CS 2336
Discrete Mathematics

Lecture 15

Graphs: Planar Graphs

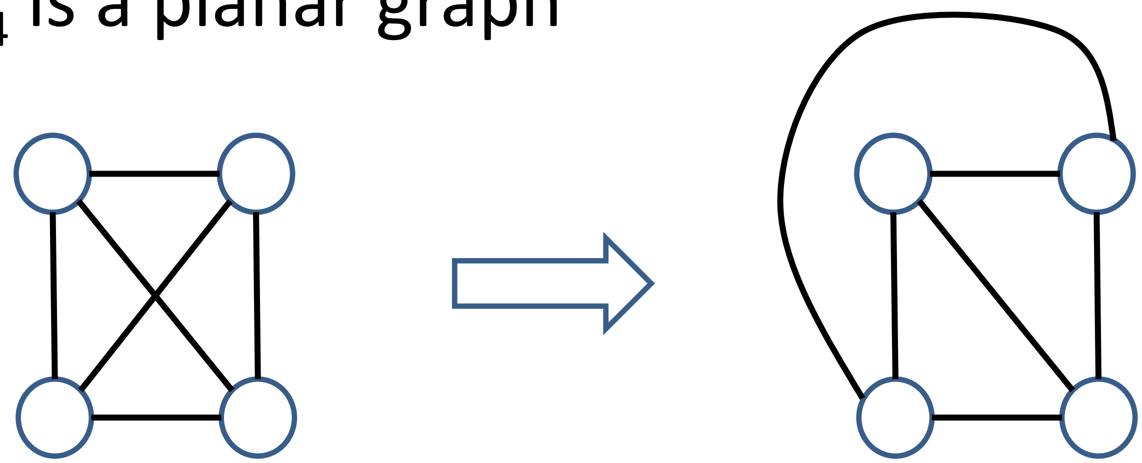
Outline

- What is a Planar Graph ?
- Euler Planar Formula
  - Platonic Solids
  - Five Color Theorem
- Kuratowski's Theorem

What is a Planar Graph ?

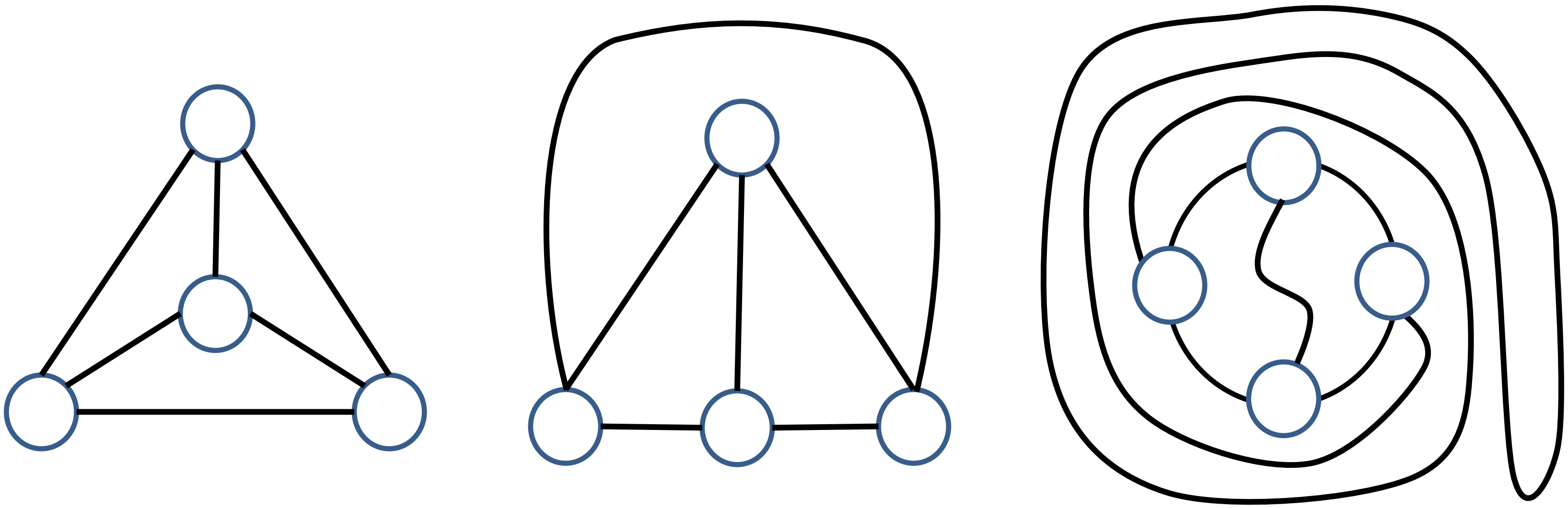
Definition : A planar graph is an undirected graph that can be drawn on a plane without any edges crossing. Such a drawing is called a planar representation of the graph in the plane.

