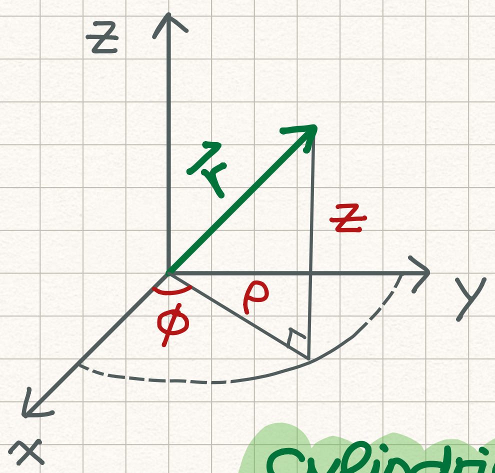
Curvilinear Coordinates



The "del" $\vec{\nabla}$ operator takes rather different forms in the curvilinear coordinates

Cylindrical coordinates

$$\boxed{\begin{aligned} x &= \rho\cos\phi \\ y &= \rho\sin\phi \\ z &= z \end{aligned}}$$

The position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$$

$$\vec{a}_\rho = \frac{\partial\vec{r}}{\partial\rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\vec{a}_\phi = \frac{\partial\vec{r}}{\partial\phi} = -\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}$$

$$\vec{a}_z = \frac{\partial\vec{r}}{\partial z} = \hat{k}$$

$\vec{a}_\rho, \vec{a}_\phi, \vec{a}_z$ are the base vectors for (ρ, ϕ, z)

normalize to unit vectors:

$$\hat{e}_\rho = \vec{a}_\rho = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$$

$$\hat{e}_\phi = \frac{1}{\rho}\vec{a}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$$\hat{e}_z = \vec{a}_z = \hat{k}$$

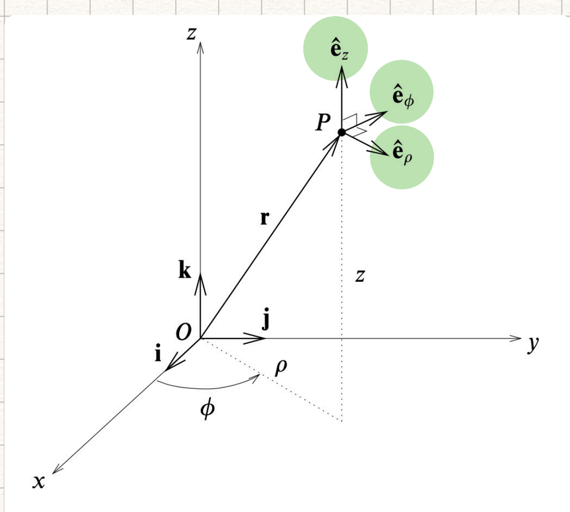


Figure 10.7 Cylindrical polar coordinates ρ, ϕ, z.

$\{\hat{e}_\rho, \hat{e}_\phi, \hat{e}_z\}$ orthonormal basis

The infinitesimal displacement $d\vec{r}$ is

$$d\vec{r} = \frac{\partial \vec{r}}{\partial \rho} d\rho + \frac{\partial \vec{r}}{\partial \phi} d\phi + \frac{\partial \vec{r}}{\partial z} dz$$

$$= d\rho\, \vec{a}_\rho + d\phi\, \vec{a}_\phi + dz\, \vec{a}_z$$

$$= d\rho\, \hat{e}_\rho + \rho\, d\phi\, \hat{e}_\phi + dz\, \hat{e}_z$$

The element of arc length ds is

$$(ds)^2 = d\vec{r} \cdot d\vec{r} = (d\rho)^2 + \rho^2 (d\phi)^2 + (dz)^2$$

NOTE. In general curvilinear coordinates q_i,

$$\vec{a}_i = \frac{\partial \vec{r}}{\partial q_i} \quad \leftarrow \text{not necessarily orthogonal.}$$

$$(ds)^2 = d\vec{r} \cdot d\vec{r} = \sum_i \frac{\partial \vec{r}}{\partial q_i} dq_i \cdot \sum_j \frac{\partial \vec{r}}{\partial q_j} dq_j$$

$$= \sum_{ij} \underbrace{\frac{\partial \vec{r}}{\partial q_i} \cdot \frac{\partial \vec{r}}{\partial q_j}}_{\text{metric tensor } g_{ij} = \vec{a}_i \cdot \vec{a}_j} dq_i\, dq_j$$

