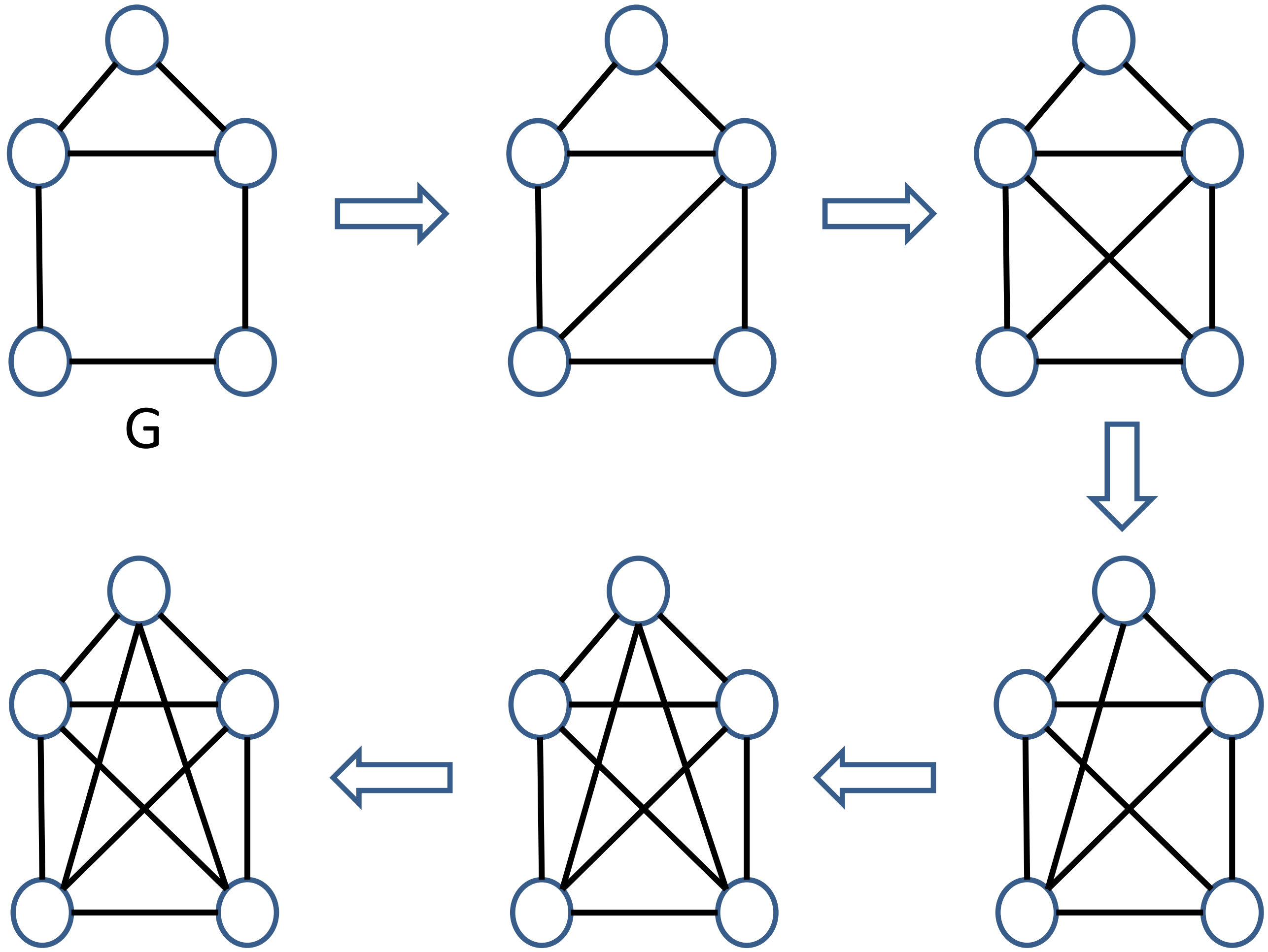
- Ex :



G Hamilton closure

Hamilton Paths and Circuits

- Ex :



Hamilton closure

Hamilton Paths and Circuits

Theorem [Bondy and Chvátal (1976)] :

A graph G contains a Hamilton circuit ⇔ its Hamilton closure contains a Hamilton circuit

- The "only if" case is trivial
- For the "if" case, we can prove it by contradiction
- However, we shall give the proof a bit later, as we are now ready to talk about the two corollaries

Hamilton Paths and Circuits

- Let G be a simple graph with $n \geq 3$ vertices

Corollary [Dirac (1952)] :

