Divergence Theorem

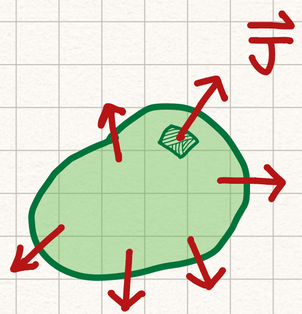
Recall the definition of divergence :

$$\boxed{\vec{\nabla} \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}}$$

The divergence theorem states

$$\boxed{\oint_{\partial V} \vec{J} \cdot d\vec{a} = \int_{V} \vec{\nabla} \cdot \vec{J} \, d\tau}$$

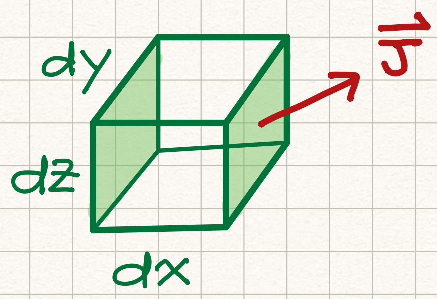
There are two ways to compute the flux through the surface ∂V :



Flux $\rightarrow \oint_{\partial V} \vec{J} \cdot d\vec{a}$

Flux $\rightarrow \int_{V} \vec{\nabla} \cdot \vec{J} \, d\tau$

Let's prove the theorem by **Mr. Cube**



$$\oint_{\partial V} \vec{J} \cdot d\vec{a}$$

= Flux (x axis)
+ Flux (y axis)
+ Flux (z axis)

$$\text{Flux (x axis)} = J_x(x+dx)\,dydz - J_x(x)\,dydz = \frac{\partial J_x}{\partial x}\,dxdydz$$

Similarly, one can compute the fluxes along the y, z axes.

$$\text{Flux (y axis)} = \frac{\partial J_y}{\partial y}\,dydxdz$$

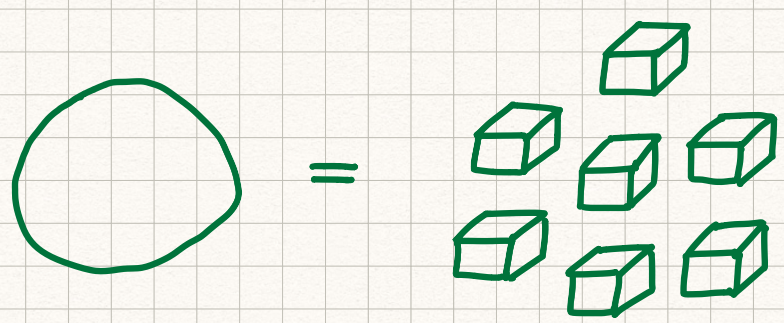
$$\text{Flux (z axis)} = \frac{\partial J_z}{\partial z}\,dzdxdy$$

Note that the volume integral for Mr. Cube is rather simple.

$$\int_V \vec{\nabla}\cdot\vec{J}\,d\tau = \vec{\nabla}\cdot\vec{J}\,dxdydz$$

It is clear that both integrals equal,

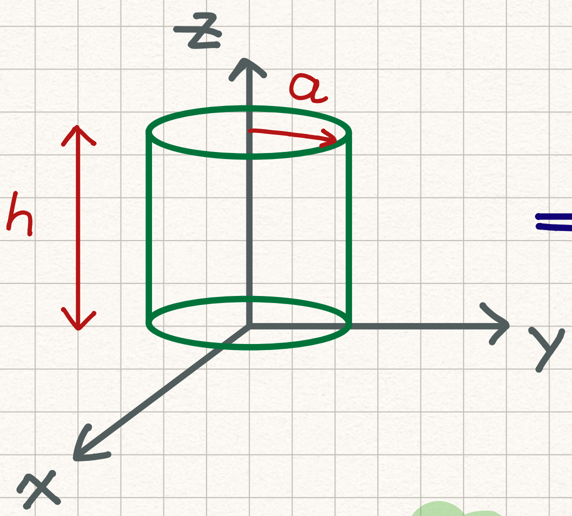
$$\int_{\partial V} \vec{J}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{J}\,d\tau$$



The volume can be decomposed by many Mr. Cubes.

