Solution to Homework Assignment No. 3

1. The corresponding characteristic equation is

$$\begin{aligned} & r^2+4r+8=0 \\ \Rightarrow\ & r=-2+2j,-2-2j \\ \Rightarrow\ & r=2\sqrt{2}e^{j(3\pi/4)},2\sqrt{2}e^{-j(3\pi/4)}. \end{aligned}$$

Hence the general solution is

$$a_n=\beta_1 2^{3n/2}\cos(3n\pi/4)+\beta_2 2^{3n/2}\sin(3n\pi/4).$$

For initial conditions,

$$\begin{aligned} & 0=a_0=\beta_1 \\ & 2=a_1=2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\beta_1+\frac{1}{\sqrt{2}}\beta_2\right) \\ \Rightarrow\ & \beta_1=0,\beta_2=1. \end{aligned}$$

Therefore, $a_n=2^{3n/2}\sin(3n\pi/4)$, for $n\geq 0$.

2. The characteristic equation of the associated homogeneous recurrence relation is

$$\begin{aligned} & r^2-6r+9=0 \\ \Rightarrow\ & (r-3)^2=0 \\ \Rightarrow\ & r=3,3. \end{aligned}$$

Hence the general solution to the associated homogeneous recurrence relation is

$$a_n=\alpha_1 3^n+\alpha_2 n3^n.$$

Let the trial sequence for a particular solution to the nonhomogeneous recurrence relation be $p_n=B_0 2^n+B_1 n^2 3^n$. Then

$$\begin{aligned} & \left[B_0 2^{n+2}+B_1(n+2)^2 3^{n+2}\right]-6\left[B_0 2^{n+1}+B_1(n+1)^2 3^{n+1}\right]+9(B_0 2^n+B_1 n^2 3^n) \\ & =3\cdot 2^n+7\cdot 3^n \\ \Rightarrow\ & (4-12+9)B_0 2^n+(9(n+2)^2-18(n+1)^2+9n^2)B_1 3^n=3\cdot 2^n+7\cdot 3^n \\ \Rightarrow\ & B_0 2^n+18B_1 3^n=3\cdot 2^n+7\cdot 3^n \\ \Rightarrow\ & B_0=3,B_1=\frac{7}{18}. \end{aligned}$$

Therefore, $p_n=3\cdot 2^n+(7/18)n^2 3^n$ is a particular solution to the nonhomogeneous recurrence relation. Hence the general solution to the nonhomogeneous recurrence relation is

$$a_n=\alpha_1 3^n+\alpha_2 n3^n+3\cdot 2^n+\frac{7}{18}n^2 3^n.$$

For initial conditions,

$$\begin{aligned}
& 1 = a_0 = \alpha_1 + 3 \\
& 4 = a_1 = 3\alpha_1 + 3\alpha_2 + 6 + \frac{7}{6} \\
\Rightarrow\ & \alpha_1 = -2, \alpha_2 = \frac{17}{18}.
\end{aligned}$$

Therefore, $a_n = -2 \cdot 3^n + (17/18)n3^n + 3 \cdot 2^n + (7/18)n^2 3^n$, for $n \geq 0$.

3. Observing $a_1 = 1^3$, $a_2 = 1^3 + 2^3$, $a_3 = 1^3 + 2^3 + 3^3$, ..., we have the recurrence relation

$$a_{n+1} - a_n = (n+1)^3, \text{ for } n \geq 1 \text{ with } a_1 = 1.$$

The associated homogeneous recurrence relation (HRR) is $a_{n+1} - a_n = 0$, which gives the characteristic equation

$$r - 1 = 0 \Rightarrow r = 1.$$

So the general solution to the associated HRR is $a_n = \alpha_1$. Let the trial sequence to the nonhomogeneous recurrence relation (NRR) be $B_4n^4 + B_3n^3 + B_2n^2 + B_1n$. Then

$$\begin{aligned}
& B_4(n+1)^4 + B_3(n+1)^3 + B_2(n+1)^2 + B_1(n+1) - (B_4n^4 + B_3n^3 + B_2n^2 + B_1n) \\
& = (n+1)^3 \\
\Rightarrow\ & B_4(4n^3 + 6n^2 + 4n + 1) + B_3(3n^2 + 3n + 1) + B_2(2n + 1) + B_1 = (n+1)^3 \\
\Rightarrow\ & B_4 = 1/4, B_3 = 1/2, B_2 = 1/4, B_1 = 0.
\end{aligned}$$

The general solution to the NRR is $a_n = (1/4)n^4 + (1/2)n^3 + (1/4)n^2 + \alpha_1$. For the initial condition, we have

$$\begin{aligned}
& a_1 = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} + \alpha_1 = 1 \\
\Rightarrow\ & \alpha_1 = 0.
\end{aligned}$$

Therefore,

$$a_n = (1/4)n^4 + (1/2)n^3 + (1/4)n^2 = \left(\frac{n(n+1)}{2}\right)^2, \text{ for } n \geq 1.$$

