Matrix 2

Matrix has tons of interesting properties. Here we are going to introduce several important ones ☺

(1) Trace $\mathrm{Tr}\,A$

(2) Determinant $\det A$

(3) Inverse A^{-1}

Let's start with the simplest : Trace.

$$\mathrm{Tr}\,A = \sum_i A_{ii} = A_{11} + A_{22} + \cdots + A_{NN}$$

Lemma. $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB)$

$$\mathrm{Tr}(ABC) = \sum_i (ABC)_{ii} = \sum_{ijk} A_{ij} B_{jk} C_{ki}$$

$$= \sum_{ijk} B_{jk} C_{ki} A_{ij} = \sum_j (BCA)_{jj}$$

$$= \mathrm{Tr}(BCA)$$

Similarly, $\mathrm{Tr}(ABC) = \sum_{ijk} A_{ij} B_{jk} C_{ki}$

$$= \sum_{ijk} C_{ki} A_{ij} B_{jk} = \mathrm{Tr}(CAB)$$

Determinant

Determinant is a headache to teach !!! The main reason is Levi-Civita tensor in its definition

Take 3×3 matrix as a working example.

$$\det A = |A| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

$$= \left(A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32}\right)$$

$$- \left(A_{11}A_{23}A_{32} + A_{12}A_{21}A_{33} + A_{13}A_{22}A_{31}\right)$$

Observations: (123), (231), (312) $+$

(132), (213), (321) $-$

can be written as $(-1)^P$ where P is the permutation time to go back to (123)

(231) $\rightarrow$ (132) $\rightarrow$ (123) $P = 2$ even

(321) $\rightarrow$ (123) $P = 1$ odd

It shall be clear that there are $N!$ permutations

$3! = 6$

The determinant of the matrix is defined as

$$\det A = \sum_{\text{perm}} (-1)^{P} A_{1P_1} A_{2P_2} A_{3P_3}$$

all possible $N!$ permutations

$(P_1 P_2 P_3) = (123), \cdots$

The definition of $\det A$ leads to Laplace expansion of the determinant:

$$\det A = A_{11} C_{11} + A_{12} C_{12} + A_{13} C_{13}$$

(minor)

M_{11}

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \qquad C_{11} = (-1)^{1+1} \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix}$$

M_{12}

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \qquad C_{12} = (-1)^{1+2} \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix}$$

In general, the cofactor C_{ij} is

$$C_{ij} = (-1)^{i+j} M_{ij}$$

example rotation matrix $R_z(\theta)$ in 3D

$$|R_z(\theta)| = \begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

choose the 3rd row for Laplace expansion.

$$= 0 \cdot \begin{vmatrix} \vdots & \vdots \end{vmatrix} - 0 \begin{vmatrix} \vdots & \vdots \end{vmatrix}$$

$$+1 \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$$\to \quad |R_z(\theta)| = \cos^2\theta - (-\sin^2\theta) = 1.$$

norm invariance under $R_z(\theta)$!

Properties of Determinant

(1) $|A^T| = |A|$

$$\det A = \sum_{\text{perm}} (-1)^P A_{1P_1} A_{2P_2} A_{3P_3}$$

$$= \sum_{\text{perm}} (-1)^Q A_{Q_1 1} A_{Q_2 2} A_{Q_3 3}$$

(2) $|A^{\dagger}| = |A|^{*}$

easy to see $\quad |A^{\dagger}| = |A^{*}| = |A|^{*}$ ✓

(3) $|A| = -|A'|$ — interchange 2 rows/columns

$$\det A' = \sum_{\text{perm}} (-1)^{p'} A'_{1p_1'} A'_{2p_2'} A'_{3p_3'}$$

$$= \sum_{\text{perm}} (-1)^{p'} A_{1p_1'} A_{3p_2'} A_{2p_3'}$$

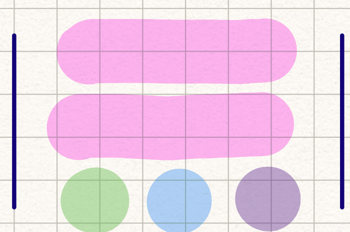
$$= \sum_{\text{perm}} (-1)^{p'} A_{1p_1'} A_{2p_3'} A_{3p_2'} \qquad (p_1 p_3 p_2) = -(p_1 p_2 p_3)$$

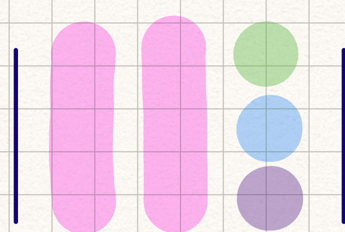
$$= -\sum_{\text{perm}} (-1)^{p} A_{1p_1} A_{2p_2} A_{3p_3}$$

$$= -\det A$$ YES!!

