Solution to Homework Assignment No. 1

1.

p	q	r	$\neg p$	$p \Rightarrow q$	$q \Rightarrow r$	$p \Rightarrow r$	$p \Leftrightarrow q$	$\neg(p \Leftrightarrow q)$	$\neg p \Leftrightarrow q$	$p \wedge q$	$(p \wedge q) \Rightarrow r$
0	0	0	1	1	1	1	1	0	0	0	1
0	0	1	1	1	1	1	1	0	0	0	1
0	1	0	1	1	0	1	0	1	1	0	1
0	1	1	1	1	1	1	0	1	1	0	1
1	0	0	0	0	1	0	0	1	1	0	1
1	0	1	0	0	1	1	0	1	1	0	1
1	1	0	0	1	0	0	1	0	0	1	0
1	1	1	0	1	1	1	1	0	0	1	1

$(p \Rightarrow r) \vee (q \Rightarrow r)$	$q \vee r$	$p \Rightarrow (q \vee r)$	$(p \Rightarrow q) \wedge (p \Rightarrow r)$
1	0	1	1
1	1	1	1
1	1	1	1
1	1	1	1
1	0	0	0
1	1	1	0
0	1	1	0
1	1	1	1

(a) $\neg(p \Leftrightarrow q)$ and $\neg p \Leftrightarrow q$ are logically equivalent.

(b) $(p \wedge q) \Rightarrow r$ and $(p \Rightarrow r) \vee (q \Rightarrow r)$ are logically equivalent.

(c) $p \Rightarrow (q \vee r)$ and $(p \Rightarrow q) \wedge (p \Rightarrow r)$ are not logically equivalent.

2. (a) If $A = \{1,2,3,4,5\}$ and $B = \{4,5,6,7,8\}$, then $A \bigtriangleup B = \{1,2,3,6,7,8\}$.

(b) We have

$$\begin{aligned}
A \bigtriangleup B &= (A \cup B) - (A \cap B) \qquad \text{(by definition)} \\
&= (A \cup B) \cap \big(\overline{A \cap B}\big) \\
&= (A \cup B) \cap \big(\overline{A} \cup \overline{B}\big) \\
&= \big(A \cap (\overline{A} \cup \overline{B})\big) \cup \big(B \cap (\overline{A} \cup \overline{B})\big) \\
&= \big((A \cap \overline{A}) \cup (A \cap \overline{B})\big) \cup \big((B \cap \overline{A}) \cup (B \cap \overline{B})\big) \\
&= \big(A \cap \overline{B}\big) \cup \big(B \cap \overline{A}\big) = (A - B) \cup (B - A).
\end{aligned}$$

3. *Induction basis:* For $n = 1$, we have

$$\begin{aligned}\sum_{i=1}^{1} \frac{F_{i-1}}{2^i} &= \frac{F_0}{2^1} \\ &= 0 \\ &= 1 - \frac{2}{2} \\ &= 1 - \frac{F_{1+2}}{2^1}.\end{aligned}$$

Induction step: Assume that this formula is true for $n = k$, i.e.,

$$\sum_{i=1}^{k} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{k+2}}{2^k}.$$

Then, for $n = k + 1$,

$$\begin{aligned}\sum_{i=1}^{k+1} \frac{F_{i-1}}{2^i} &= \sum_{i=1}^{k} \frac{F_{i-1}}{2^i} + \frac{F_{(k+1)-1}}{2^{k+1}} \\ &= 1 - \frac{F_{k+2}}{2^k} + \frac{F_k}{2^{k+1}} \\ &= 1 - \frac{2F_{k+2} - F_k}{2^{k+1}} \\ &= 1 - \frac{(F_{k+2} - F_k) + F_{k+2}}{2^{k+1}} \\ &= 1 - \frac{F_{k+1} + F_{k+2}}{2^{k+1}} \\ &= 1 - \frac{F_{k+3}}{2^{k+1}} \\ &= 1 - \frac{F_{(k+1)+2}}{2^{k+1}}.\end{aligned}$$

Therefore, by mathematical induction, for all $n \geq 1$,

$$\sum_{i=1}^{n} \frac{F_{i-1}}{2^i} = 1 - \frac{F_{n+2}}{2^n}.$$

4. The induction step fails when $k = 0$. In this case, in the denominator $a^{k-1} = a^{-1}$, and the exponent of a is not a nonnegative integer, which violates the condition of the induction hypothesis that j is a nonnegative integer.