- Ex : K_4 is a planar graph



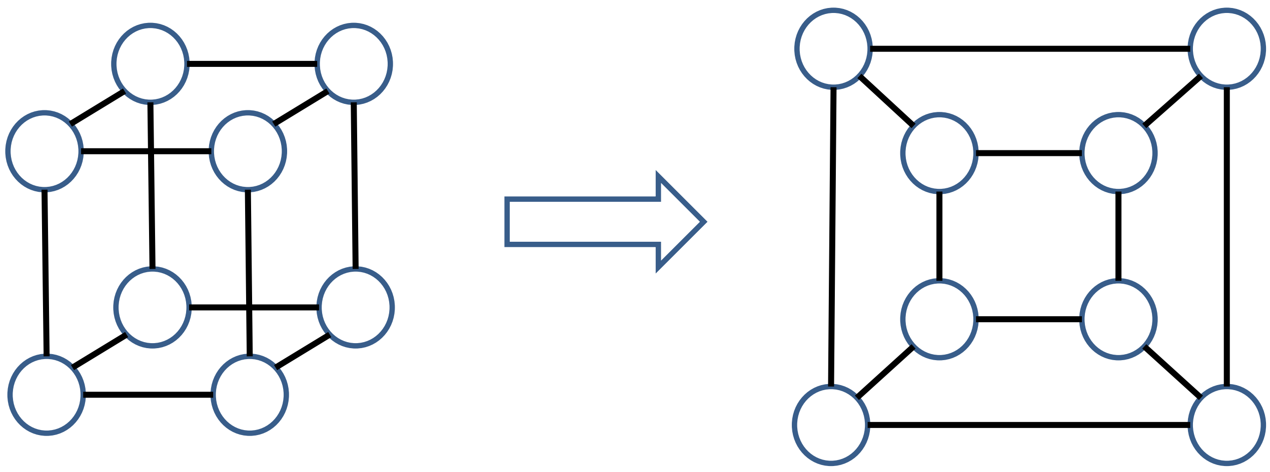
Examples of Planar Graphs

- Ex : Other planar representations of K_4



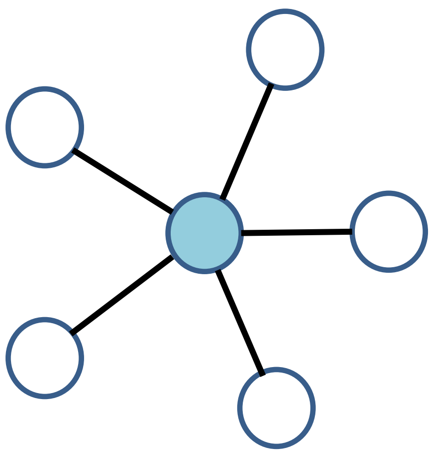
Examples of Planar Graphs

- Ex : Q_3 is a planar graph

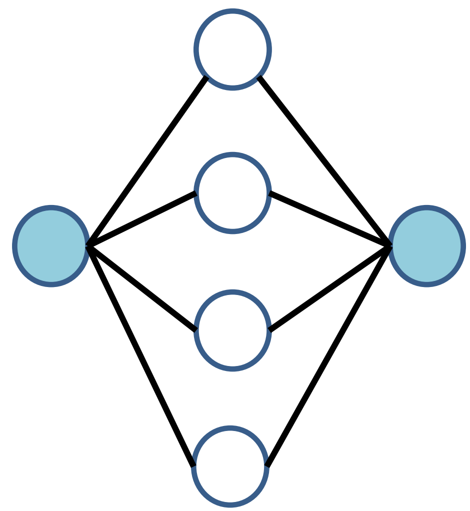


Examples of Planar Graphs

- Ex : $K_{1,n}$ and $K_{2,n}$ are planar graphs for all n



$K_{1,5}$

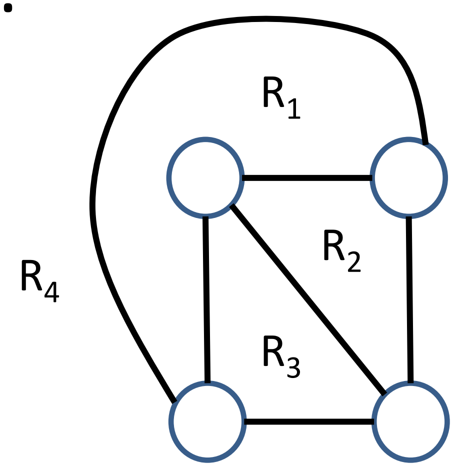


$K_{2,4}$

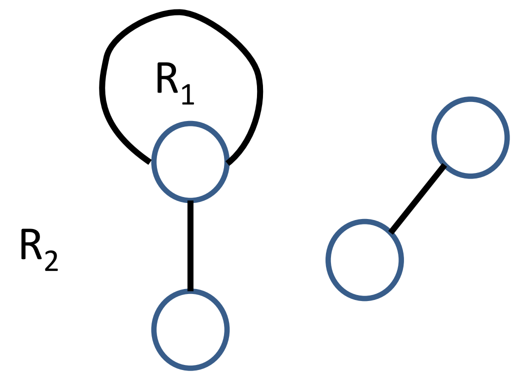
Euler's Planar Formula

Definition : A planar representation of a graph splits the plane into regions, where one of them has infinite area and is called the infinite region.

- Ex :



4 regions
(R_4 = infinite region)



2 regions
(R_2 = infinite region)

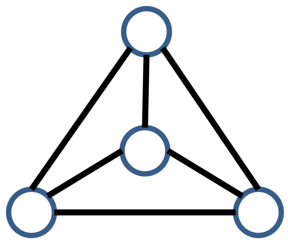
Euler's Planar Formula

- Let G be a connected planar graph, and consider a planar representation of G. Let

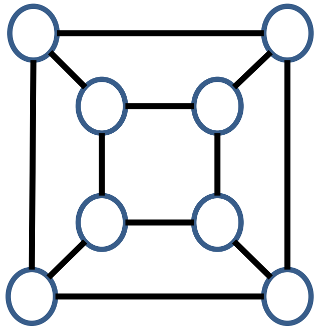
  V = # vertices, E = # edges, F = # regions.

Theorem : $V + F = E + 2.$

- Ex :