If the degree of each vertex in G is at least n/2, then G contains a Hamilton circuit

Corollary [Ore (1960)] :

If for any pair of non-adjacent vertices u and v,

$$\deg(u) + \deg(v) \geq n,$$

then G contains a Hamilton circuit

Hamilton Paths and Circuits

- Proof of Dirac's and Ore's Theorems :

  It is easy to verify that

  (i) if the degree of each vertex is at least n/2, or

  (ii) if for any pair of non-adjacent vertices u and v,

$$\deg(u) + \deg(v) \geq n,$$

  ➔ G's Hamilton closure is a complete graph K_n

  ➔ When $n \geq 3$, K_n has a Hamilton circuit

  ➔ Bondy-Chvátal implies that there will be a Hamilton circuit in G

Hamilton Paths and Circuits

- Next, we shall give the proof of the “if case” of Bondy-Chvátal’s Theorem
- Proof (“if case”):

  Suppose on the contrary that

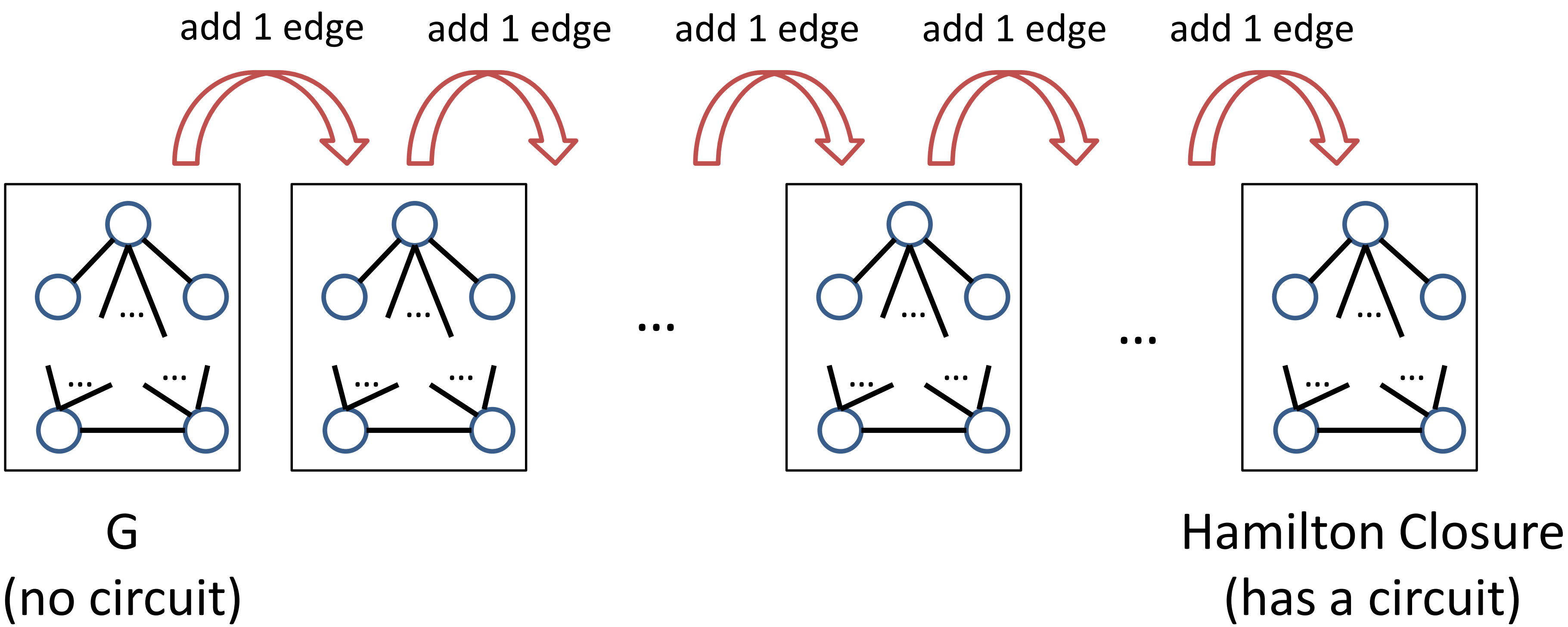
  (i) G does not have a Hamilton circuit, but

  (ii) G’s Hamilton closure has a Hamilton circuit.

