Solution to Final Examination

1. (a) The line graph $L(K_5)$ is given in Fig. 1.

(b) Yes, since $L(K_5)$ is connected and every vertex in $L(K_5)$ has degree 6.

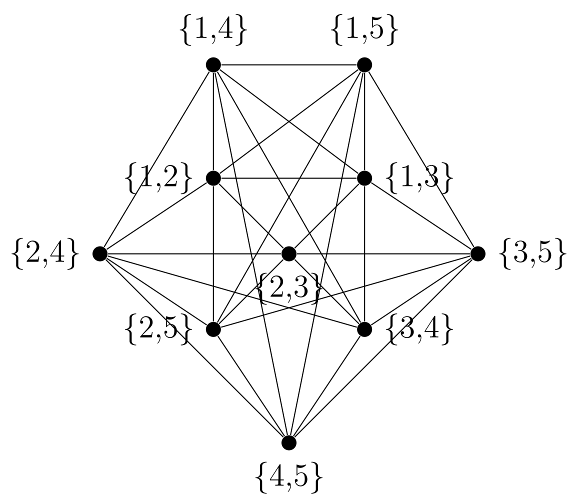

Figure 1: $L(K_5)$ for Problem 1.(a).

2. (a) $lm + ln + mn$.

(b) 3.

(c) Yes, $K_{1,1,6}$ is planar. A planar embedding of $K_{1,1,6}$ is shown in Fig. 2. While $K_{1,2,3}$ contains a subgraph homeomorphic to $K_{3,3}$ as shown in Fig. 3, $K_{1,2,3}$ is nonplanar.

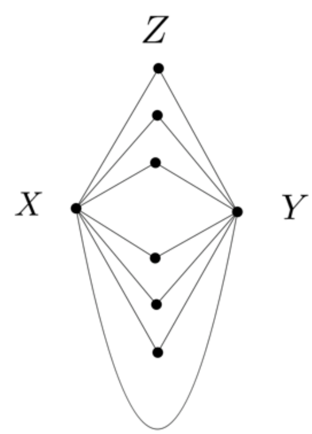

Figure 2: A planar embedding of $K_{1,1,6}$ in Problem 2.(c).

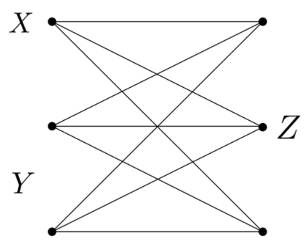

Figure 3: A subgraph of $K_{1,2,3}$ in Problem 2.(c).

(d) For $l = m = 1$ and $n \geq 1$, $K_{1,1,n}$ is planar and a planar embedding is shown in Fig. 4. For $l = 1$ and $m = n = 2$, $K_{1,2,2}$ is planar and a planar embedding is shown in Fig. 5. For $l = m = n = 2$, $K_{2,2,2}$ is planar and a planar embedding is shown in Fig. 6. All other $K_{l,m,n}$'s contain a subgraph homeomorphic to $K_{3,3}$ and are nonplanar.

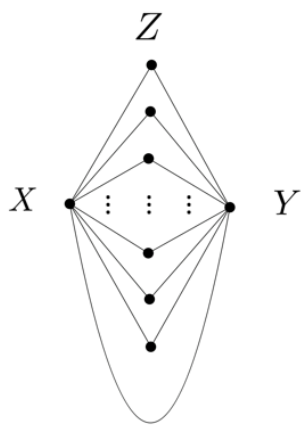

Figure 4: A planar embedding of $K_{1,1,n}$ in Problem 2.(d).

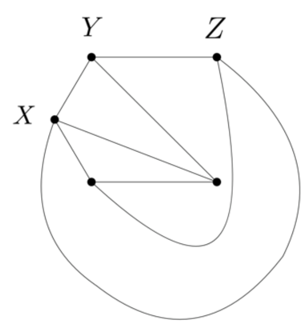

Figure 5: A planar embedding of $K_{1,2,2}$ in Problem 2.(d).