V = 4, F = 4, E = 6



V = 8, F = 6, E = 12

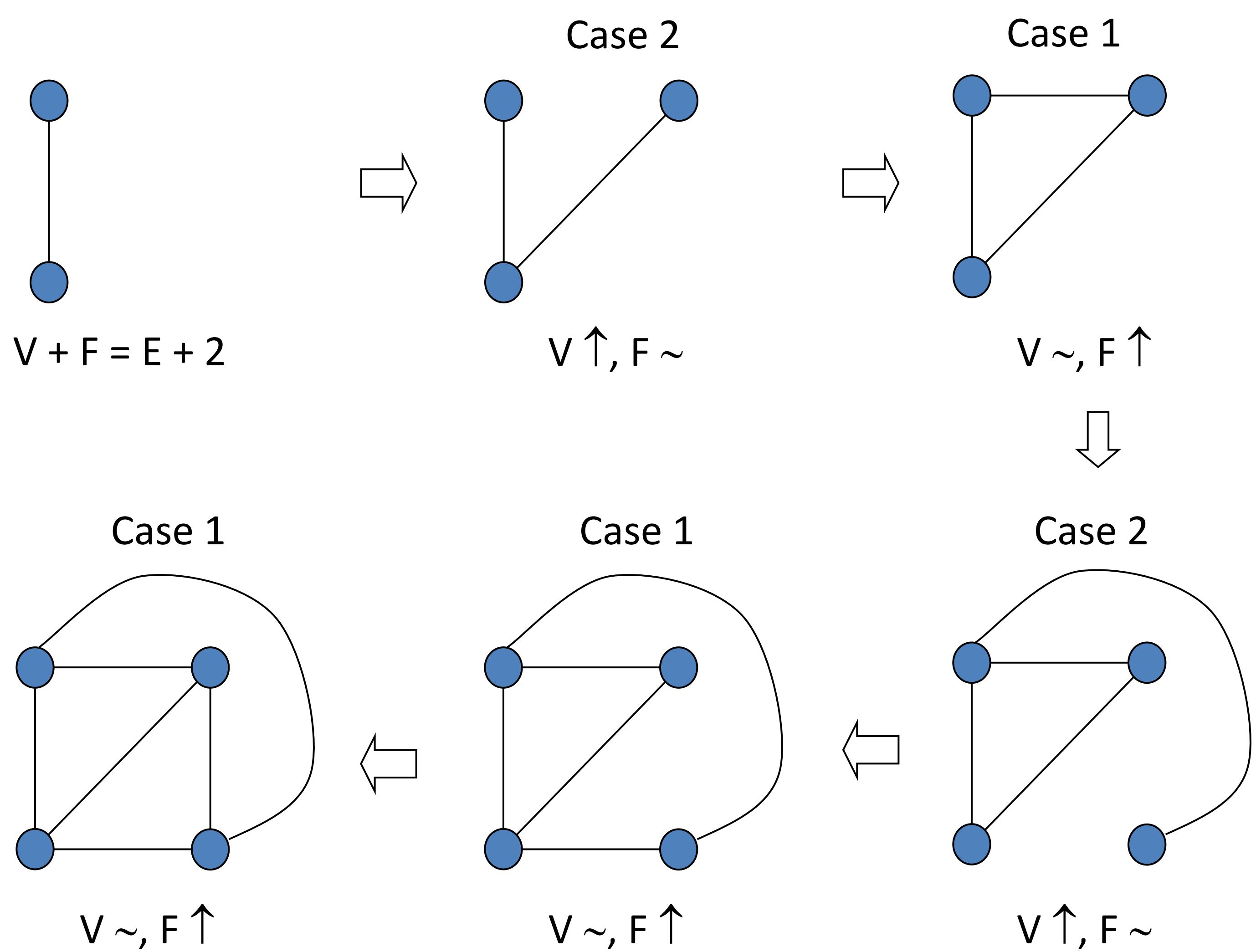
Euler’s Planar Formula

- Proof Idea :
  - Add edges one by one, so that in each step, the subgraph is always connected
  - Use induction to show that the formula is always satisfied for each subgraph
  - For the new edge that is added, it either joins :

    (1) two existing vertices $\rightarrow$ V~, F$\uparrow$

    (2) one existing + one new vertex $\rightarrow$ V~, F$\uparrow$

Euler’s Planar Formula



Euler's Planar Formula

- Let G be a connected simple planar graph with

  V = # vertices, E = # edges.

$$\boxed{\text{Corollary : If } V \geq 3, \text{ then } E \leq 3V - 6.}$$

- Proof : Each region is surrounded by at least 3 edges (how about the infinite region?)

  ➔ $3F \leq$ total edges $= 2E$

  ➔ $E + 2 = V + F \leq V + 2E/3$

  ➔ $E \leq 3V - 6$

Euler's Planar Formula

Theorem : K_5 and $K_{3,3}$ are non-planar.

- Proof :

(1) For K_5, $V = 5$ and $E = 10$

➔ $E > 3V - 6$ ➔ non-planar

(2) For $K_{3,3}$, $V = 6$ and $E = 9$.

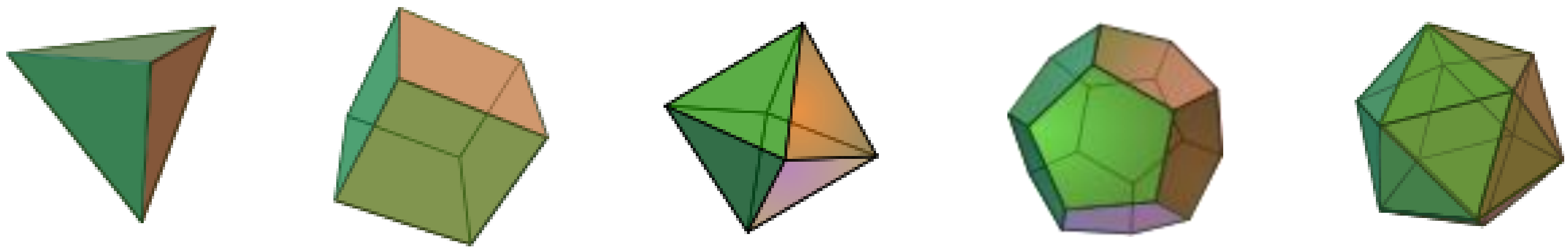
➔ If it is planar, each region is surrounded by at least 4 edges (why?)