(4) $|\lambda A| = \lambda^{N} |A|$

(5) $|A| = 0$ if $A_{i\cdot} = A_{j\cdot}$ or $A_{\cdot i} = A_{\cdot j}$

 $= 0$

 $= 0$

(6) Adding a constant multiple of one R/C to another R/C doesn't change the determinant.

$$\begin{vmatrix} \cos\theta - \lambda\sin\theta & -\sin\theta & 0 \\ \sin\theta + \lambda\cos\theta & \cos\theta & 0 \\ 0 + \lambda 0 & 0 & 1 \end{vmatrix}$$

$$= +1 \begin{vmatrix} \cos\theta - \lambda\sin\theta & -\sin\theta \\ \sin\theta + \lambda\cos\theta & \cos\theta \end{vmatrix}$$

$$= \cos^2\theta \cancel{-\lambda\sin\theta\cos\theta} - (-\sin^2\theta \cancel{-\lambda\cos\theta\sin\theta})$$

$$= \cos^2\theta + \sin^2\theta = 1$$ — det $R_z(\theta)$ remains the same ☺

(7) $|AB| = |A||B|$ for square matrices.

It can be generalized to multiple products

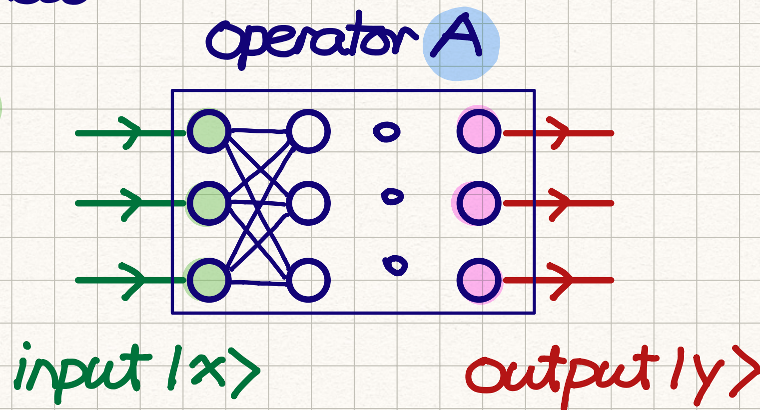
$$|A \cdot B \cdots Z| = |A||B| \cdots |Z|$$

easy to use,
hard to prove...

Inverse of a matrix

In many scientific questions, $A|x\rangle = |y\rangle$, one would like to know the inverse operator

$$A^{-1}|y\rangle = |x\rangle$$



If A is a linear operator, this can be done easily as long as $\det A \neq 0$!

$\det A = 0 \longrightarrow$ singular matrix

$\det A \neq 0 \longrightarrow$ non-singular matrix

Suppose A is non-singular $\longrightarrow A^{-1}$ exists.

$$A^{-1}|y\rangle = |x\rangle \longrightarrow A^{-1}A|x\rangle = |x\rangle$$

← arbitrary ☺

thus, $A^{-1}A = \mathbb{1}$.

$$A|x\rangle = |y\rangle \longrightarrow AA^{-1}|y\rangle = |y\rangle$$

thus, $AA^{-1} = \mathbb{1}$

The inverse operator A^{-1} in matrix form

$$\boxed{(A^{-1})_{ij} = \frac{1}{|A|} C_{ji}}$$

example rotation matrix $R_z(\theta)$ in 3D

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(R^{-1})_{11} = (-1)^{1+1} \begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} = \cos\theta$$

$$(R^{-1})_{12} = (-1)^{1+2} \begin{vmatrix} -\sin\theta & 0 \\ 0 & 1 \end{vmatrix} = \sin\theta$$

$$R^{-1} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = R^T$$

rotations: $R^T R = R R^T = \mathbb{1}$ orthogonal matrix

Proof is straightforward ☺

$$(A^{-1}A)_{ij} = \sum_k (A^{-1})_{ik} A_{kj} = \frac{1}{|A|} \sum_k C_{ki} A_{kj}$$

(1) $i=j$ case

$$(A^{-1}A)_{ii} = \frac{1}{|A|} \sum_k A_{ki} C_{ki} = \frac{1}{|A|} |A| = 1.$$

(2) $i \neq j$ case

Replacing i^{th} column by j^{th} column

Because 2 columns $(i^{th}$ & $j^{th})$ are the same,

$A'_{ki} = A_{kj}$

$$|A'| = 0$$

$$\rightarrow \quad |A'| = \sum_k A'_{ki} C'_{ki} = \sum_k A_{kj} C_{ki} = 0$$

Combine both results together ~

$$(A^{-1}A)_{ij} = \delta_{ij} \quad \rightarrow \quad A^{-1}A = \mathbb{1}.$$

Similarly, one can show that $AA^{-1} = \mathbb{1}$.