Solution to Homework Assignment No. 2

1. (a) Since $(a,b) \in R_1$ but $(b,a) \notin R_1$, R_1 is not symmetric. Therefore, R_1 is not an equivalence relation.

(b) We can check that the following conditions hold:

1. Reflexive: $(x,x) \in R_2, \forall x \in A$.
2. Symmetric: $(x,y) \in R_2 \Rightarrow (y,x) \in R_2, \forall x,y \in A$.
3. Transitive: $(x,y) \in R_2$ and $(y,z) \in R_2 \Rightarrow (x,z) \in R_2, \forall x,y,z \in A$.

Therefore, R_2 is an equivalence relation, and the corresponding equivalence classes are $\{a,b,c\}$ and $\{d\}$.

(c) We can check that the following conditions hold:

1. Reflexive: $(x,x) \in R_3, \forall x \in A$.
2. Symmetric: $(x,y) \in R_3 \Rightarrow (y,x) \in R_3, \forall x,y \in A$.
3. Transitive: $(x,y) \in R_3$ and $(y,z) \in R_3 \Rightarrow (x,z) \in R_3, \forall x,y,z \in A$.

Therefore, R_3 is an equivalence relation, and the corresponding equivalence classes are $\{a\}, \{b\}, \{c\}$, and $\{d\}$.

2. (a) We can check that the following conditions hold:

1. Reflexive: $(x,x) \in R_1, \forall x \in A$.
2. Antisymmetric: $(x,y) \in R_1$ and $(y,x) \in R_1 \Rightarrow x = y, \forall x,y \in A$.
3. Transitive: $(x,y) \in R_1$ and $(y,z) \in R_1 \Rightarrow (x,z) \in R_1, \forall x,y,z \in A$.

Therefore, R_1 is a partial order, and the corresponding Hasse diagram is shown in Fig. 1.

(b) Since $(a,b) \in R_2$ and $(b,a) \in R_2$ but $a \neq b$, R_2 is not antisymmetric. Therefore, R_2 is not a partial order.

(c) We can check that the following conditions hold:

1. Reflexive: $(x,x) \in R_3, \forall x \in A$.
2. Antisymmetric: $(x,y) \in R_3$ and $(y,x) \in R_3 \Rightarrow x = y, \forall x,y \in A$.
3. Transitive: $(x,y) \in R_3$ and $(y,z) \in R_3 \Rightarrow (x,z) \in R_3, \forall x,y,z \in A$.

Therefore, R_3 is a partial order, and the corresponding Hasse diagram is shown in Fig. 2.

3. (a) Since $|A \times A| = 4 \cdot 4 = 16$, the number of different relations on A is $2^{16} = 65536$.

(b) Recall that there is a one-to-one correspondence between the set of equivalence relations on A and the set of partitions of A. Equivalently, we compute the number of partitions of A. There are totally 15 different partitions of A:

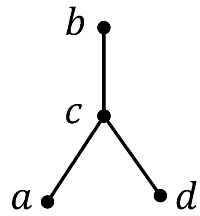

Figure 1: Hasse diagram for R_1.

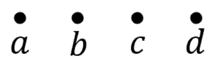

Figure 2: Hasse diagram for R_3.

- 1 partition of this type: $\{b_1\}, \{b_2\}, \{b_3\}, \{b_4\}$
- $\binom{4}{1} = 4$ partitions of this type: $\{b_1\}, \{b_2, b_3, b_4\}$
- $\binom{4}{2}/2 = 3$ partitions of this type: $\{b_1, b_2\}, \{b_3, b_4\}$
- $\binom{4}{2} = 6$ partitions of this type: $\{b_1, b_2\}, \{b_3\}, \{b_4\}$
- 1 partition of this type: $\{b_1, b_2, b_3, b_4\}$

where $b_i \in A$, for $i = 1, 2, 3, 4$, and b_i's are distinct. Therefore, there are 15 equivalence relations on A.

4. The corresponding Hasse diagram is shown in Fig. 3.

(a) a, b, c.

(b) None.

(c) e.

(d) a, b, c, d.

(e) d.

5. (a) The number of ways is $\binom{8}{2,2,2,2} = \frac{8!}{2!2!2!2!} = 2520$.

(b) Substituting the second equation into the first, we obtain

$$x_1 + x_3 + x_5 = 15 - 5 = 10$$
$$x_2 + x_4 + x_6 = 5.$$

The number of nonnegative integer solutions to $x_1 + x_3 + x_5 = 10$ is $\binom{3+10-1}{10} = 66$. The number of nonnegative integer solutions to $x_2 + x_4 + x_6 = 5$ is $\binom{3+5-1}{5} = 21$. Therefore, the total number of nonnegative integer solutions to the pair of equations is $66 \cdot 21 = 1386$.

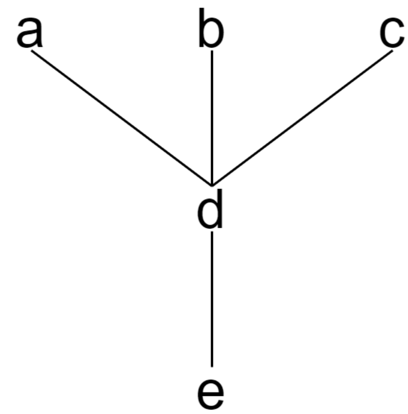

Figure 3: Hasse diagram for Problem 4.