4. The recurrence relation for this problem is

$$a_{n+1} - (1 + r/12)a_n = -D$$

with $a_0 = C$ and $a_{12N} = 0$. The associated HRR is $a_{n+1} - (1 + r/12)a_n = 0$ which gives the characteristic equation $\lambda - (1 + r/12) = 0$ with root $1 + r/12$. Hence the general solution to the associated HRR is $a_n = \alpha \cdot (1 + r/12)^n$. Try a particular solution to the NRR as A. Then $A - (1 + r/12) \cdot A = -D$, which gives $A = 12D/r$. Thus the general solution to the NRR is $a_n = \alpha(1+r/12)^n + 12D/r$. For conditions $a_0 = C$ and $a_{12N} = 0$, we have

$$a_0 = \alpha + \frac{12D}{r} = C$$
$$a_{12N} = \alpha\left(1+\frac{r}{12}\right)^{12N} + \frac{12D}{r} = 0$$

which gives

$$\left(C - \frac{12}{r}D\right)\left(1+\frac{r}{12}\right)^{12N} + \frac{12}{r}D = 0.$$

Therefore, we obtain

$$D = C \cdot \frac{(r/12)(1+r/12)^{12N}}{(1+r/12)^{12N}-1}.$$

5. Let the generating function for a_n be $A(x)$. Taking generating functions on both sides of the recurrence relation, we have

$$\sum_{n\geq 2} a_n x^n - \sum_{n\geq 2} a_{n-1}x^n - 2\cdot\sum_{n\geq 2} a_{n-2}x^n = \sum_{n\geq 2} 2^n x^n$$

which yields

$$(A(x) - a_1 x - a_0) - x(A(x) - a_0) - 2x^2 A(x) = \frac{1}{1-2x} - 1 - 2x.$$

We thus obtain

$$A(x) - 12x - 4 - xA(x) + 4x - 2x^2 A(x) = \frac{4x^2}{1-2x}$$
$$\Rightarrow (1 - x - 2x^2)A(x) = \frac{4x^2}{1-2x} + 8x + 4$$
$$\Rightarrow A(x) = \frac{4-12x^2}{(1+x)(1-2x)^2} = \frac{2/3}{(1-2x)^2} + \frac{38/9}{1-2x} + \frac{-8/9}{1+x}.$$

Therefore,

$$a_n = \frac{44}{9}2^n - \frac{8}{9}(-1)^n + \frac{2}{3}n2^n, \quad \text{for } n \geq 0.$$

6. Let the generating function for F_n and L_n be $F(x)$ and $L(x)$, respectively. Taking generating functions on both sides of the relation, we have

$$\sum_{n\geq 1} L_n x^n = \sum_{n\geq 1} F_{n+1}x^n + \sum_{n\geq 1} F_{n-1}x^n$$

which yields

$$L(x) - L_0 = \frac{F(x) - F_1 x - F_0}{x} + xF(x).$$

Recalling that

$$F(x) = \frac{x}{1 - x - x^2}$$

we thus have

$$\begin{aligned} L(x) &= \frac{(1 + x^2)F(x)}{x} + 1 \\ &= \frac{2 - x}{1 - x - x^2}. \end{aligned}$$

7. (a) Let the generating functions for a_n and b_n be $A(x)$ and $B(x)$, respectively. We have

$$\begin{aligned} A(x) - a_0 &= -2xA(x) - 4xB(x) \\ B(x) - b_0 &= 4xA(x) + 6xB(x) \end{aligned}$$

which yields

$$\begin{aligned} (1 + 2x)A(x) + 4xB(x) &= 1 \\ -4xA(x) + (1 - 6x)B(x) &= 0. \end{aligned}$$

We obtain

$$A(x) = \frac{1 - 6x}{1 - 4x + 4x^2}.$$

Then

$$A(x) = \frac{3}{1 - 2x} + \frac{-2}{(1 - 2x)^2}.$$

Hence, for $n \geq 0$,

$$a_n = 3 \cdot 2^n - 2(n + 1)2^n = 2^n - n2^{n+1}.$$

(b) From (a), we obtain

$$B(x) = \frac{4x}{1 - 4x + 4x^2}.$$

Then

$$B(x) = \frac{-2}{1 - 2x} + \frac{2}{(1 - 2x)^2}.$$

Hence, for $n \geq 0$,

$$b_n = -2 \cdot 2^n + 2(n + 1)2^n = n2^{n+1}.$$

8. (a) From Problem 7(a), we have

$$(1 - 4x + 4x^2)A(x) = 1 - 6x$$

yielding

$$a_0 = 1$$
$$a_1 - 4a_0 = -6$$
$$a_n - 4a_{n-1} + 4a_{n-2} = 0, \text{ for } n \geq 2.$$

Therefore, the recurrence relation that a_n satisfies is

$$a_n - 4a_{n-1} + 4a_{n-2} = 0, \text{ for } n \geq 2$$

with $a_0 = 1$ and $a_1 = -2$.

(b) From Problem 7(b), we have

$$(1 - 4x + 4x^2)B(x) = 4x$$

yielding

$$b_0 = 0$$
$$b_1 - 4b_0 = 4$$
$$b_n - 4b_{n-1} + 4b_{n-2} = 0, \text{ for } n \geq 2.$$

Therefore, the recurrence relation that b_n satisfies is

$$b_n - 4b_{n-1} + 4b_{n-2} = 0, \text{ for } n \geq 2$$

with $b_0 = 0$ and $b_1 = 4$.