EECS 2060 Discrete Mathematics — Chi-chao Chao

Spring 2021

Final Examination

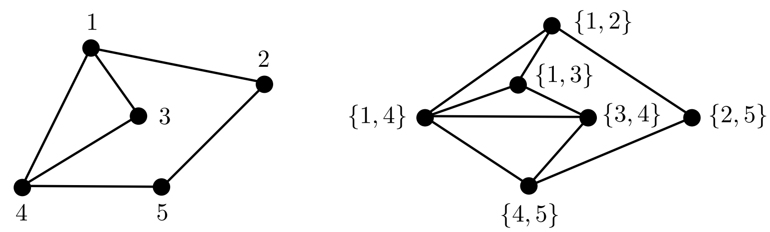
7:00pm to 10:10pm, June 25, 2021

Rules and Regulations:

This is a *remote open-book* examination. If you have any questions, no one except me can be consulted. There is a time limit of *three hours and ten minutes*. Please submit your answer in a single pdf file to iLMS by the due time. No late submission will be accepted.

Problems for Solution:

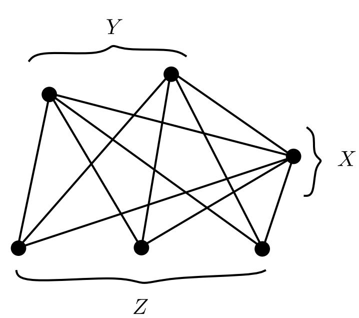
1. (10%) Given an undirected simple graph G, its *line graph* $L(G)$ is an undirected simple graph such that: (1) each vertex of $L(G)$ represents an edge of G; (2) two vertices of $L(G)$ are adjacent, i.e, linked by an edge, if and only if their corresponding edges share a common vertex in G. For example, a graph G (left) and its line graph $L(G)$ (right) are shown below.



   (a) (5%) Please draw $L(K_5)$, where K_5 is the complete graph on 5 vertices.

   (b) (5%) Does $L(K_5)$ have a Euler circuit?

2. (20%) The *complete tripartite graph* $K_{l,m,n}$ is defined as follows: There are three disjoint subsets of vertices, X, Y, and Z, with $|X| = l$, $|Y| = m$, and $|Z| = n$. Two vertices are linked by an edge if and only if they lie in different subsets. The graph $K_{1,2,3}$ is illustrated in the accompanying figure.



   (a) (4%) How many edges does $K_{l,m,n}$ have?

   (b) (4%) What is the girth of $K_{l,m,n}$?

(c) (6%) Is $K_{1,1,6}$ planar? Give a planar embedding of the graph or provide an argument that none exists. How about $K_{1,2,3}$?

(d) (6%) For what values of l, m, and n is $K_{l,m,n}$ planar, assuming $l \leq m \leq n$?

3. (a) (6%) What is the maximum number of internal vertices that a complete quaternary (i.e., $m = 4$,) tree of height 8 can have? What is the maximum number of internal vertices for a complete m-ary tree of height h?

(b) (4%) Suppose a certain binary tree's vertices are listed in preorder as A, B, D, E, C, F, G, and in inorder as D, B, E, A, F, G, C. Draw the tree.

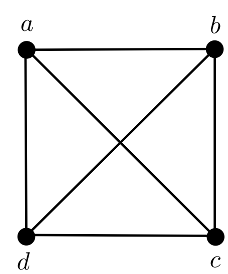
(c) (4%) Represent the following algebraic expression as a binary tree and then write the expression in Polish notation.

$$((A + B) * (C + D)) \div (((A - B) * C) + D).$$

(d) (6%) Apply merge sort to the following list. Draw the splitting and merging trees. Also find the *exact* number of comparisons used in application of the procedure.

77, 23, 82, 47, 65, 17, 97, 85, 35, 91, 61, 73, 12.

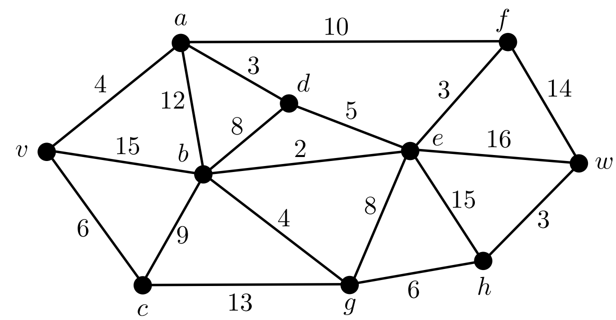
4. (10%) Consider K_4, the complete graph on 4 vertices, shown below.