➔ $F \leq \lfloor 2E/4 \rfloor = 4$

➔ $V + F \leq 10 < E + 2$ ➔ non-planar

Platonic Solids

Definition : A Platonic solid is a convex 3D shape that all faces are the same, and each face is a regular polygon



Platonic Solids

Theorem: There are exactly 5 Platonic solids

- Proof:

  Let n = # vertices of each polygon

  m = degree of each vertex

  For a platonic solid, we must have

$$nF = 2E \qquad \text{and} \qquad Vm = 2E$$

Platonic Solids

- Proof (continued):

By Euler's planar formula,

$$2E/m + 2E/n = V + F = E + 2$$

$\Rightarrow$ $1/m + 1/n = 1/2 + 1/E$ ...... (*)

Also, we need to have

$n \geq 3$ and $m \geq 3$ [from 3D shape]

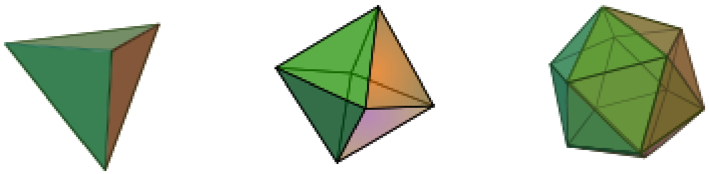
but one of them must be $= 3$ [from (*)]

Platonic Solids

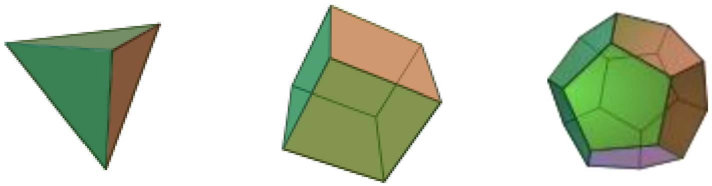
- Proof (continued):

➔ Either

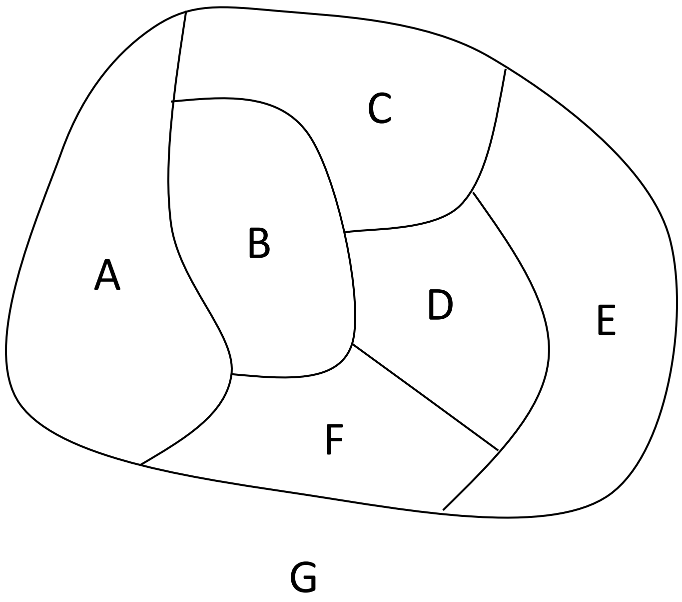
(i) n = 3 (with m = 3, 4, or 5)



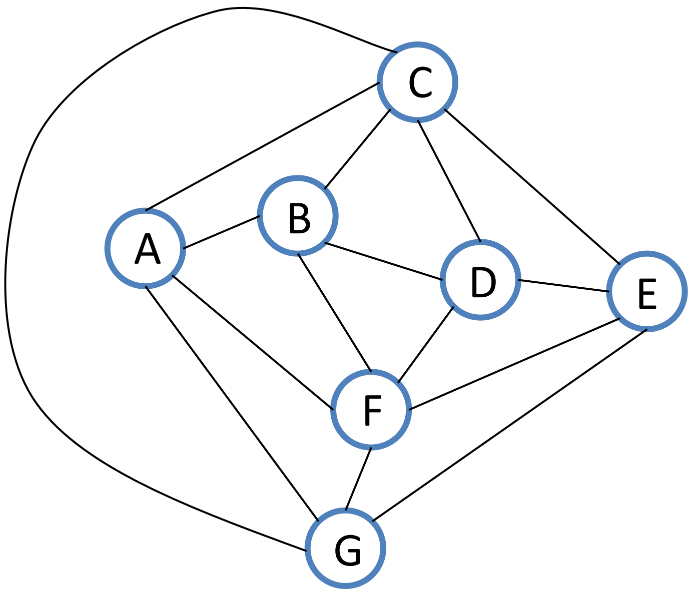
(ii) m = 3 (with n = 3, 4, or 5)



Map Coloring and Dual Graph



A Map M

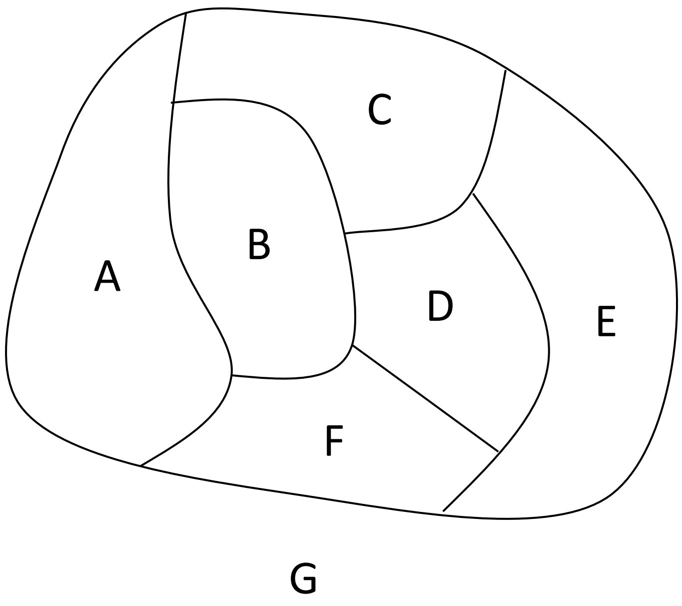


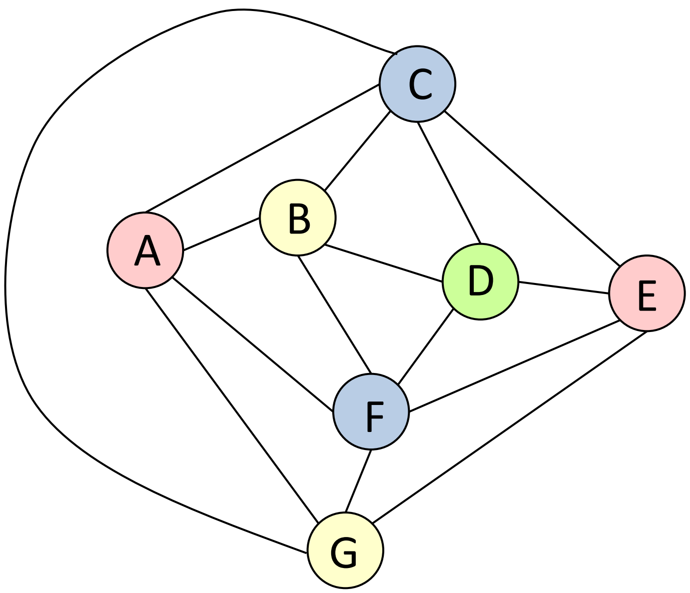
Dual Graph of M

Map Coloring and Dual Graph

Observation: A proper color of M

⇔ A proper vertex color the dual graph





Proper coloring : Adjacent regions (or vertices) have to be colored in different colors

