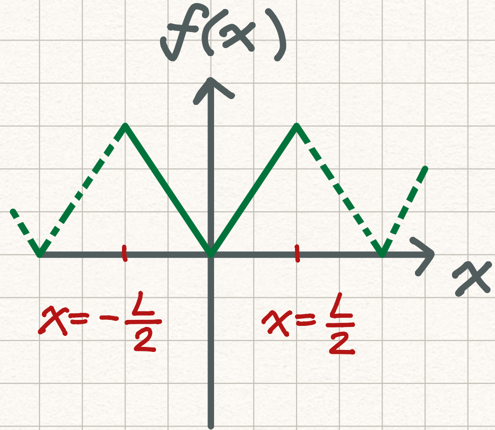
Fourier Series



A periodic function $f(x)$ can be viewed as a "vector", expanded in terms of cos/sin basis.

$$\boxed{f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{L}\right) + b_n \sin\left(\frac{2n\pi x}{L}\right)}$$

The main idea is treating $\{1, \cos(\cdot), \sin(\cdot)\}$ as the basis vectors

① The correct period : $f(x) = f(x+L)$

$\Rightarrow$ $1, \ \cos\left(\frac{2n\pi x}{L}\right), \ \sin\left(\frac{2n\pi x}{L}\right)$

② orthogonal basis :

$$\int_{-L/2}^{L/2} \cos\left(\frac{2n\pi x}{L}\right) \sin\left(\frac{2m\pi x}{L}\right) dx = 0$$

$$\int_{-L/2}^{L/2} \cos\left(\frac{2n\pi x}{L}\right) \cos\left(\frac{2m\pi x}{L}\right) dx = \begin{cases} L & n=m=0 \\ \frac{1}{2}L & n=m>0 \\ 0 & n \neq m \end{cases}$$

$$\int_{-L/2}^{L/2} \sin\left(\frac{2n\pi x}{L}\right) \sin\left(\frac{2m\pi x}{L}\right) dx = \begin{cases} \frac{1}{2}L & n=m>0 \\ 0 & n\neq m \end{cases}$$

It is clear that $\left\{ 1, \cos\left(\frac{2n\pi x}{L}\right), \sin\left(\frac{2n\pi x}{L}\right) \right\}$ are orthogonal. But, not normalized...

Fourier coefficients :

$$|f\rangle = \sum_n c_n |n\rangle$$

orthogonal, but not normalized.

$$\langle m|f\rangle = \sum_n c_n \langle m|n\rangle$$

$\delta_{mn} \cdot N_m$

Thus, the coefficient can be expressed as

$$c_m \cdot N_m = \langle m|f\rangle \Rightarrow c_m = \frac{1}{N_m} \langle m|f\rangle$$

Now, returning to Fourier Series. The inner product is nothing but "integral" ☺

$$\boxed{\begin{aligned} a_n &= \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cos\left(\frac{2n\pi x}{L}\right) dx \\ b_n &= \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin\left(\frac{2n\pi x}{L}\right) dx \end{aligned}}$$

Symmetry Consideration

Symmetry helps to simplify the Fourier series ☺

① **Cos Fourier Series** ~ $f(x) = f(-x)$

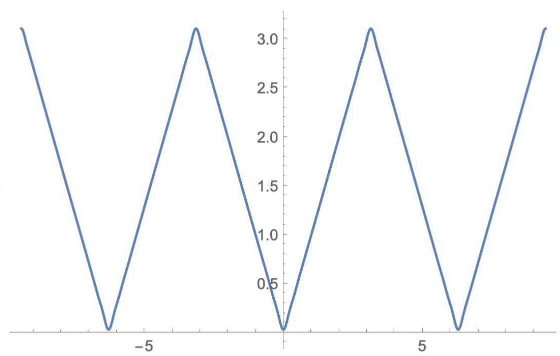
Because $f(x)$ is an even function, all b_n coefficients vanish.

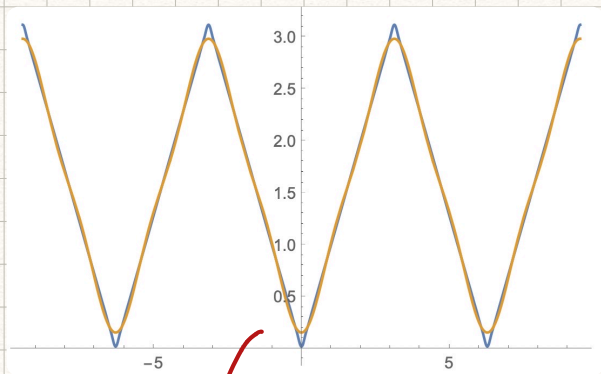
$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2n\pi x}{L}\right)$$