Q.E.D.

example : $\vec{J} = (x, y, z)$



$$\oint_{\partial V} \vec{J}\cdot d\vec{a}$$

$$= \text{Flux (top)} + \text{Flux (bottom)} + \text{Flux (side)}$$

$$\text{Flux (top)} = J_z(z=h)\cdot \pi a^2 = \pi a^2 h$$

$$\text{Flux (bottom)} = -J_z(z=0)\cdot \pi a^2 = 0$$

Note that $\hat{n} = \left(\frac{x}{a}, \frac{y}{a}, 0\right)$ on the side surface,

$$\vec{J}\cdot\hat{n} = J_x\cdot\frac{x}{a} + J_y\cdot\frac{y}{a} = \frac{1}{a}(x^2+y^2) = a.$$

$$\text{Flux (side)} = (\vec{J}\cdot\hat{n})\cdot 2\pi a h = 2\pi a^2 h.$$

Adding all fluxes together,

$$\int_{\partial V} \vec{J}\cdot d\vec{a} = \pi a^2 h + 0 + 2\pi a^2 h = \underline{3\pi a^2 h}.$$

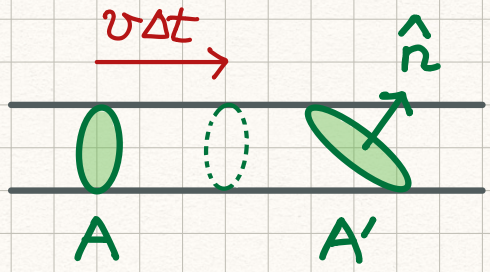
Now, try to compute the flux by $\vec{\nabla}\cdot\vec{J}$

$$\vec{\nabla}\cdot\vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 1+1+1 = 3$$

$$\int_V \vec{\nabla}\cdot\vec{J}\, d\tau = 3\cdot \pi a^2 h = \underline{3\pi a^2 h}$$ — the same

Continuity Equation

Let's review the definition of current density $\vec{J}$



total amount

$$= J \cdot \Delta t \cdot A$$

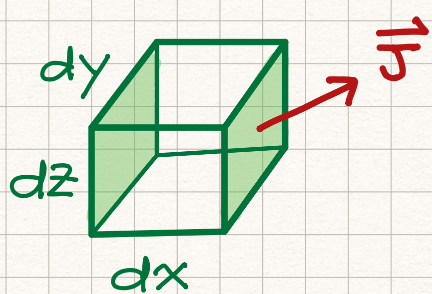
$$= \rho \, (v \Delta t A)$$

$\Rightarrow \quad J = \rho v \quad$ or in vector form $\vec{J} = \rho \vec{v}$

By comparing the current through A and A',

$$I = \vec{J} \cdot \hat{n} A = \vec{J} \cdot \vec{A}$$

— relation between I and $\vec{J}$



$$I_{out} = \frac{\partial J_x}{\partial x} dx\,dy\,dz + \frac{\partial J_y}{\partial y} dy\,dx\,dz + \frac{\partial J_z}{\partial z} dz\,dx\,dy$$

$$\Rightarrow \quad I_{out} = \vec{\nabla} \cdot \vec{J} \, dx\,dy\,dz$$

The charge changing rate inside Mr. Cube is

$$\frac{dQ}{dt} = \frac{\partial \rho}{\partial t} dx\,dy\,dz = S \, dx\,dy\,dz - I_{out}$$

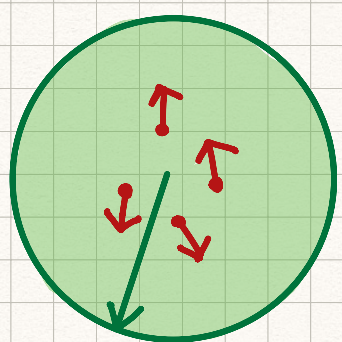
source inside

Thus, $\left[\frac{\partial \rho}{\partial t} - s + \vec{\nabla}\cdot\vec{J}\right] dx\,dy\,dz = 0$

$$\rightarrow \quad \vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = s$$

If no source is present, $s=0$,

$$\boxed{\vec{\nabla}\cdot\vec{J} + \frac{\partial \rho}{\partial t} = 0}$$ — Continuity equation



Enclose all charges by an infinitely large sphere ~

$$\int_V \vec{\nabla}\cdot\vec{J}\, d\tau + \int_V \frac{\partial \rho}{\partial t}\, d\tau = 0$$

Applying divergence theorem,

$$\int_V \vec{\nabla}\cdot\vec{J}\, d\tau = \int_{\partial V} \vec{J}\cdot d\vec{a} = 0$$

