Fourier Transform

The periodic fn $f(x) = f(x+L)$ can be expressed as the complex Fourier series,

$$\boxed{\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \\ c_n &= \frac{1}{L}\int_{-L/2}^{L/2} f(x)\, e^{-ik_n x}\, dx \end{aligned}}$$

Here, the discrete wave number $k_n = \frac{2n\pi}{L}$.

What happens in the thermodynamic limit?

i.e. $L \to \infty$ so that $\Delta k = \frac{2\pi}{L} \to 0$ and the wave number k becomes continuous

Introduce the function $\tilde{f}(k)$

$$\tilde{f}(k_n) \equiv \frac{\sqrt{2\pi}}{\Delta k} c_n = \frac{L}{\sqrt{2\pi}} c_n$$

Substitute into the coefficient relation:

$$\tilde{f}(k_n) = \frac{\cancel{L}}{\sqrt{2\pi}} \cdot \frac{1}{\cancel{L}} \int_{-L/2}^{L/2} f(x)\, e^{-ik_n x}\, dx$$

In the thermodynamic limit, $L \to \infty$,

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-ikx} dx$$

We can also rewrite the original Fourier series into integral form:

$$c_n = \frac{\Delta k}{\sqrt{2\pi}} \tilde{f}(k_n)$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \Delta k \, \tilde{f}(k_n) e^{ik_n x}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \, \tilde{f}(k) e^{ikx}$$

Collecting both relations together ~

$$\boxed{\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk \, \tilde{f}(k) e^{ikx} \\ \tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx \, f(x) e^{-ikx} \end{aligned}}$$

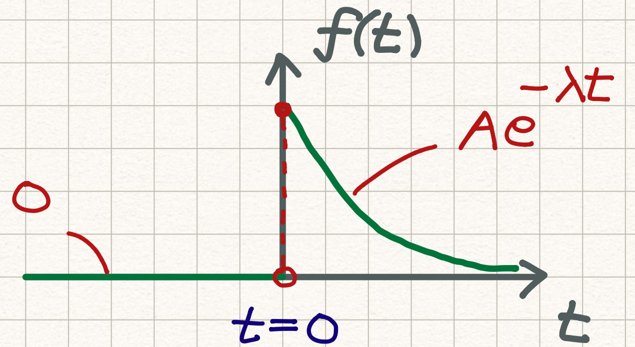
unitary matrix!

Follows Riley's symmetric convention

Example The duality between (x, k) can be applied to (t, ω) as well.

$$\boxed{\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega\, \tilde{f}(\omega)\, e^{-i\omega t} \\ \tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt\, f(t)\, e^{i\omega t} \end{aligned}}$$

sign difference!



$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, e^{i\omega t}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} A\, e^{-\lambda t + i\omega t}\, dt$$

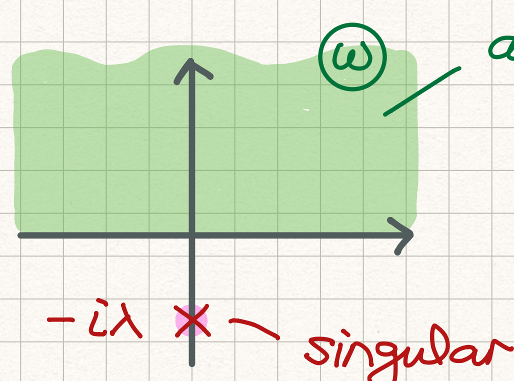
$$\tilde{f}(\omega) = \frac{A}{\sqrt{2\pi}} \frac{-1}{\lambda - i\omega}\, e^{-(\lambda - i\omega)t} \Big|_0^\infty$$

no contribution

$$= \frac{-A}{\sqrt{2\pi}\,(\lambda - i\omega)} \cdot (0 - 1) = \frac{A}{\sqrt{2\pi}\, i} \frac{-1}{(\omega + i\lambda)}$$

$\tilde{f}(\omega)$ is analytic in the upper half plane.

→ causality at work!



Uncertainty Principle

Consider a particle described by the wave fn,

$$\psi(x) = A\, e^{-\frac{1}{4}\frac{x^2}{\Delta_x^2}}$$

A is normalization constant.