Five Color Theorem

- Appel and Haken (1976) showed that every planar graph can be 4 colored

  (Proof is tedious, has 1955 cases and many subcases)

- Here, we shall show that :

Theorem : Every planar graph can be 5 colored.

- The above theorem implies that every map can be 5 colored (as its dual is planar)

Five Color Theorem

- Proof :

  We assume the graph has at least 5 vertices. Else, the theorem will immediately follow.

  Next, in a planar graph, we see that there must be a vertex with degree at most 5.

  Else,

$$2E = \text{total degree} \geq 3V$$

which contradicts with the fact $E \leq 3V - 6$.

Five Color Theorem

- Proof (continued) :

  Let v be a vertex whose degree is at most 5.

  Now, assume inductively that all planar graphs with n – 1 vertices can be colored in 5 colors

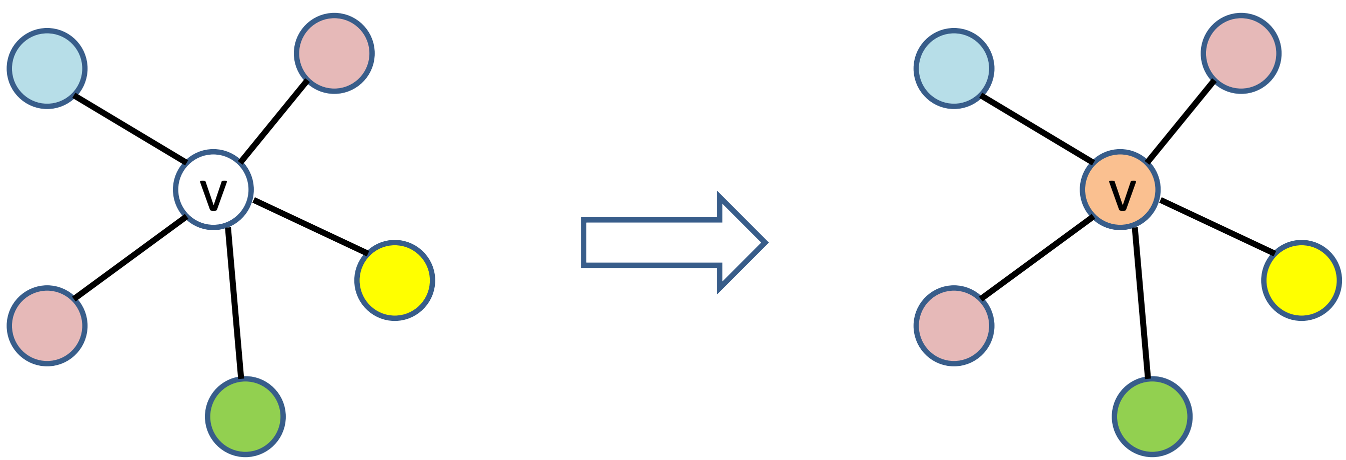
  ➔ Thus if v is removed, we can color the graph properly in 5 colors

  What if we add back v to the graph now ??

Five Color Theorem

- Proof (continued) :

  Case 1 : Neighbors of v uses at most 4 colors

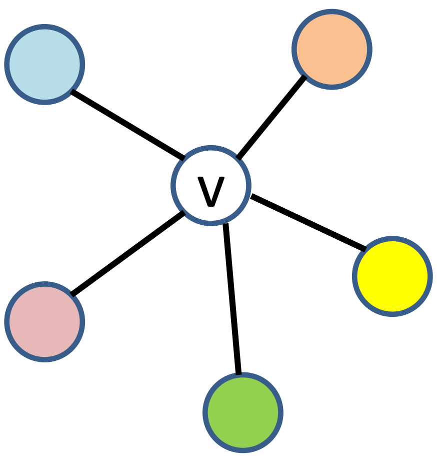


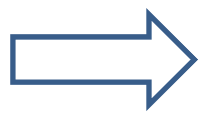
there is a 5th color for v

Five Color Theorem

- Proof (continued) :

  Case 2 : Neighbors of v uses up all 5 colors



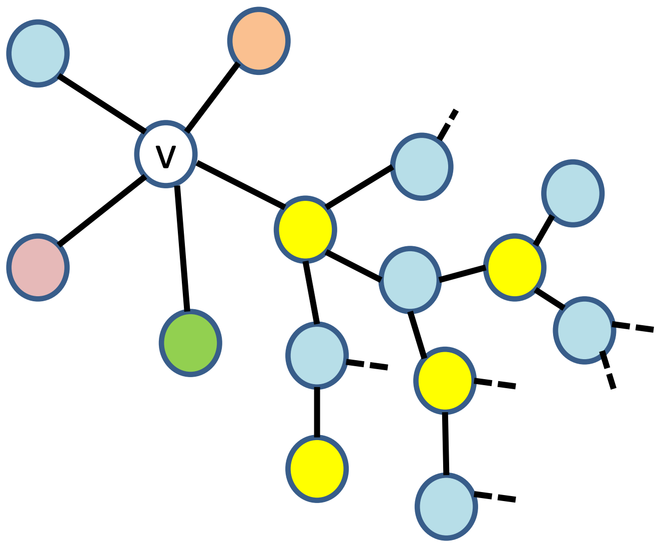


Can we save 1 color, by coloring the yellow neighbor in blue ?

