Solution to Homework Assignment No. 4

1. (a) Let $A(x)$ be the generating function for a_n. Since the possible outcomes of rolling a die are $1, 2, \ldots, 6$ and four dice are rolled, we have

$$A(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^4.$$

(b) Let $B(x)$ be the generating function for b_n. Since the possible even outcomes of a die are $2, 4, 6$ and the possible odd outcomes of a die are $1, 3, 5$, we have

$$B(x) = (x + x^3 + x^5)^2(x^2 + x^4 + x^6)^2.$$

2. (a) The generating function for $p(n \mid \text{each part appears an even number of times})$ is given by

$$(1 + x^2 + x^4 + \cdots)(1 + x^4 + x^8 + \cdots)(1 + x^6 + x^{12} + \cdots)\cdots = \prod_{i=1}^{\infty} \frac{1}{1 - x^{2i}}.$$

(b) The generating function for $p(n \mid \text{each part is even})$ is given by

$$(1 + x^2 + x^4 + \cdots)(1 + x^4 + x^8 + \cdots)(1 + x^6 + x^{12} + \cdots)\cdots = \prod_{i=1}^{\infty} \frac{1}{1 - x^{2i}}.$$

3. Let $P_a(x)$ be the generating function for the number of partitions of n where no part is divisible by 4 and $P_b(x)$ be the generating function for the number of partitions of n where no no even part is repeated. We have

$$\begin{aligned}
P_a(x) &= (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^5 + x^{10} + \cdots)\cdots \\
&= \frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^5)(1-x^6)(1-x^7)(1-x^9)\cdots} \\
&= \frac{(1-x^4)(1-x^8)(1-x^{12})\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)(1-x^8)(1-x^9)\cdots} \\
&= \frac{(1+x^2)(1-x^2)(1+x^4)(1-x^4)(1+x^6)(1-x^6)\cdots}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)(1-x^8)(1-x^9)\cdots} \\
&= \frac{(1+x^2)(1+x^4)(1+x^6)\cdots}{(1-x)(1-x^3)(1-x^5)(1-x^7)\cdots} \\
&= (1 + x + x^2 + \cdots)(1 + x^2)(1 + x^3 + x^6 + \cdots)(1 + x^4)(1 + x^5 + x^{10} + \cdots)(1 + x^6)\cdots \\
&= P_b(x).
\end{aligned}$$

Therefore, the numbers of partitions of the two kinds are equal.

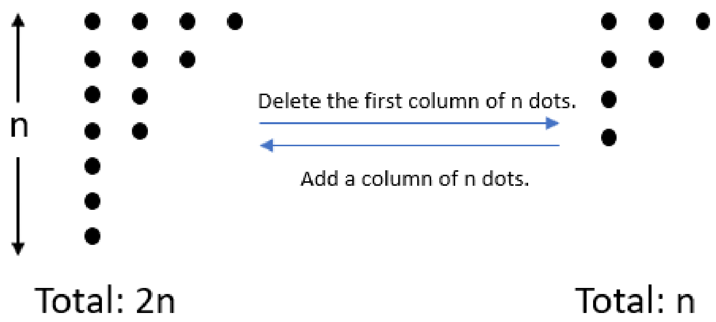

Figure 1: Ferrers graphs for Problem 4.

4. The corresponding Ferrers graphs are shown in Fig. 1. In the Ferrers graph for a partition of $2n$ in which there are n parts, if the first column of n dots is removed, we get the Ferrers graph for a partition of n. Conversely, if a column of n dots is added to the Ferrers graph for a partition of n, we obtain the Ferrers graph for a partition of $2n$ in which there are n parts. Therefore, there is a one-to-one correspondence between the sets of partitions of the two kinds, so they have the same cardinality.

5. There is only one loop in the procedure, and in the loop we have one addition and one multiplication for each iteration. Hence there are totally n additions and n multiplications, which are both $O(n)$.

6. (a) Yes, the two graphs are isomorphic. An isomorphism f is given as $f(u_1) = v_5, f(u_2) = v_2, f(u_3) = v_3, f(u_4) = v_6, f(u_5) = v_4, f(u_6) = v_1$.

(b) No, the two graphs are not isomorphic. Note that there is a vertex v_6 with degree 4 in the graph on the right-hand side but there are no such vertices in the graph on the left-hand side.

7. (a) Since each edge contributes exactly one to the in degree of its terminal vertex and exactly one to the out degree of its initial vertex, we therefore have

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

(b) The adjacency matrix for this graph is

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

where the rows (from top to bottom) and the columns (from left to right) are indexed in the order a, b, c, d, e. The in degree of a vertex is the corresponding column sum of the adjacency matrix, and thus

$$\deg^-(a) = 2, \deg^-(b) = 2, \deg^-(c) = 3, \deg^-(d) = 2, \deg^-(e) = 2.$$

The out degree of a vertex is the corresponding row sum of the adjacency matrix, and thus

$$\deg^+(a) = 3, \deg^+(b) = 1, \deg^+(c) = 2, \deg^+(d) = 2, \deg^+(e) = 3.$$

Since there are 11 edges, we have

$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = 11 = |E|.$$

8. (a) The degrees of the vertices of $\overline{G}$ are $n-1-d_1, n-1-d_2, \ldots, n-1-d_n$.

(b) The number of edges for the complete graph K_n with n vertices is $(1/2)n(n-1)$. Since the sum of the numbers of edges of G and $\overline{G}$ is equal to the number of edges of K_n, we have

$$9 + 6 = 15 = (1/2)n(n-1).$$

Therefore, $n = 6$.