(a) (4%) Find the depth-first spanning tree rooted at vertex a. (Follow the alphabetical order of vertices in case of a tie.)

(b) (6%) How many nonisomorphic spanning trees are there for K_4? How many nonidentical (though some may be isomorphic) spanning trees are there for K_4?

5. (15%) Consider the weighted simple graph given below.



(a) (5%) Use Dijkstra's algorithm to find a tree of shortest paths from vertex v to all the other vertices.

(b) (5%) Find a minimal spanning tree.

(c) (5%) Find a maximal spanning tree among all the spanning trees that do not include the edge of weight 12.

6. (a) (5%) Consider two partitions of the set $S = \{1, 2, 3, \ldots, 16\}$:

$$S = A_1 \cup A_2 \cup A_3 \cup A_4 = B_1 \cup B_2 \cup B_3 \cup B_4$$

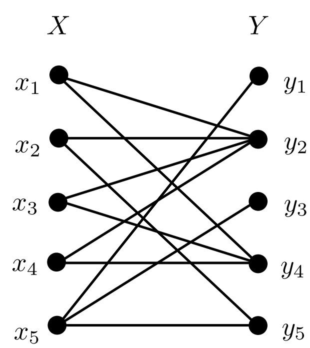
where $A_1 = \{1, 5, 9, 13\}$, $A_2 = \{2, 6, 10, 14\}$, $A_3 = \{3, 7, 11, 15\}$, $A_4 = \{4, 8, 12, 16\}$, and $B_1 = \{1, 2, 3, 5, 7, 11, 13\}$, $B_2 = \{4, 6, 9, 10, 14, 15\}$, $B_3 = \{8, 12\}$, $B_4 = \{16\}$. Is it possible to select four distinct numbers from S such that there is a representative for each A_i, $i = 1, 2, 3, 4$, and a representative for each B_j, $j = 1, 2, 3, 4$? (*Hint:* Construct a bipartite graph $G = (X \cup Y, E)$ with $X = \{a_1, a_2, a_3, a_4\}$ and $Y = \{b_1, b_2, b_3, b_4\}$ such that there is an edge $e \in E$ linking a_i and b_j if $A_i \cap B_j \neq \emptyset$.)

(b) (5%) Suppose that we are given two partitions of a set S:

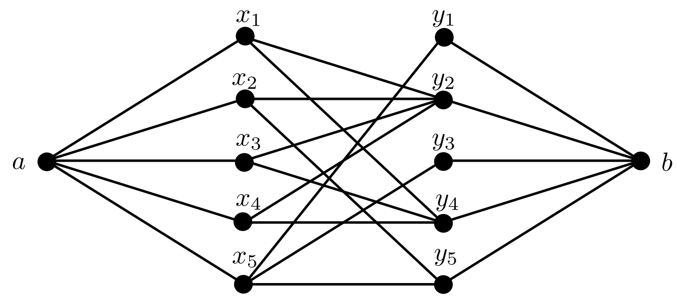
$$S = A_1 \cup A_2 \cup \cdots \cup A_n = B_1 \cup B_2 \cup \cdots \cup B_n.$$

A *simultaneous system of distinct representatives (sSDR)* is a set $\{s_1, s_2, \ldots, s_n\}$ of distinct elements of S such that each subset of either partition contains one s_i. Prove that there is an sSDR if and only if the union of any k subsets A_i's is not contained in the union of fewer than k subsets B_j's, for $k = 1, 2, \ldots, n-1$.

7. (a) (5%) Consider the following bipartite graph $G = (X \cup Y, E)$. Find a maximal matching by using the Hungarian algorithm.



(b) (5%) Change the graph (modified from the bipartite graph in (a)) shown below into a network so that a maximum flow in the network corresponds to a maximal matching in the bipartite graph in (a). (*Hint:* You need to add direction on each edge and then assign appropriate capacity on each edge.)



(c) (5%) Find a maximum flow in the network obtained in (b) by using the Edmonds-Karp algorithm.