Five Color Theorem

- Proof ("Case 2" continued):

Can we color the yellow neighbor in blue ?

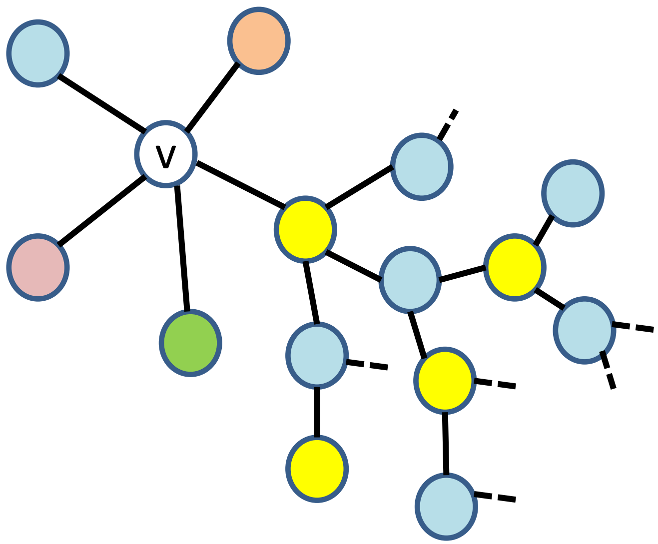


First, we check if the yellow neighbor can connect to the blue neighbor by a "switching" yellow-blue path

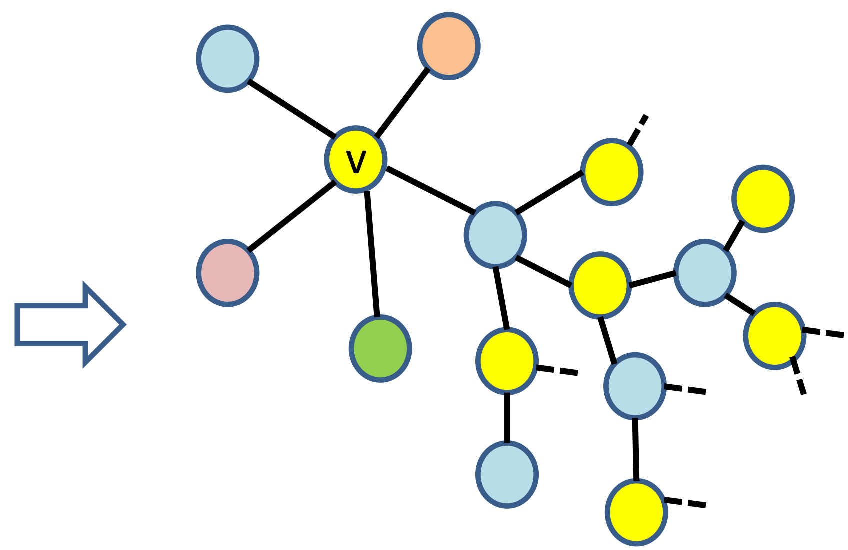
Five Color Theorem

- Proof ("Case 2" continued):

Can we color the yellow neighbor in blue ?



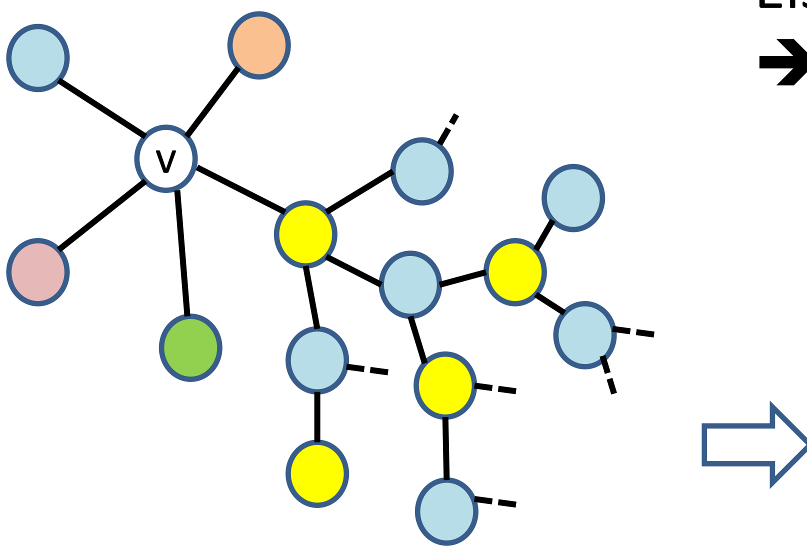
If not, we perform "switching" and thus save one color for v



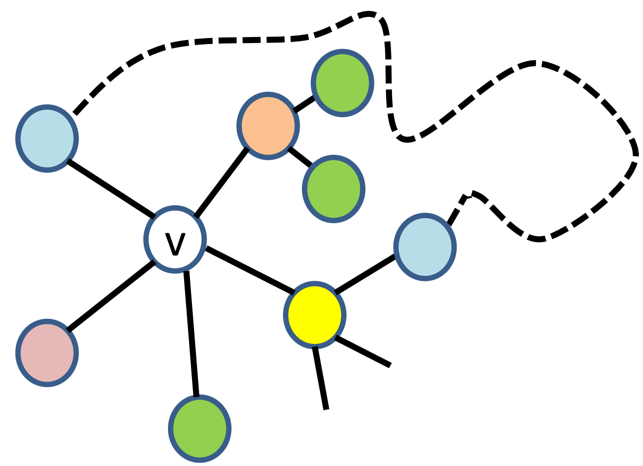
Five Color Theorem

- Proof ("Case 2" continued):

Can we color the yellow neighbor in blue ?