$$\boxed{(ds)^2 = \sum_{ij} g_{ij}\, dq_i\, dq_j}$$

describes the curvature of the space

For the cylindrical,

$$g_{ij} = 0, \quad i \neq j$$

$$g_{ij} = h_i^2\, \delta_{ij}$$

$$h_\rho = 1, \quad h_\phi = \rho, \quad h_z = 1$$

Spherical Coordinates

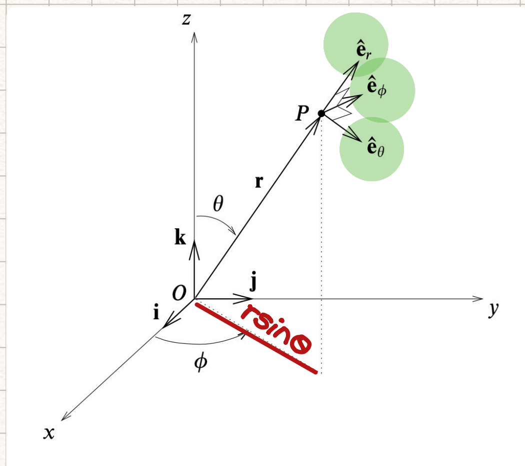


Figure 10.9 Spherical polar coordinates r, θ, ϕ.

$$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}}$$

$$\begin{aligned}\vec{r} &= x\hat{i} + y\hat{j} + z\hat{k} \\ &= r\sin\theta\cos\phi\,\hat{i} \\ &\quad + r\sin\theta\sin\phi\,\hat{j} \\ &\quad + r\cos\theta\,\hat{k}\end{aligned}$$

The base vectors $\vec{a}_i$ are

$$\vec{a}_r = \frac{\partial\vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\vec{a}_\theta = \frac{\partial\vec{r}}{\partial\theta} = r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}$$

$$\vec{a}_\phi = \frac{\partial\vec{r}}{\partial\phi} = -r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}$$

$$|\vec{a}_r| = 1, \quad |\vec{a}_\theta| = r, \quad |\vec{a}_\phi| = r\sin\theta$$

normalize to unit vectors,

$$\hat{e}_r = \vec{a}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{e}_\theta = \frac{1}{r}\vec{a}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{e}_\phi = \frac{1}{r\sin\theta}\vec{a}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

$\{\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi\}$ orthonormal basis

The element of arc length ds is

$$(ds)^2 = \sum_{ij} g_{ij}\, dq_i\, dq_j \quad — \quad g_{ij} = \vec{a}_i \cdot \vec{a}_j$$

metric tensor

$$= \sum_i h_i^2 (dq_i)^2 \quad — \quad h_i = |\vec{a}_i|$$

scale factor

$$= (dr)^2 + r^2 (d\theta)^2 + r^2 \sin^2\theta\, (d\phi)^2$$

scale factors $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$

singular @ $r=0$ (for h_θ)

singular @ $r=0$, @ $\theta = 0, \pi$ (for h_ϕ)

volume element

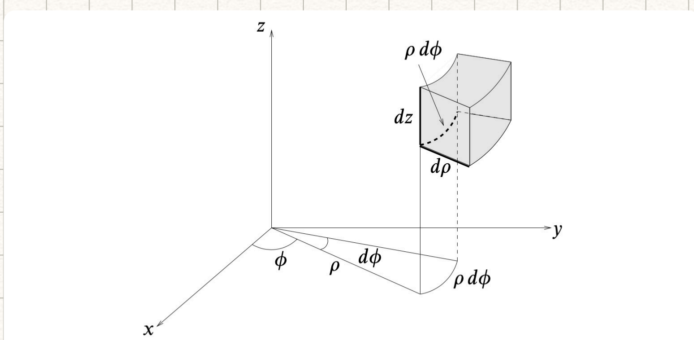


Figure 10.8 The element of volume in cylindrical polar coordinates is given by $\rho\, d\rho\, d\phi\, dz$.