3. (a) The maximum number of internal vertices for a complete quaternary tree of height 8 is $1+4+4^2+\cdots+4^7 = 21845$. The maximum number of internal vertices for a

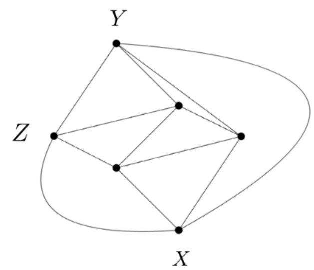

Figure 6: A planar embedding of $K_{2,2,2}$ in Problem 2.(d).

complete m-ary tree of height h is $1 + m + m^2 + \cdots + m^{h-1} = (m^h - 1)/(m-1)$.

(b) The binary tree is shown in Fig. 7.

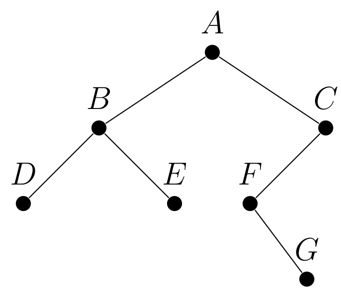

Figure 7: The binary tree for Problem 3.(b).

(c) The corresponding binary tree is shown in Fig. 8. The Polish notation is:

$$\div * + AB + CD + * - ABCD.$$

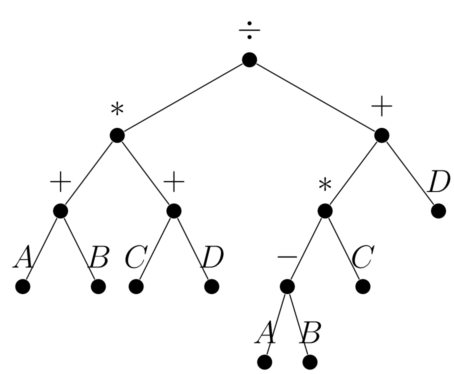

Figure 8: The corresponding binary tree for Problem 3.(c).

(d) The splitting and merging trees are shown in Fig. 9. The exact number of comparisons is 35.

4. (a) The depth-first spanning tree is given in Fig. 10.