6. (a) By Binomial Theorem, we have

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{n} x^n.$$

Taking derivative on both sides, we obtain

$$\begin{aligned} n(1+x)^{n-1} &= \binom{n}{1} + 2\binom{n}{2} x + \cdots + n\binom{n}{n} x^{n-1} \\ &= \sum_{k=1}^{n} \binom{n}{k} k x^{k-1}. \end{aligned}$$

(b) From (a), let $x = 1$ and we have

$$\begin{aligned} n2^{n-1} &= \sum_{k=1}^{n} \binom{n}{k} k \cdot 1^{k-1} \\ &= \sum_{k=1}^{n} k \binom{n}{k}. \end{aligned}$$

(c) Consider that there are n people. We want to select a committee and select a leader of the committee. One way is to choose the leader first and then select the remaining committee members from the rest $n-1$ people. There are n ways to choose the leader and 2^{n-1} ways for the remaining members. So there are totally $n2^{n-1}$ different ways, which is the result on the left-hand side of the equality. Another way is to select all the members of the committee first and then choose the leader from the selected members. Let there be k members in the committee, for $1 \le k \le n$. Given k, there are $\binom{n}{k}$ ways to choose the committee members and k ways for the leader. Hence the total number of ways is $\sum_{k=1}^{n} k\binom{n}{k}$, which is exactly the result on the right-hand side of the equality.

7. Let $n = \prod_{i=1}^t p_i^{e_i}$, where p_i's are distinct primes and $e_i \geq 1$, $i = 1, 2, \ldots, t$. Then

$$\begin{aligned}\phi(n) &= \phi\left(\prod_{i=1}^t p_i^{e_i}\right) \\ &= \left|\left\{m : 1 \leq m \leq n, \gcd\left(m, \prod_{i=1}^t p_i^{e_i}\right) = 1\right\}\right| \\ &= \left|\{m : 1 \leq m \leq n, p_i \nmid m, \quad \text{for } i = 1, 2, \ldots, t\}\right|.\end{aligned}$$

Let $A_i = \{m : 1 \leq m \leq n, p_i \mid m\}$, for $i = 1, 2, \ldots, t$. We have

$$\begin{aligned}\phi(n) &= \left|\bigcap_{i=1}^t \overline{A_i}\right| \\ &= n - \left|\bigcup_{i=1}^t A_i\right| \\ &= n - \left(\sum_{i=1}^t |A_i| - \sum_{1\leq i<j\leq t} |A_i \cap A_j| + \sum_{1\leq i<j<k\leq t} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{t-1}\left|\bigcap_{i=1}^t A_i\right|\right) \\ &= n - \left(\sum_{i=1}^t \frac{n}{p_i} - \sum_{1\leq i<j\leq t} \frac{n}{p_ip_j} + \sum_{1\leq i<j<k\leq t} \frac{n}{p_ip_jp_k} - \cdots + (-1)^{t-1}\frac{n}{p_1p_2\cdots p_t}\right) \\ &= n\left(1 - \sum_{i=1}^t \frac{1}{p_i} + \sum_{1\leq i<j\leq t} \frac{1}{p_ip_j} - \sum_{1\leq i<j<k\leq t} \frac{1}{p_ip_jp_k} + \cdots + (-1)^t\frac{1}{p_1p_2\cdots p_t}\right) \\ &= n\prod_{i=1}^t \left(1 - \frac{1}{p_i}\right).\end{aligned}$$

8. Note that $|S_n| = n!$. By the principle of inclusion and exclusion, we obtain

$$\begin{aligned}d_n &= |\overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}| \\ &= |S_n| - |A_1 \cup A_2 \cup \cdots \cup A_n| \\ &= n! - \alpha_1 + \alpha_2 + \cdots + (-1)^n\alpha_n\end{aligned}$$

where $\alpha_1 = |A_1| + |A_2| + \cdots + |A_n|$, $\alpha_2 = |A_1 \cap A_2| + |A_1 \cap A_3| + \cdots + |A_{n-1} \cap A_n|$, $\ldots$, $\alpha_n = |A_1 \cap A_2 \cap \cdots \cap A_n|$. We have

$$\begin{gathered}|A_i| = (n-1)!, \text{ for } 1 \leq i \leq n \\ |A_i \cap A_j| = (n-2)!, \text{ for } 1 \leq i < j \leq n \\ \vdots \\ |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_r}| = (n-r)!, \text{ for } 1 \leq i_1 < i_2 < \cdots < i_r \leq n,\ 1 \leq r \leq n.\end{gathered}$$

Therefore,

$$\begin{aligned} d_n &= n! - \sum_{i=1}^{n} \binom{n}{i} (-1)^{i-1} (n-i)! \\ &= n! - \sum_{i=1}^{n} \frac{n!}{i!} (-1)^{i-1} \\ &= n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}. \end{aligned}$$