The probability distribution of its position x is captured by the probability density fn:

$$P(x) = |\psi(x)|^2 = A^2 e^{-\frac{1}{2}\frac{x^2}{\Delta_x^2}}$$

$$\int_{-\infty}^{+\infty} P(x)\, dx = 1 \longrightarrow A^2 = \frac{1}{\sqrt{2\pi}\,\Delta_x}$$

The probability density fn is normal distribution

$$\boxed{P(x) = \frac{1}{\sqrt{2\pi}\,\Delta_x}\, e^{-\frac{1}{2}\frac{x^2}{\Delta_x^2}}}$$

average $\mu = \langle x \rangle = 0$

standard deviation $\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \Delta_x$

Δx by physicists...

One can apply Fourier transform to compute the wave fn $\tilde{\psi}(k)$ in the momentum space.

$$\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\ \psi(x)\, e^{-ikx}$$

$$= \frac{A}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dx\ e^{-\frac{1}{4}\frac{x^2}{\Delta_x^2} - ikx}$$

working on the exponent...

$$-\frac{1}{4\Delta_x^2}\left[x^2 + 4i\Delta_x^2 kx + (2i\Delta_x^2 k)^2\right] + \frac{1}{4\Delta_x^2}(2i\Delta_x^2 k)^2$$

$$= -\frac{1}{4\Delta_x^2}(x + 2i\Delta_x^2 k)^2 - \Delta_x^2 k^2$$

$$\tilde{\psi}(k) = \frac{A}{\sqrt{2\pi}} e^{-\Delta_x^2 k^2} \underbrace{\int_{-\infty}^{+\infty} dx\ e^{-\frac{1}{4\Delta_x^2} x'^2}}_{= \sqrt{4\pi}\,\Delta_x}$$

$$\rightarrow \boxed{\tilde{\psi}(k) = \sqrt{2} A \Delta_x\, e^{-\Delta_x^2 k^2}}$$ — still a Gaussian after F.T. ☺

The probability density $P(k)$ in the momentum space is

$$P(k) = |\tilde{\psi}(k)|^2 = 2A^2\Delta_x^2\, e^{-2\Delta_x^2 k^2}$$

$$= \frac{2\Delta_x}{\sqrt{2\pi}} e^{-2\Delta_x^2 k^2}$$

Compare with the standard form :

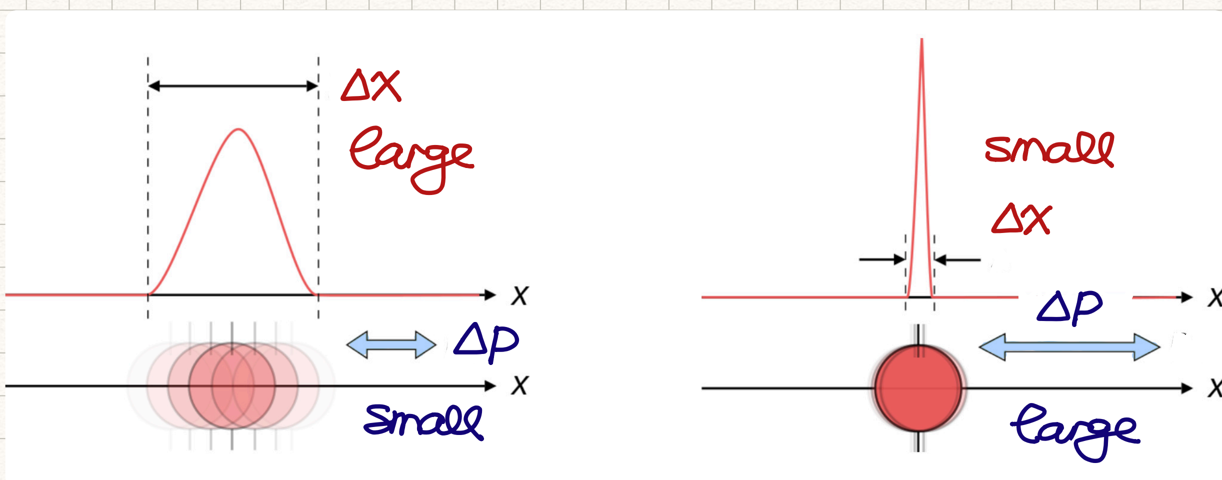
$$\boxed{\rho(k) = \frac{1}{\sqrt{2\pi}\,\Delta_k}\, e^{-\frac{1}{2}\frac{k^2}{\Delta_k^2}}}$$