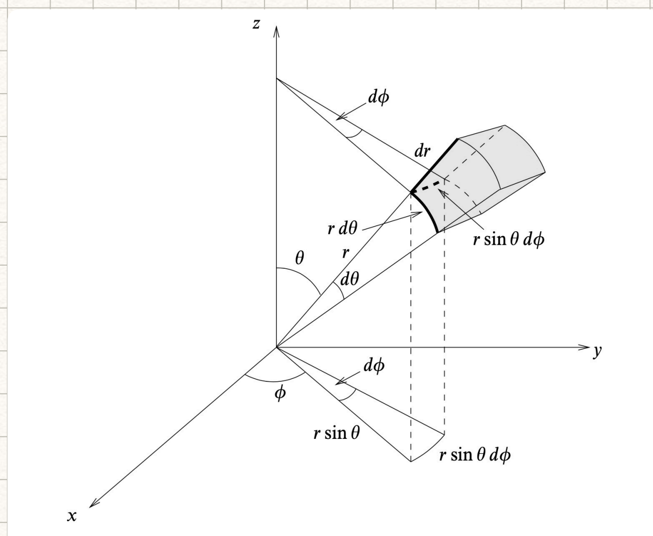
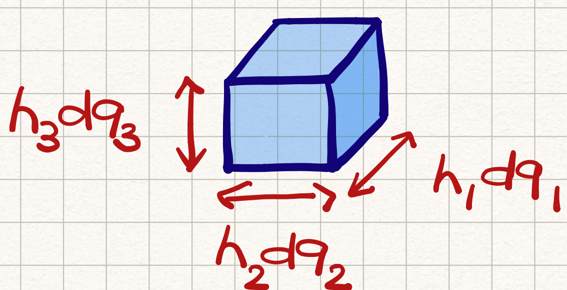


Figure 10.10 The element of volume in spherical polar coordinates is given by $r^2 \sin\theta\, dr\, d\theta\, d\phi$.



Mr. Cube

$$d\tau = h_1 h_2 h_3\, dq_1\, dq_2\, dq_3$$

Gradient

By definition, $d\Phi = \vec{\nabla}\Phi \cdot d\vec{r}$

By partial differentiation, $d\Phi = \sum_{i=1}^{3} \frac{\partial \Phi}{\partial q_i} dq_i$

Set $dq_2 = dq_3 = 0$, — holding q_2, q_3 constant.

$$d\vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 = \vec{a}_1 dq_1 = \hat{e}_1 h_1 dq_1$$

Thus, $d\Phi = \vec{\nabla}\Phi \cdot \hat{e}_1 h_1 dq_1 = \frac{\partial \Phi}{\partial q_1} dq_1$

$$\rightarrow \boxed{\vec{\nabla}\Phi \cdot \hat{e}_1 = \frac{1}{h_1} \frac{\partial \Phi}{\partial q_1}}$$

Similarly, one can work out the other comp.

$$\vec{\nabla}\Phi = \frac{1}{h_1}\frac{\partial \Phi}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial \Phi}{\partial q_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial \Phi}{\partial q_3}\hat{e}_3$$

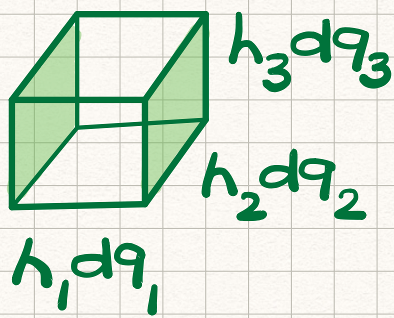
OR, in the operator form ~

$$\boxed{\vec{\nabla} = \hat{e}_1 \frac{1}{h_1}\frac{\partial}{\partial q_1} + \hat{e}_2 \frac{1}{h_2}\frac{\partial}{\partial q_2} + \hat{e}_3 \frac{1}{h_3}\frac{\partial}{\partial q_3}}$$

Divergence

$$\int_{\partial V} \vec{J}\cdot d\vec{a} = \int_V \vec{\nabla}\cdot\vec{J}\, d\tau$$

Mr. Cube



Flux along q_1 direction:

$$\Phi_1 = J_1 h_2 h_3 (q_1 + dq_1)\, dq_2 dq_3 - J_1 h_2 h_3 (q_1)\, dq_2 dq_3 = \frac{\partial}{\partial q_1}(h_2 h_3 J_1)\, dq_1 dq_2 dq_3$$

$$\int_{\partial V} \vec{J}\cdot d\vec{a} = \Phi_1 + \Phi_2 + \Phi_3$$

$$= \left[\frac{\partial}{\partial q_1}(h_2 h_3 J_1) + \frac{\partial}{\partial q_2}(h_3 h_1 J_2) + \frac{\partial}{\partial q_3}(h_1 h_2 J_3)\right] dq_1 dq_2 dq_3$$

The volume integral is simple ~

$$\int_V \vec{\nabla}\cdot\vec{J}\, d\tau = \vec{\nabla}\cdot\vec{J}\cdot h_1 h_2 h_3\, dq_1 dq_2 dq_3$$