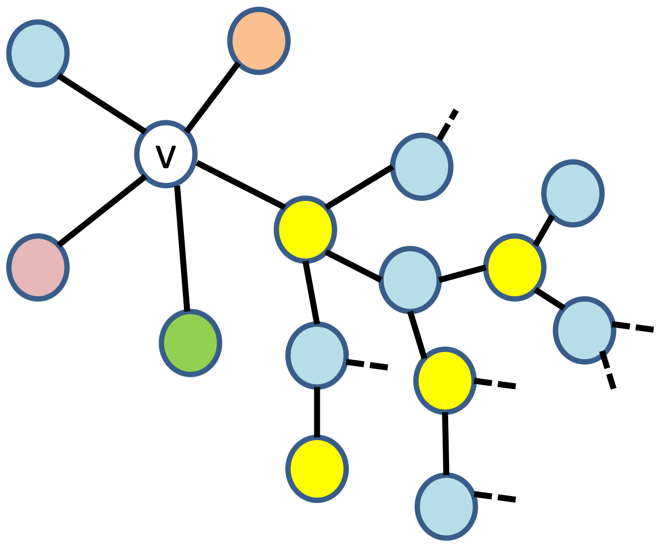
Else, they are connected
➔ orange and green cannot be connected by "switching path



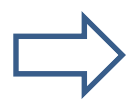
Five Color Theorem

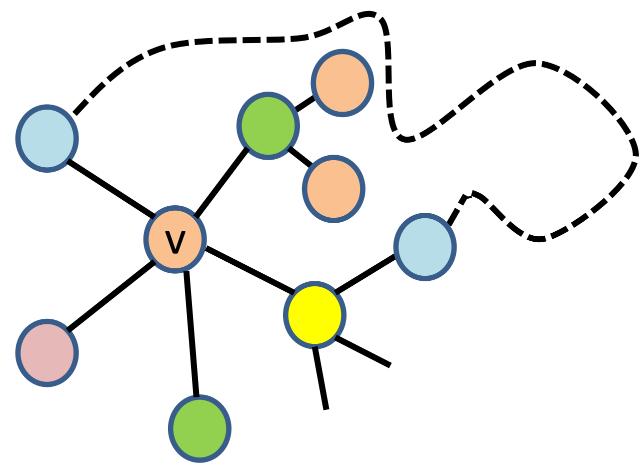
- Proof ("Case 2" continued):

We color the orange neighbor in green !



➔ we can perform "switching" (orange and green) to save one color for v

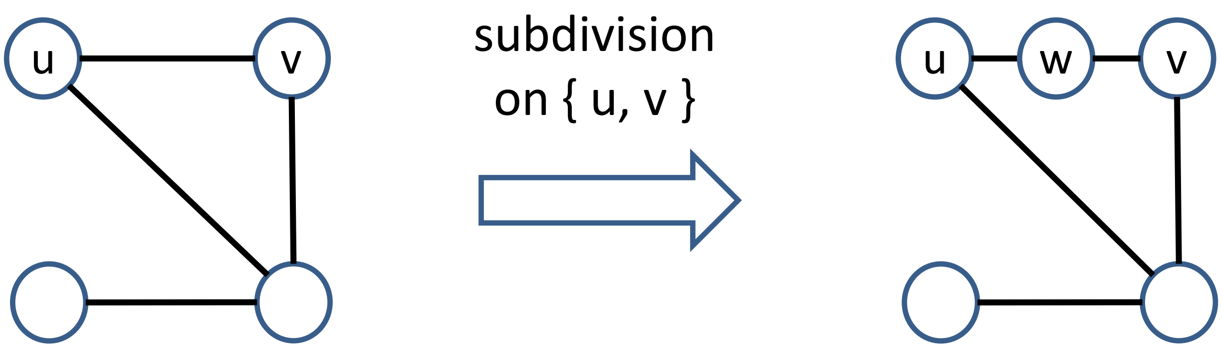




Kuratowski's Theorem

Definition : A subdivision operation on an edge { u, v } is to create a new vertex w, and replace the edge by two new edges { u, w } and { w, v }.

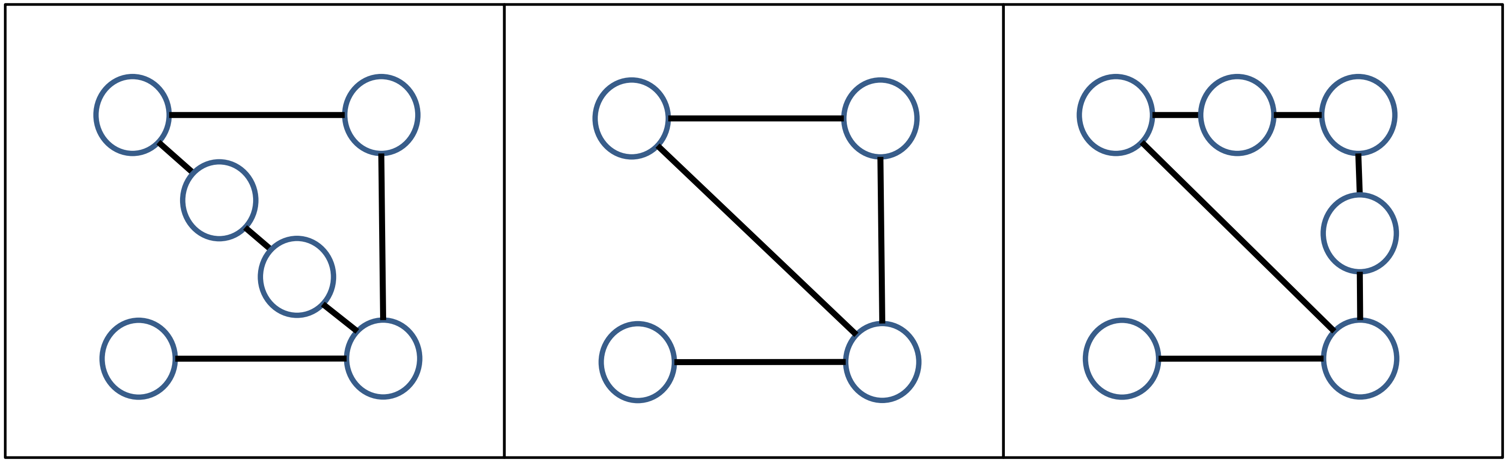
- Ex :



Kuratowski's Theorem

Definition : Graphs G and H are homeomorphic if both can be obtained from the same graph by a sequence of subdivision operations.