  Then, consider the sequence of graphs obtained by adding one edge each time when we produce the Hamilton closure from G

Hamilton Paths and Circuits

- Proof ("if case" continued):

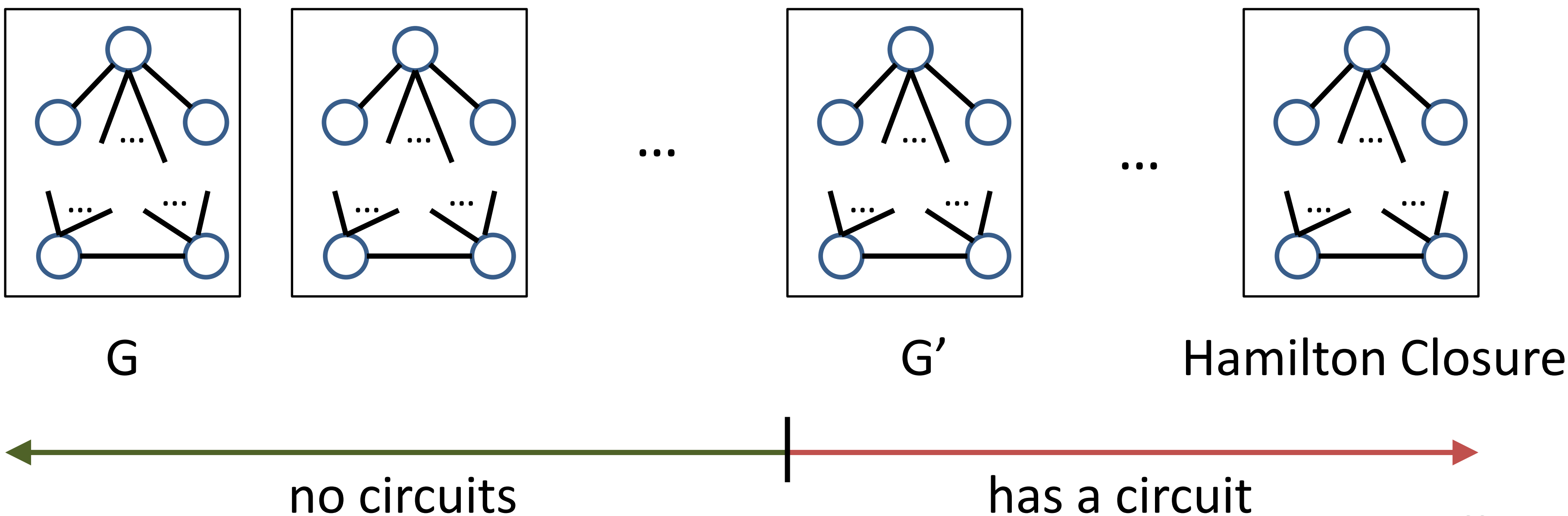


Hamilton Paths and Circuits

- Proof ("if case" continued):

  Let G' be the first graph in the sequence that contains a Hamilton circuit

  Let { u, v } be the edge added to produce G'



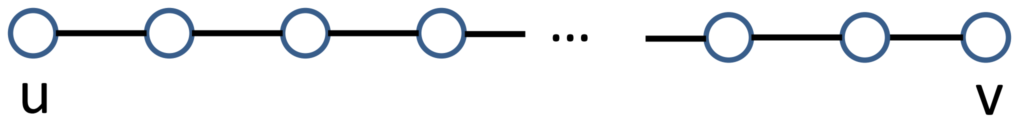
Hamilton Paths and Circuits

- Proof ("if case" continued):

  Now, we show that the graph before G' must also contain a Hamilton circuit, which immediately will cause a contradiction.

  Consider the graph before adding { u, v } to G'.

  It must contain a Hamilton path from u to v (why?)



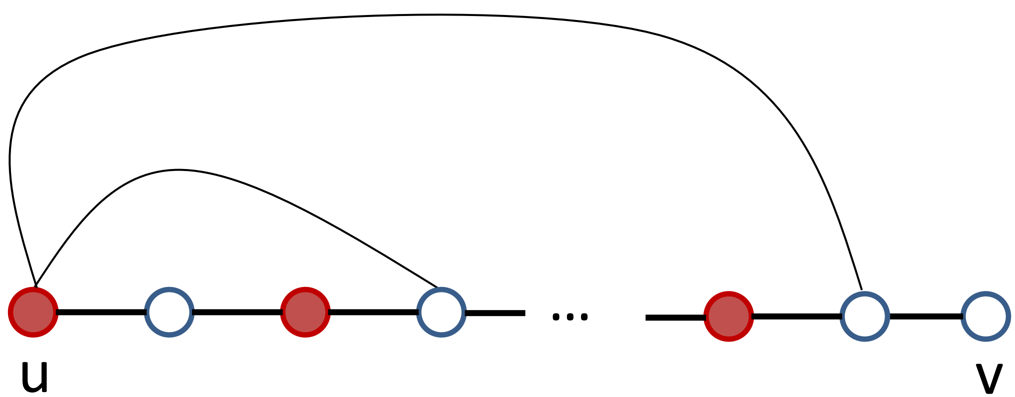
Hamilton Paths and Circuits

- Proof ("if case" continued):