5. (a) It is false. Consider a counterexample shown in Fig. 1, where $g \circ f$ is surjective but f is not.

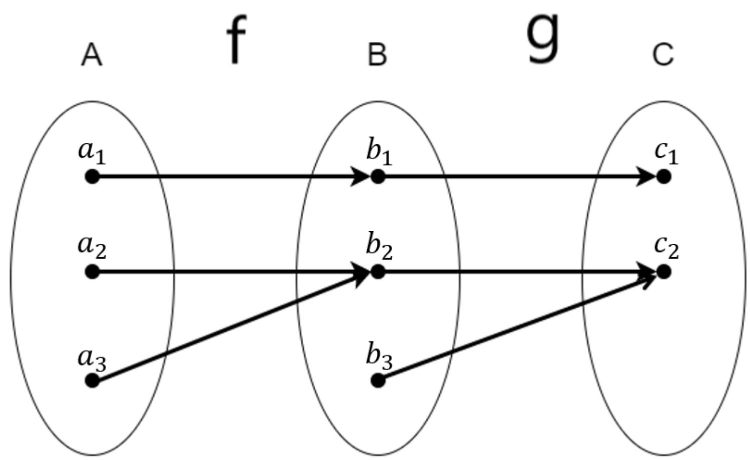

Figure 1: A counterexample for Problem 5.(a).

(b) It is true.
Proof: Suppose $g \circ f$ is surjective. Hence for all $c \in C$, there exists $a \in A$ such that $(g \circ f)(a) = g(f(a)) = c$. Therefore, for all $c \in C$, there exists $b = f(a) \in B$ such that $g(b) = c$, which yields that g is surjective.

6. (a) Suppose f has a left inverse, and then there exists a function $l : B \to A$ such that $(l \circ f)(a) = a$ for all $a \in A$. Hence, for $a_1, a_2 \in A$,

$$\begin{aligned} & f(a_1) = f(a_2) \\ & \Rightarrow l\left(f(a_1)\right) = l\left(f(a_2)\right) \\ & \Rightarrow (l \circ f)(a_1) = (l \circ f)(a_2) \\ & \Rightarrow a_1 = a_2. \end{aligned}$$

Therefore, f is injective.

(b) Let $l' : B \to A$ be a function. Consider $b \in B$. If $b \in f(A)$, since f is injective, there exists a unique $a \in A$ such that $f(a) = b$. In this case, we define $l'(b) = a$. If $b \in B - f(A)$, then we define $l'(b)$ to be any arbitrary element in A. Thus for all $a \in A$, let $b = f(a)$. Then $b \in f(A)$ and $l'(b) = a$. Hence $(l' \circ f)(a) = l'(f(a)) = l'(b) = a$. Therefore, l' is a left inverse of f.

7. Consider the 100 subsets $\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots, \{199, 200\}$ of $S = \{1, 2, \ldots, 200\}$. If 101 different numbers are selected from S, then by the pigeonhole principle, there must be at least one subset whose elements are both selected, which means that at least two chosen numbers are consecutive.

8. Consider the $n+1$ subsets $\{1, 2n+1\}$, $\{2, 2n\}$, $\{3, 2n-1\}$, $\ldots$, $\{n, n+2\}$, $\{n+1\}$ of S. For any subset of size $n+2$, by the pigeonhole principle, it must contain two elements from the same two-element subset whose members sum to $2n+2$.