Solution to Homework Assignment No. 6

1. (a) The shortest path from b to g is b, c, d, h, g.

(b) A tree of shortest paths from vertex a to all the other vertices is given in Fig. 1.

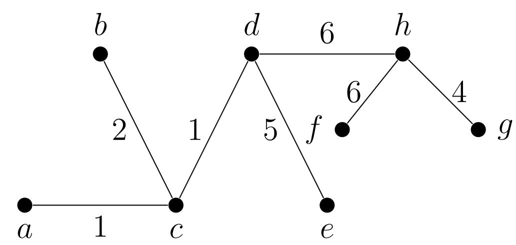

Figure 1: A tree of shortest paths from vertex a for Problem 1.(b).

2. A counterexample is given in Fig. 2, where the edge with minimum weight is $\{v_1, v_2\}$, which is not included in the shortest path from v_0 to v_1 or v_2.

Figure 2: A counterexample for Problem 2.

3. A minimal spanning tree is shown in Fig. 3.

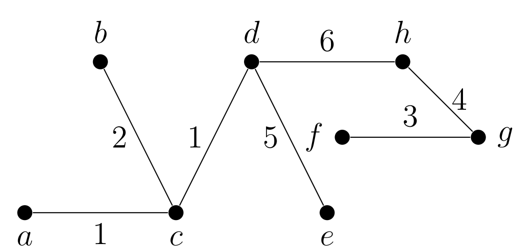

Figure 3: A minimal spanning tree for Problem 3.

4. (a) **procedure** MODIFIED KRUSKAL

$T := \varnothing$

for $i = 1$ to $|V| - 1$ **do**
$e :=$ an edge of maximum weight not in T that does not form a cycle when added to T
add e to T
end for
end procedure

(b) A maximal spanning tree is shown in Fig. 4.

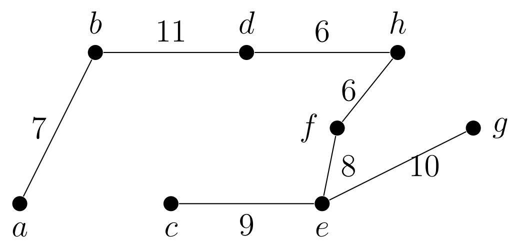

Figure 4: A maximal spanning tree for Problem 4.(b).

5. Consider a bipartite graph $G = (X \cup Y, E)$ where X is the set of girls and Y is the set of boys. There is an edge in E linking $x \in X$ and $y \in Y$ if girl x likes boy y. For any $A \subseteq X$, the number of edges incident to vertices in A is $4 \cdot |A|$, while the number of edges incident to vertices in $R(A)$ is $4 \cdot |R(A)|$. Since every edge incident to a vertex in A must be an edge incident to a vertex in $R(A)$, we have $4 \cdot |A| \leq 4 \cdot |R(A)|$, implying $|A| \leq |R(A)|$. By Hall's theorem, there exists a complete matching.

6. (a) This problem can be considered as finding a complete matching for the bipartite graph shown in Fig. 5. A complete matching is found and shown in Fig. 6. Hence, $(6, 5, 2, 4, 1)$ is an SDR.

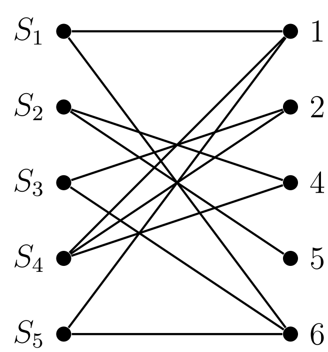

Figure 5: A bipartite graph for Problem 6.(a).

(b) Since the 5 sets $\{a, m\}, \{a, r, e\}, \{m, a, e\}, \{m, e\}, \{r, a, m\}$ include only the 4 elements a, e, m, r, by Hall's Theorem, a complete matching is not possible for the associated bipartite graph. Hence, no SDR exists.

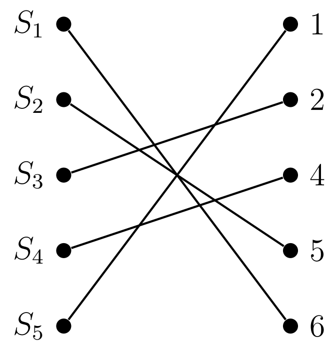

Figure 6: A complete matching for Problem 6.(a).

S	$\{s\}$	$\{s,a\}$	$\{s,b\}$	$\{s,c\}$	$\{s,a,b\}$	$\{s,a,c\}$	$\{s,b,c\}$	$\{s,a,b,c\}$
T	$\{a,b,c,t\}$	$\{b,c,t\}$	$\{a,c,t\}$	$\{a,b,t\}$	$\{c,t\}$	$\{b,t\}$	$\{a,t\}$	$\{t\}$
cap(S,T)	7	9	8	11	7	12	9	7

Table 1: All the cuts and corresponding capacities for Problem 7.(a).

7. (a) All the cuts (S,T) and corresponding capacities cap(S,T) are shown in Table 1.

(b) The maximum value of a flow from s to t is equal to the minimum capacity of a cut separating s and t. Therefore, the maximum value of a flow from s to t is 7.

8. (a) The value of the flow f is the outflow of the source s. Therefore, the value of the flow f is 6.

(b) A maximum flow f_1 for this network is given by

$$f_1(s,a) = f_1(f,i) = f_1(i,t) = 6$$
$$f_1(s,b) = f_1(s,c) = f_1(c,f) = 4$$
$$f_1(b,a) = f_1(b,f) = 2$$
$$f_1(a,d) = f_1(d,g) = f_1(g,t) = 8$$

with all other $f_1(x,y) = 0$. Hence its value is $f_1(s,a) + f_1(s,b) + f_1(s,c) = 14$.

(c) A minimum cut (S,T) for this network is given by

$$S = \{s,a,b,c,d,e,f,g,h\}$$
$$T = \{i,t\}.$$