  Also, since we are connecting u and v in G',

  $$\deg(u) + \deg(v) \geq n$$

  Consider all the nodes connected by u, and we mark their 'left' neighbors in red



Hamilton Paths and Circuits

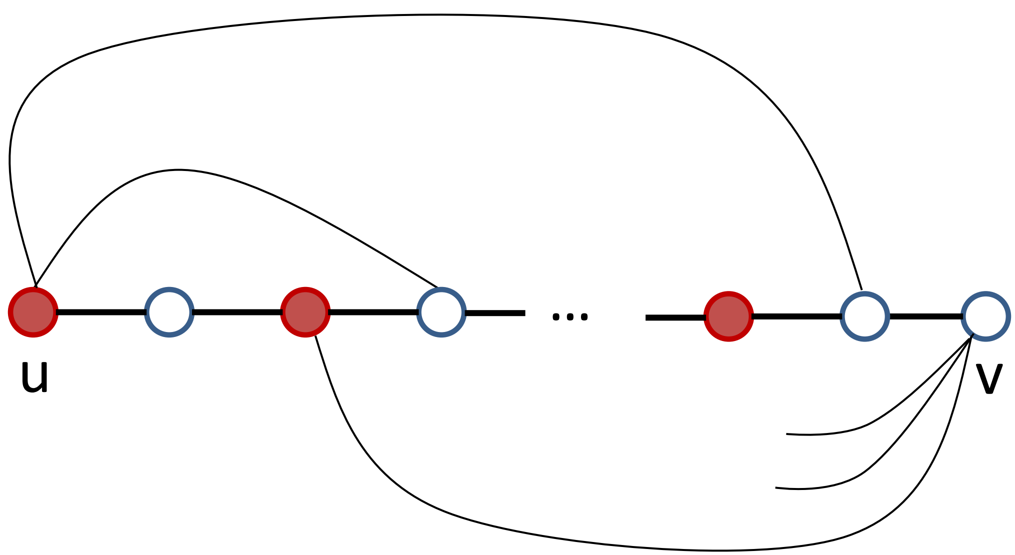
- Proof ("if case" continued):

  Since

  (i) v does not connect to u nor itself, and

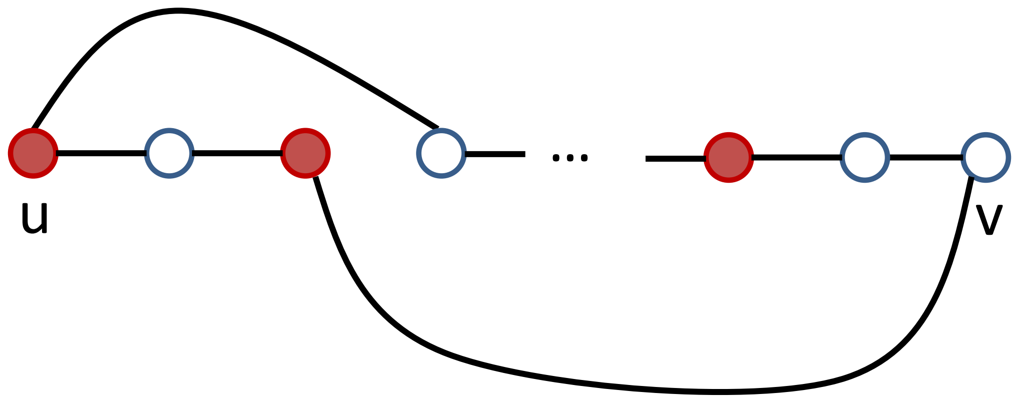
  (ii) deg(u) + deg(v) $\geq$ n

  ➔ v must connect to some red node (why?)



Hamilton Paths and Circuits

- Proof ("if case" continued):
  - ➔ We get a Hamilton circuit, even without connecting u and v !



  - ➔ This contradicts with the choice of G', and the theorem is thus correct