In[3]:=
```
FourierCosSeries[x, x, 20]
Plot[%, {x, -3 Pi, 3 Pi}]
```

Out[3]=
$$\frac{\pi}{2} - \frac{4 \cos[x]}{\pi} - \frac{4 \cos[3x]}{9\pi} - \frac{4 \cos[5x]}{25\pi} - \frac{4 \cos[7x]}{49\pi} - \frac{4 \cos[9x]}{81\pi} - \frac{4 \cos[11x]}{121\pi} - \frac{4 \cos[13x]}{169\pi} - \frac{4 \cos[15x]}{225\pi} - \frac{4 \cos[17x]}{289\pi} - \frac{4 \cos[19x]}{361\pi}$$

Out[4]=





only the first 3 terms get the fn rather well.

$$f(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2} \cos(nx)$$

$-\pi \leq x \leq \pi$ (setting $L = 2\pi$)

$a_0 = \pi$

$$a_n = \begin{cases} -\frac{4}{\pi} \cdot \frac{1}{n^2}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

In[41]:=
```
FourierCosCoefficient[Abs[x], x, n]
DiscretePlot[Abs[%], {n, 0, 10}, PlotRange → All,
 PlotStyle → AbsolutePointSize[10]]
```

Out[41]=
$$\begin{cases} \pi & n == 0 \\ -\frac{(-1+(-1)^n)^2}{n^2 \pi} & \text{True} \end{cases}$$

Out[42]=

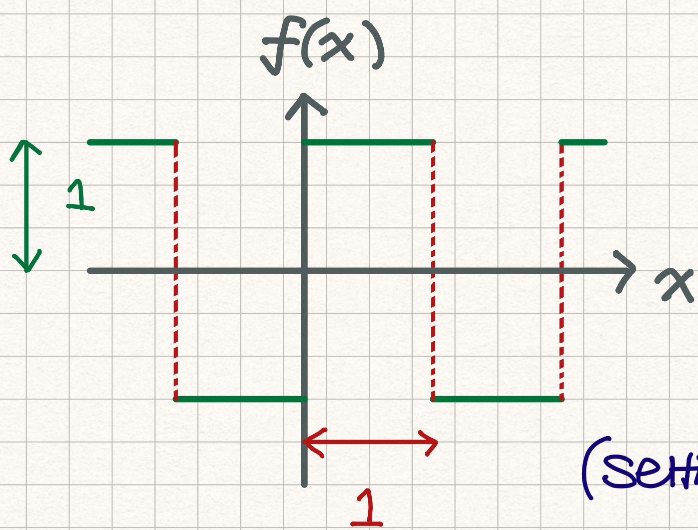
Why ?

a_n decreases very fast

① sin Fourier Series ~ $f(x) = -f(-x)$

All a_n coefficients (including a_0) are zero. The Fourier series simplifies

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{2n\pi x}{L}\right)$$



$$f(x) = \frac{4}{\pi} \sum_{n\,\text{odd}} \frac{\sin(n\pi x)}{n}$$

$$b_n = \begin{cases} \frac{4}{\pi} \cdot \frac{1}{n}, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

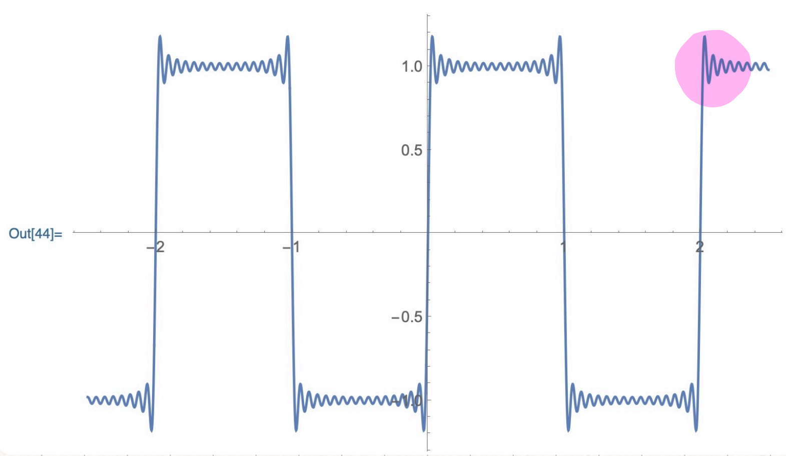
(Setting $L=2$)