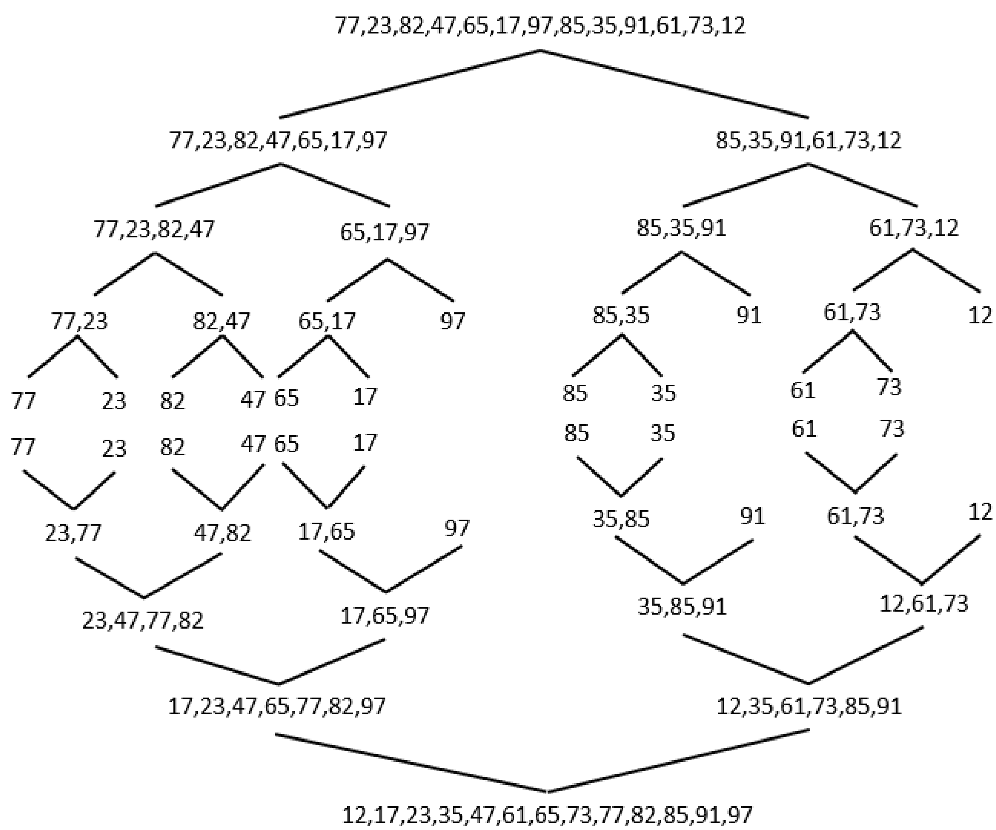

Figure 9: Splitting and merging trees for Problem 3.(d).

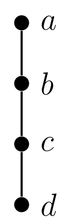

Figure 10: Depth-first spanning tree for Problem 4.(a).

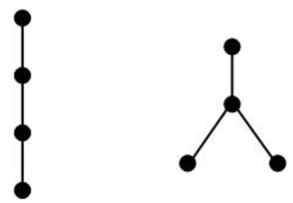

Figure 11: Nonisomorphic spanning trees for Problem 4.(b).

(b) There are two nonisomorphic spanning trees, as shown in Fig. 11. For the first type of isomorphic spanning trees shown on the left of Fig. 11, there are $4!/2 = 12$ nonidentical spanning trees. For the second type of isomorphic spanning trees shown on the right of Fig. 11, there are 4 nonidentical spanning trees. Therefore, there are totally $12 + 4 = 16$ nonidentical spanning trees.

5. (a) A tree of shortest paths from vertex v to all the other vertices is shown in Fig. 12.

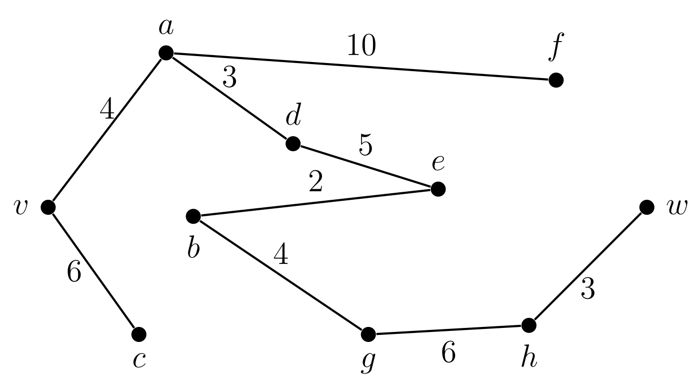

Figure 12: A tree of shortest paths from vertex v to all the other vertices for Problem 5.(a).

(b) A minimal spanning tree is given in Fig. 13.

(c) A desired maximal spanning tree is shown in Fig. 14.

6. (a) Construct the bipartite graph $G = (X \cup Y, E)$ with $X = \{a_1, a_2, a_3, a_4\}$ and $Y = \{b_1, b_2, b_3, b_4\}$ such that there is an edge $e \in E$ linking a_i and b_j if $A_i \cap B_j = \emptyset$, shown in Fig. 15. This problem can be considered as finding a complete matching

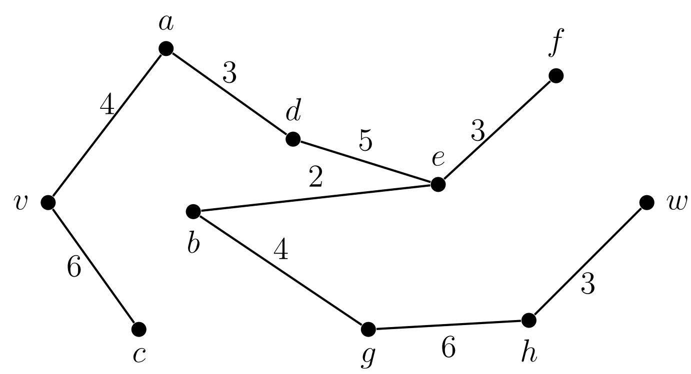

Figure 13: A minimal spanning tree for Problem 5.(b).