$\Delta_k = \frac{1}{2\Delta_x}$ OR $\Delta_x \Delta_k = \frac{1}{2}$

Making use of Einstein- de Broglie relation

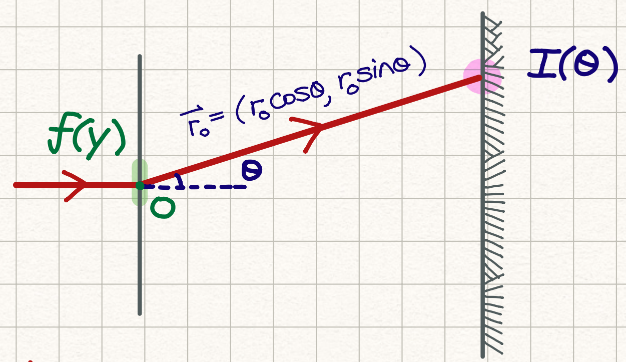
$p = \hbar k \longrightarrow \Delta p = \hbar \Delta k = \hbar \Delta_k$

So, $\Delta x \Delta p = \frac{1}{2}\hbar$ Heisenberg's uncertainty principle



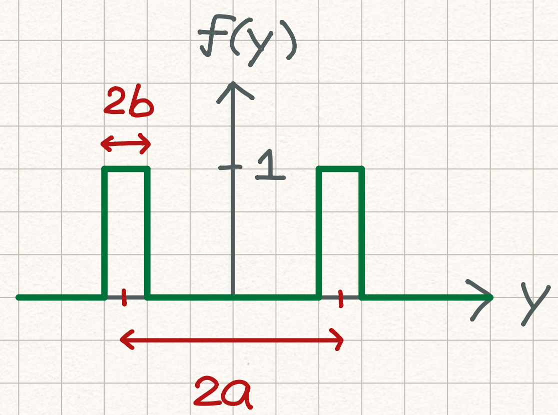
Fraunhofer Diffraction

Consider the Fraunhofer diffraction through some optical aperture described by $f(y)$:



$\vec{k}_{in} = (k, 0) = (\frac{2\pi}{\lambda}, 0)$

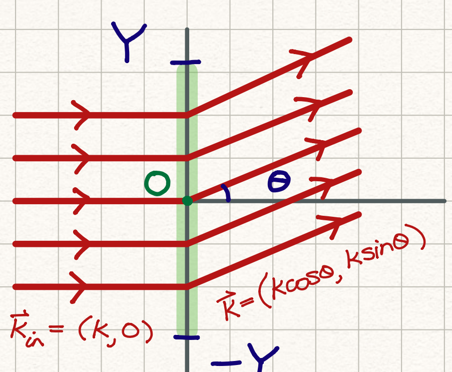
double-slit experiment ~



It is rather interesting that the intensity $I(\theta)$ is given by

$$\boxed{I(\theta) = \frac{2\pi}{r_0^2} \left| \hat{f}(k\sin\theta) \right|^2}$$

So, diffraction just performs the Fourier transform of the optical aperture $f(y)$ ☺



$$A(\theta) = \int_{-\infty}^{+\infty} dy\, f(y) \frac{e^{i\vec{k}\cdot\vec{r}}}{r}$$

- $\vec{r} = \vec{r}_0 - y\hat{y} = (r_0\cos\theta, r_0\sin\theta - y)$
- $r = |\vec{r}|$ is the distance ☺

Because $r_0 >> Y$, one can approximate $r \approx r_0$.

$$A(\theta) = \frac{e^{i\vec{k}\cdot\vec{r_0}}}{r_0} \int_{-\infty}^{+\infty} dy\, f(y)\, e^{-ik\sin\theta y}$$

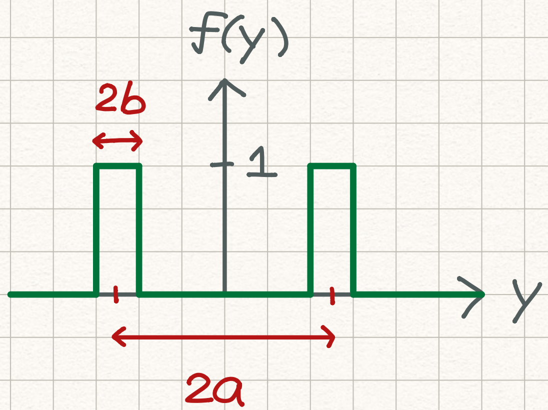
$$= \frac{e^{i\vec{k}\cdot\vec{r_0}}}{r_0} \sqrt{2\pi}\, \tilde{f}(k\sin\theta)$$