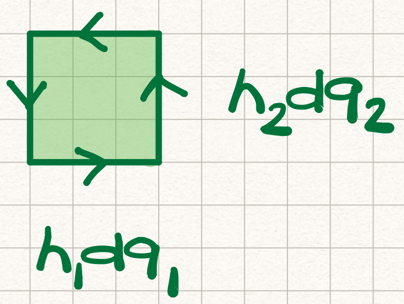
By comparison, it's easy to see that

$$\boxed{\vec{\nabla}\cdot\vec{J} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(h_2 h_3 J_1) + \frac{\partial}{\partial q_2}(h_1 h_3 J_2) + \frac{\partial}{\partial q_3}(h_1 h_2 J_3)\right]}$$

Curl

$$\oint_{\partial S} \vec{V}\cdot d\vec{r} = \int_S \vec{\nabla}\times\vec{V}\cdot d\vec{S}$$

Ms Loop



Consider the circulation in the q_1-q_2 plane. It contains all contributions from the 4 segments:

$$\oint_{\partial S} \vec{V}\cdot d\vec{r} = h_2 V_2(q_1+dq_1)\,dq_2 - h_2 V_2(q_1)\,dq_2 - h_1 V_1(q_2+dq_2)\,dq_1 + h_1 V_1(q_2)\,dq_1$$

$$= \left[\frac{\partial}{\partial q_1}(h_2 V_2) - \frac{\partial}{\partial q_2}(h_1 V_1)\right] dq_1 dq_2$$

The surface integral can be expressed as

$$\int_S \vec{\nabla}\times\vec{V}\cdot d\vec{S} = (\vec{\nabla}\times\vec{V})\cdot\hat{e}_3\; h_1 h_2\, dq_1 dq_2$$

By comparison, the curl $(\vec{\nabla}\times\vec{V})_3$ is

$$(\vec{\nabla}\times\vec{V})\cdot\hat{e}_3 = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial q_1}(h_2 V_2) - \frac{\partial}{\partial q_2}(h_1 V_1)\right]$$

working all comp.

$$\boxed{\vec{\nabla}\times\vec{V} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 V_1 & h_2 V_2 & h_3 V_3 \end{vmatrix}}$$

Summary

$$
\begin{aligned}
\nabla\Phi &= \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1}\hat{\mathbf{e}}_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2}\hat{\mathbf{e}}_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}\hat{\mathbf{e}}_3 \\
\nabla\cdot\mathbf{a} &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_2h_3a_1) + \frac{\partial}{\partial u_2}(h_3h_1a_2) + \frac{\partial}{\partial u_3}(h_1h_2a_3)\right] \\
\nabla\times\mathbf{a} &= \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{\mathbf{e}}_1 & h_2\hat{\mathbf{e}}_2 & h_3\hat{\mathbf{e}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1a_1 & h_2a_2 & h_3a_3 \end{vmatrix} \\
\nabla^2\Phi &= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\Phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right]
\end{aligned}
$$

Table 10.4 Vector operators in orthogonal curvilinear coordinates u_1, u_2, u_3. Φ is a scalar field and $\mathbf{a}$ is a vector field.

cylindrical

$h_\rho = 1, \; h_\phi = \rho, \; h_z = 1$

$$
\begin{aligned}
\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z \\
\nabla\cdot\mathbf{a} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho a_\rho) + \frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi} + \frac{\partial a_z}{\partial z} \\
\nabla\times\mathbf{a} &= \frac{1}{\rho}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \rho\hat{\mathbf{e}}_\phi & \hat{\mathbf{e}}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\phi} & \dfrac{\partial}{\partial z} \\ a_\rho & \rho a_\phi & a_z \end{vmatrix} \\
\nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}
\end{aligned}
$$

Table 10.2 Vector operators in cylindrical polar coordinates; Φ is a scalar field and $\mathbf{a}$ is a vector field.

Spherical

$h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta$

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi$$

$$\nabla\cdot\mathbf{a} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\ a_\theta) + \frac{1}{r\sin\theta}\frac{\partial a_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{a} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & r\sin\theta\,\hat{\mathbf{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial\theta} & \dfrac{\partial}{\partial\phi} \\ a_r & ra_\theta & r\sin\theta\ a_\phi \end{vmatrix}$$

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Table 10.3 Vector operators in spherical polar coordinates; Φ is a scalar field and $\mathbf{a}$ is a vector field.