$\vec{J}$ vanishes on the $R\to\infty$ sphere

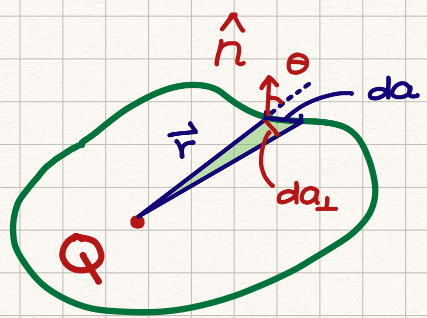
In consequence, $\int_V \frac{\partial \rho}{\partial t}\, d\tau = \frac{d}{dt}\int_V \rho\, d\tau = 0$

$$\rightarrow \quad \frac{dQ}{dt} = 0 \qquad Q(t) = \text{const.}$$

Charge conservation

Gauss' Law

solid angle $d\Omega \equiv \frac{1}{r^2} da_\perp$



$$d\Omega = \frac{1}{r^2} \cos\theta \, da$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r}$$

$$\oint_{\partial V} \vec{E} \cdot d\vec{a} = \oint_{\partial V} \frac{Q}{4\pi\epsilon_0} \frac{1}{r^2} \underbrace{\hat{r} \cdot \hat{n}}_{\cos\theta} da$$

$$= \frac{Q}{4\pi\epsilon_0} \oint_{\partial V} \frac{1}{r^2} \cos\theta \, da$$

$$= \frac{Q}{4\pi\epsilon_0} \oint_{\partial V} d\Omega = \frac{Q}{\cancel{4\pi}\epsilon_0} \cdot \cancel{4\pi}$$

$$\rightarrow \boxed{\oint_{\partial V} \vec{E} \cdot d\vec{a} = \frac{Q}{\epsilon_0}}$$

The charges can be viewed as "sources" or "sinks" of the electric flux Φ_E

$Q = \int_V \rho \, d\tau \rightarrow \oint_{\partial V} \vec{E} \cdot d\vec{a} = \int_V \frac{\rho}{\epsilon_0} d\tau$

By divergence theorem, $\int_V \left(\vec{\nabla} \cdot \vec{E} - \frac{\rho}{\epsilon_0} \right) d\tau = 0$

$$\rightarrow \vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$ Maxwell eq.

point charge

$\odot\!\odot$?!

Let's compute the divergence of $\vec{E}$ by a point charge Q @ the origin.

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r} = \frac{Q}{4\pi\epsilon_0}\left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right)$$

$$\vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

$\frac{\partial r}{\partial x} = \frac{x}{r}$

$$= \frac{Q}{4\pi\epsilon_0}\left\{ \frac{r^3 - 3r^2\frac{x}{r}x}{r^6} + \frac{r^3 - 3r^2\frac{y}{r}y}{r^6} + \frac{r^3 - 3r^2\frac{z}{r}z}{r^6} \right\}$$

$$= \frac{Q}{4\pi\epsilon_0 r^5}\left\{ r^2 - 3x^2 + r^2 - 3y^2 + r^2 - 3z^2 \right\}$$

$$= 0 \quad ¡$$

so... $\int_V \vec{\nabla}\cdot\vec{E}\, d\tau = 0$ for a point charge.

But, according to Gauss' law:

$$\int_{\partial V} \vec{E}\cdot d\vec{a} = \frac{Q}{\epsilon_0} \longrightarrow \int_V \vec{\nabla}\cdot\vec{E}\, d\tau = \frac{Q}{\epsilon_0}$$

It turns out that $\vec{\nabla}\cdot\vec{E}$ is not really zero

$$\vec{\nabla}\cdot\vec{E} = \frac{Q}{\epsilon_0}\,\delta(x)\delta(y)\delta(z)$$

This is quite reasonable because the charge density $\rho(x,y,z) = 0$ except @ $\vec{r} = 0$.

NOTE $\quad \vec{E} = -\nabla V, \quad V = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$

$$\vec{\nabla}\cdot\vec{E} = -\vec{\nabla}\cdot\vec{\nabla} V = -\vec{\nabla}^2 V = -\frac{Q}{4\pi\epsilon_0}\nabla^2\left(\frac{1}{r}\right)$$

By comparison, one can see that

$$\underbrace{-\frac{1}{4\pi}\nabla^2\left(\frac{1}{r}\right)}_{\text{Coulomb potential}} = \underbrace{\delta(x)\delta(y)\delta(z)}_{\text{Dirac }\delta\text{-function}}$$