The intensity in the direction θ is then given by

$$I(\theta) = |A(\theta)|^2 = \frac{2\pi}{r_0^2} \left| \tilde{f}(k\sin\theta) \right|^2$$

double-slit experiment ~

set $q = k\sin\theta$



$$\tilde{f}(q) = \frac{1}{\sqrt{2\pi}} \int_{-a-b}^{-a+b} dx\; 1 \cdot e^{-iqx}$$

$$+ \frac{1}{\sqrt{2\pi}} \int_{a-b}^{a+b} dx\; 1 \cdot e^{-iqx}$$

$$\tilde{f}(q) = \frac{-1}{\sqrt{2\pi}\, iq} \left[e^{-iq(-a+b)} - e^{-iq(-a-b)} + e^{-iq(a+b)} - e^{-iq(a-b)} \right]$$

$$= \frac{+1}{\sqrt{2\pi}\, \cancel{i}q} \left[+2\cancel{i} \sin(qb)\, e^{iqa} + 2\cancel{i} \sin(qb)\, e^{-iqa} \right]$$

$$= \frac{4\cos(qa)\sin(qb)}{\sqrt{2\pi}\, q}$$

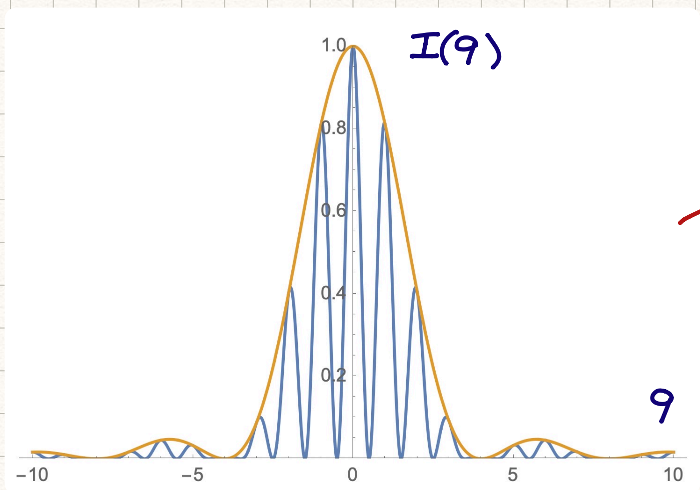
Finally, the intensity $I(\theta)$ is

$$\boxed{I(\theta) = \frac{16b^2}{r_0^2}\cdot\frac{\sin^2(qb)}{(qb)^2}\cdot\cos^2(qa)}$$

distance effect

slit-width effect

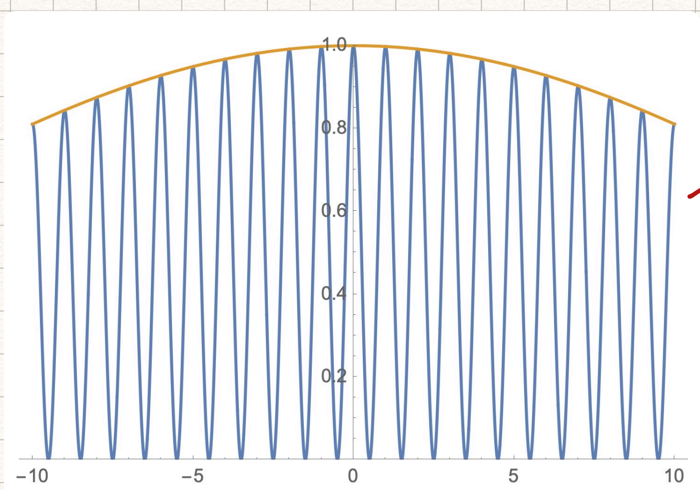
double-slit interference!



$$q = k\sin\theta = \frac{2\pi}{\lambda}\sin\theta$$

$b = \frac{1}{4}a$

slit width is comparable to the distance between the two slits.



$b = \frac{1}{40}a$

the slit width is negligibly small

Here the maximal intensity is set to unity. It is not the case when the slit width is changed in realistic experiments