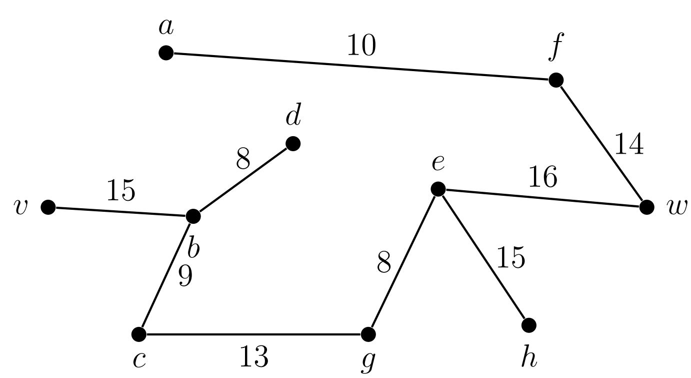

Figure 14: A desired maximal spanning tree for Problem 5.(c).

for the bipartite graph G. If we let $A = \{a_1, a_2, a_3\}$, then $R(A) = \{b_1, b_2\}$. Since $|A| = 3 > 2 = |R(A)|$, by Hall's theorem, a complete matching is not possible for G. Hence, it is not possible to select four distinct numbers from S such that there is a representative for each $A_i, i = 1, 2, 3, 4$, and a representative for each $B_j, j = 1, 2, 3, 4$.

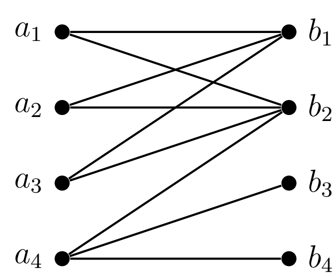

Figure 15: Bipartite graph G for Problem 6.(a).

(b) The bipartite graph $G = (X \cup Y, E)$ is constructed with $X = \{a_1, a_2, \ldots, a_n\}$ and $Y = \{b_1, b_2, \ldots, b_n\}$ such that there is an edge $e \in E$ linking a_i and b_j if $A_i \cap B_j = \emptyset$. Finding an sSDR is equivalent to finding a complete matching for

G. The union of any k subsets A_i's is not contained in the union of fewer than k subsets B_j's, for $k = 1, 2, \ldots, n-1$, if and only if $|R(A)| \geq |A|$, for all $A \subseteq X$. (Note that $A_1 \cup A_2 \cup \cdots \cup A_n = B_1 \cup B_2 \cup \cdots \cup B_n$.) By Hall's theorem, G has a complete matching if and only if $|R(A)| \geq |A|$, for all $A \subseteq X$. Hence, there is an sSDR if and only if the union of any k subsets A_i's is not contained in the union of fewer than k subsets B_j's, for $k = 1, 2, \ldots, n-1$.

7. (a) A maximal matching is shown in Fig. 16.

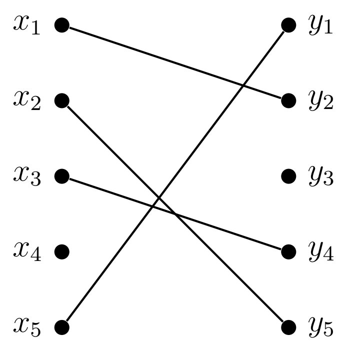

Figure 16: A maximal matching for Problem 7.(a).

(b) We add the direction from left to right on each edge and assign capacity 1 on each edge to obtain the desired network.

(c) A maximum flow f is given by

$$f(a, x_1) = f(a, x_2) = f(a, x_3) = f(a, x_5) = 1$$
$$f(x_1, y_2) = f(x_2, y_5) = f(x_3, y_4) = f(x_5, y_1) = 1$$
$$f(y_1, b) = f(y_2, b) = f(y_4, b) = f(y_5, b) = 1$$

with all other $f(x, y) = 0$.