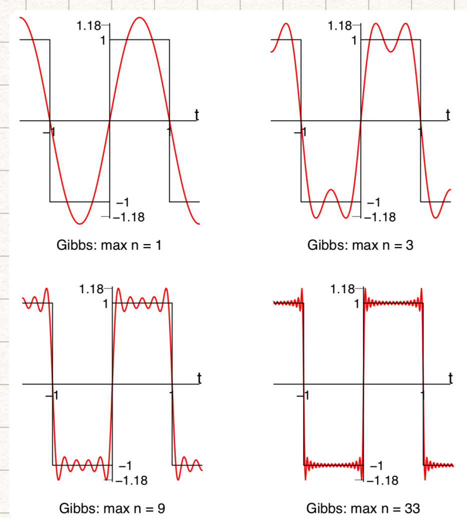
Gibbs phenomenon @ discontinuity

```
In[43]:= FourierSinSeries[1, x, 31, FourierParameters → {1, π}]
         Plot[%, {x, -2.5, 2.5}]
```

$$\text{Out[43]=}\ \frac{4\,\text{Sin}[\pi x]}{\pi} + \frac{4\,\text{Sin}[3\pi x]}{3\pi} + \frac{4\,\text{Sin}[5\pi x]}{5\pi} + \frac{4\,\text{Sin}[7\pi x]}{7\pi} + \frac{4\,\text{Sin}[9\pi x]}{9\pi} + \frac{4\,\text{Sin}[11\pi x]}{11\pi} + \frac{4\,\text{Sin}[13\pi x]}{13\pi} + \frac{4\,\text{Sin}[15\pi x]}{15\pi} + \frac{4\,\text{Sin}[17\pi x]}{17\pi} + \frac{4\,\text{Sin}[19\pi x]}{19\pi} + \frac{4\,\text{Sin}[21\pi x]}{21\pi} + \frac{4\,\text{Sin}[23\pi x]}{23\pi} + \frac{4\,\text{Sin}[25\pi x]}{25\pi} + \frac{4\,\text{Sin}[27\pi x]}{27\pi} + \frac{4\,\text{Sin}[29\pi x]}{29\pi} + \frac{4\,\text{Sin}[31\pi x]}{31\pi}$$

Out[44]=





9% of the jump

Complex Fourier Series

We can also choose the complex basis $e^{ik_n x}$

$$e^{ik_n x} = \cos(k_n x) + i\sin(k_n x) \qquad k_n = \frac{2n\pi}{L}$$

$$= \cos\left(\frac{2n\pi x}{L}\right) + i\sin\left(\frac{2n\pi x}{L}\right)$$

The periodic fn $f(x) = f(x+L)$ can be expressed as the complex Fourier series,

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \qquad c_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-ik_n x}\, dx}$$

- relations between (a_n, b_n) & c_n

$$c_n = \frac{1}{2}(a_n - ib_n)$$

$$c_{-n} = \frac{1}{2}(a_n + ib_n) \qquad n = 0, 1, 2, \cdots$$

- If $f(x)$ is real, a_n, b_n are real.

$\rightarrow$ $c_{-n} = c_n^*$

Parseval's Theorem

Consider two periodic fns $f(x)$ and $g(x)$ with the same period L. One can express their "inner product" by summation of the Fourier coefficients

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \qquad g(x) = \sum_{n=-\infty}^{\infty} d_n e^{ik_n x}$$

$$\underbrace{\langle f | g \rangle}_{\text{inner product}} = \int_{-L/2}^{L/2} f^*(x) g(x)\, dx$$

$$= \sum_n c_n^* \overbrace{\int_{-L/2}^{L/2} g(x) e^{-ik_n x}\, dx}^{L \cdot d_n}$$

$$= L \sum_n c_n^* d_n$$

NOTE

Constructing vectors from the Fourier coefficients

$$\vec{F} = \sqrt{L}\,(\cdots, c_{-1}, c_0, c_1, \cdots)$$

$$\vec{G} = \sqrt{L}\,(\cdots, d_{-1}, d_0, d_1, \cdots)$$

$$\Rightarrow \langle f | g \rangle = \vec{F}^* \cdot \vec{G}$$

Setting $f=g$, the above relation becomes

$$\boxed{\frac{1}{L}\int_{-L/2}^{L/2} |f|^2 dx = \sum_n |c_n|^2}$$

Parseval's theorem

average : $\langle \cdot \rangle = \frac{1}{L}\int_{-L/2}^{L/2} \cdot \, dx$

The spatial average of $|f|^2$ can be written as summation of all Fourier-mode contributions.

$$\langle |f|^2 \rangle = \sum_n |c_n|^2$$