EECS 2060 Discrete Mathematics — Chi-chao Chao
Spring 2021

Midterm Examination No. 2
7:00pm to 10:00pm, May 14, 2021

Problems for Solution:

1. (10%) Solve the recurrence relation

$$a_n - 4a_{n-1} + 3a_{n-2} = 2^n + n + 3, \qquad n \geq 2$$

with $a_0 = 1$ and $a_1 = 4$.

2. (a) (5%) Show that $A(x) = x(1+x)/(1-x)^3$ is the generating function for the sequence $a_n = n^2$, $n \geq 0$.

   (b) (5%) Define

$$s_n = a_0 + a_1 + \cdots + a_n, \quad \text{for } n \geq 0.$$

   Show that the generating function $S(x)$ for s_n is given by

$$S(x) = \frac{A(x)}{1-x}$$

   where $A(x)$ is the generating function for a_n ($n \geq 0$).

   (c) (10%) Let

$$s_n = \sum_{i=0}^{n} i^2, \quad \text{for } n \geq 0.$$

   First use the results in (a) and (b) to find the generating function $S(x)$ for s_n. Then find a formula for s_n.

3. (15%) Consider the following system of homogeneous recurrence relations:

$$\begin{aligned} a_n &= 2a_{n-1} + 6b_{n-1} - 3c_{n-1} \\ b_n &= 4b_{n-1} - c_{n-1} \\ c_n &= 2c_{n-1} \end{aligned}$$

for $n \geq 1$, with $a_0 = 0$, $b_0 = 0$, and $c_0 = 1$.

   (a) (5%) Find the generating function $A(x)$ for a_n.

   (b) (5%) Find the homogeneous recurrence relation that a_n satisfies (with appropriate initial conditions).

   (c) (5%) Solve for a_n.

4. (10%) Let $a_0 = 1$ and a_n be the number of unordered selections in which n letters are selected from the alphabet $\{\alpha, \beta, \gamma\}$, repetitions allowed, with the constraint that the letter α must be selected an even number of times.

   (a) (2%) Find a_1 and a_2.

   (b) (4%) Find the generating function $A(x)$ for a_n.

   (c) (4%) From (b) find an explicit expression for a_n.

5. (15%) In this problem we consider partitions of a nonnegative integer n.

   (a) (5%) Find the generating function for $p(n \,|\, \text{only even parts can occur more than once})$.

   (b) (5%) Find the generating function for $p(n \,|\, \text{each part is a multiple of 3})$.

   (c) (5%) Show that the number of partitions of n in which each part is 1 or 2 is equal to the number of partitions of $n+3$ which have exactly two distinct parts.

6. (10%) In each of the following pseudocode program segments, the complexity function $f(n)$ is defined to be the number of times the statement $sum := sum + 1$ is executed. Determine which one has the best (least) big-O form for $f(n)$. (Note that $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.)

   (a)
```
begin
sum := 0
for i = 1 to n do
    for j = 1 to n do
        sum := sum + 1
end
```

   (b)
```
begin
sum := 0
i := n
while i > 0 do
    begin
    sum := sum + 1
    i := ⌊i/2⌋
    end
end
```

   (c)
```
begin
sum := 0
for i = 1 to n do
    begin
    j := n
    while j > 0 do
        begin
        sum := sum + 1
```

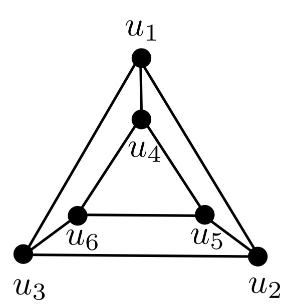
$j := \lfloor j/2 \rfloor$
end
end
end

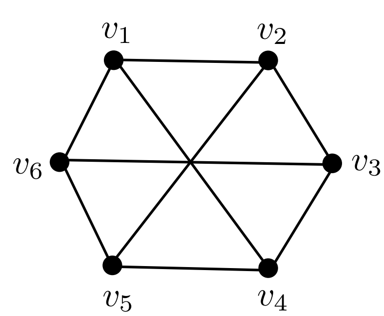
7. (a) (5%) Consider an undirected graph with vertex set $\{a, b, c, d, e, f, g, h, i, j\}$ and adjacency matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where the rows (from top to bottom) and the columns (for left to right) are indexed in the order $a, b, c, d, e, f, g, h, i, j$. Find the number of components of this graph.

(b) (6%) Determine whether the following pair of graphs is isomorphic. Exhibit an isomorphism or provide an argument that none exists.





(c) (9%) Is it possible that the following lists are the degrees of all the vertices of a simple graph with 5 vertices? If so, draw a corresponding graph. Otherwise, explain why it is not possible.

i. 3, 3, 3, 3, 4.
ii. 2, 2, 2, 2, 3.
iii. 0, 1, 2, 3, 4.