- Ex : The following graphs are all homeomorphic :



Kuratowski's Theorem

- In 1930, the Polish mathematician Kuratowski proved the following theorem :

Theorem :

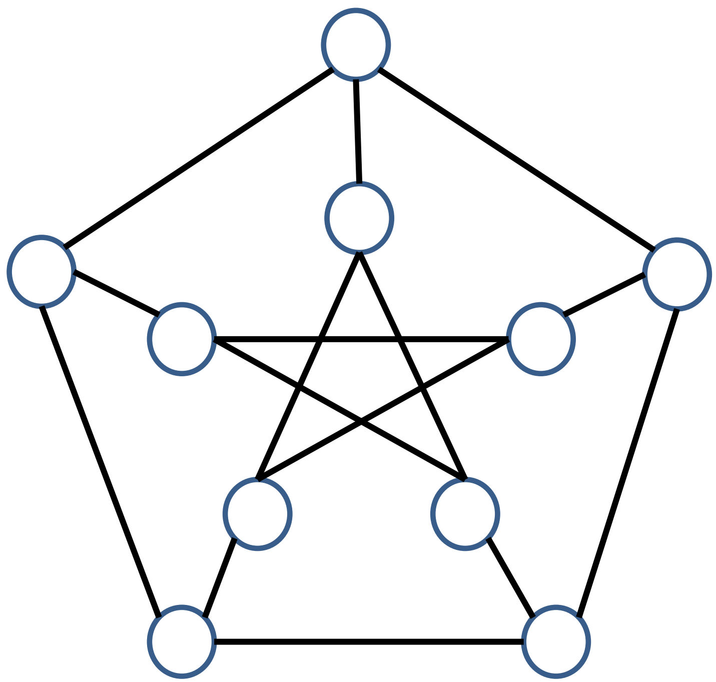
Graph G is non-planar

$\Leftrightarrow$ G has a subgraph homeomorphic to K_5 or $K_{3,3}$

- The "if" case is easy to show (how?)
- The "only if" case is hard (I don't know either …)

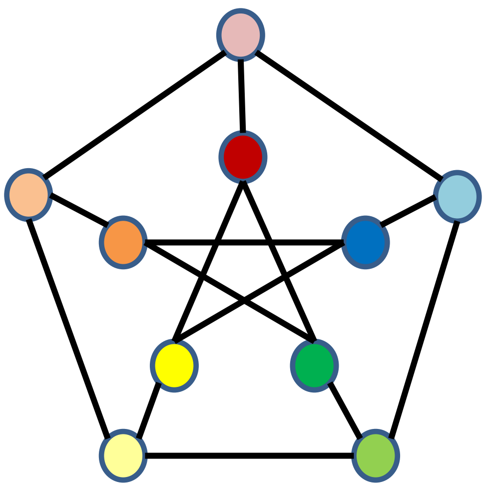
Kuratowski’s Theorem

- Ex : Show that the Petersen graph is non-planar.

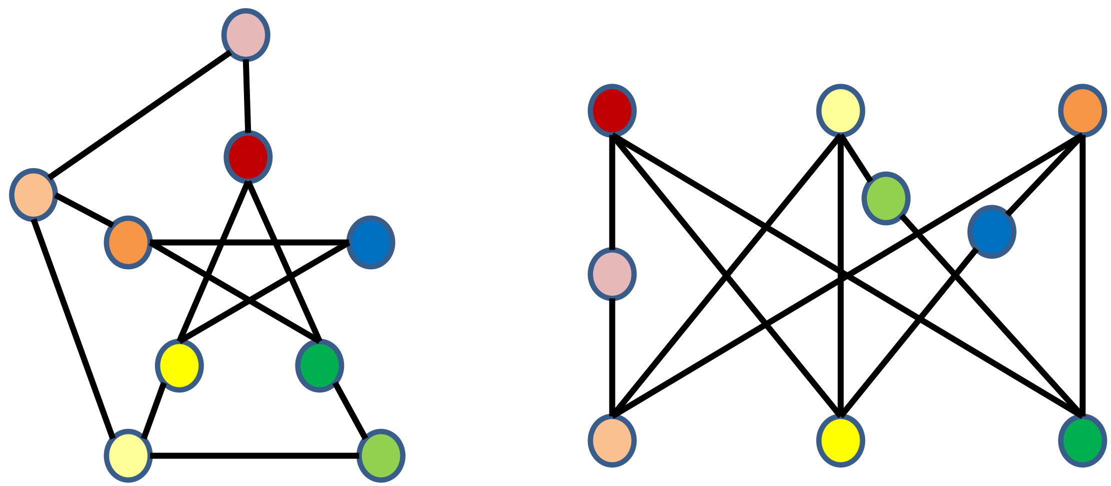


Kuratowski’s Theorem

- Proof :



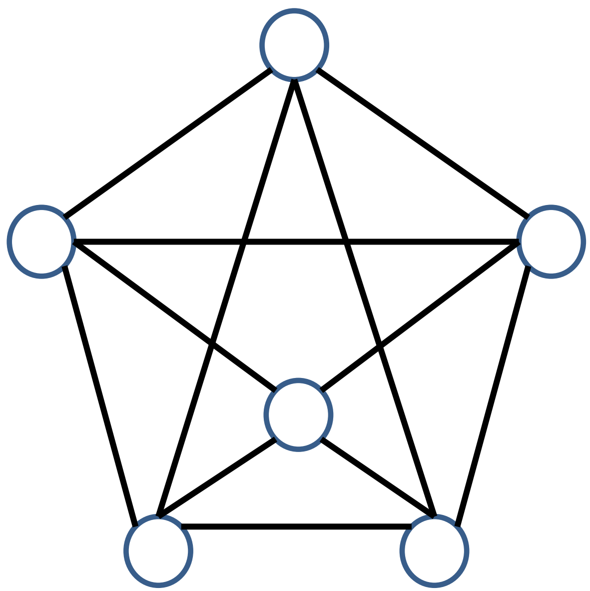
Petersen Graph



Subgraph homeomorphic to $K_{3,3}$

Kuratowski's Theorem

- Ex : Is the following graph planar or non-planar ?



Kuratowski's Theorem

- Ans : Planar