Solution to Midterm Examination No. 2

1. The characteristic equation of the associated homogeneous recurrence relation is

$$\begin{aligned} & r^2 - 4r + 3 = 0 \\ \Rightarrow \quad & r = 1, 3. \end{aligned}$$

Hence the general solution is

$$a_n = \alpha_1 1^n + \alpha_2 3^n = \alpha_1 + \alpha_2 3^n.$$

Let the trial sequence for a particular solution to the nonhomogeneous recurrence relation be $p_n = B_2 2^n + B_1 n^2 + B_0 n$. Then

$$\begin{aligned} & \left[B_2 2^n + B_1 n^2 + B_0 n\right] - 4\left[B_2 2^{n-1} + B_1(n-1)^2 + B_0(n-1)\right] \\ & +3\left[B_2 2^{n-2} + B_1(n-2)^2 + B_0(n-2)\right] = 2^n + n + 3 \\ \Rightarrow \quad & \left(1 - 2 + \frac{3}{4}\right) B_2 2^n + \left[(8-12)B_1 + (1-4+3)B_0\right] n + \left[(-4+12)B_1 + (4-6)B_0\right] \\ & = 2^n + n + 3 \\ \Rightarrow \quad & -\frac{1}{4} B_2 2^n - 4B_1 n + (8B_1 - 2B_0) = 2^n + n + 3 \\ \Rightarrow \quad & B_2 = -4, B_1 = -\frac{1}{4}, B_0 = -\frac{5}{2}. \end{aligned}$$

Therefore, $p_n = -4 \cdot 2^n - (1/4)n^2 - (5/2)n$ is a particular solution to the nonhomogeneous recurrence relation. Hence the general solution to the nonhomogeneous recurrence relation is

$$a_n = \alpha_1 + \alpha_2 3^n - 4 \cdot 2^n - \frac{1}{4}n^2 - \frac{5}{2}n.$$

For initial conditions,

$$\begin{aligned} & 1 = a_0 = \alpha_1 + \alpha_2 - 4 \\ & 4 = a_1 = \alpha_1 + 3\alpha_2 - 8 - \frac{1}{4} - \frac{5}{2} \\ \Rightarrow \quad & \alpha_1 = \frac{1}{8}, \alpha_2 = \frac{39}{8}. \end{aligned}$$

Therefore, $a_n = (1/8) + (39/8) \cdot 3^n - 4 \cdot 2^n - (1/4)n^2 - (5/2)n$, for $n \geq 0$.

2. (a) We have

$$\begin{aligned} A(x) = \frac{x(1+x)}{(1-x)^3} &= (x + x^2)(1-x)^{-3} = (x + x^2) \sum_{n \geq 0} \binom{-3}{n} (-x)^n \\ &= (x + x^2) \sum_{n \geq 0} \binom{n+2}{2} x^n. \end{aligned}$$

Hence the coefficient of x^n in $A(x)$ is

$$\begin{aligned} a_n &= \binom{n-1+2}{2} + \binom{n-2+2}{2} = \binom{n+1}{2} + \binom{n}{2} \\ &= \frac{(n+1)n}{2} + \frac{n(n-1)}{2} = n^2, \quad \text{for } n \geq 0. \end{aligned}$$

(b) We have

$$\begin{aligned} S(x) = \sum_{n\geq 0} s_n x^n &= \sum_{n\geq 0} \left(\sum_{k=0}^{n} a_k \right) x^n \\ &= \sum_{n\geq 0} \sum_{k=0}^{n} a_k x^n \\ &= \sum_{k\geq 0} \sum_{n\geq k} a_k x^n \\ &= \sum_{k\geq 0} a_k x^k \sum_{n\geq k} x^{n-k} \\ &= \sum_{k\geq 0} a_k x^k \sum_{k'\geq 0} x^{k'} \quad \text{(by letting } k' = n-k\text{)} \\ &= \frac{A(x)}{1-x}. \end{aligned}$$

(c) From (a), we have

$$A(x) = \frac{x(1+x)}{(1-x)^3}.$$

From (b), we have

$$S(x) = \frac{A(x)}{1-x}.$$

We then obtain

$$S(x) = \frac{x(1+x)}{(1-x)^4}.$$

We have

$$\begin{aligned} S(x) &= \frac{x(1+x)}{(1-x)^4} = (x+x^2)(1-x)^{-4} = (x+x^2) \sum_{n\geq 0} \binom{-4}{n} (-x)^n \\ &= (x+x^2) \sum_{n\geq 0} \binom{n+3}{3} x^n. \end{aligned}$$

Hence the coefficient of x^n in $S(x)$ is

$$\begin{aligned} s_n &= \binom{n-1+3}{3} + \binom{n-2+3}{3} = \binom{n+2}{3} + \binom{n+1}{3} \\ &= \frac{(n+2)(n+1)n}{3!} + \frac{(n+1)n(n-1)}{3!} = \frac{n(n+1)(2n+1)}{6}, \quad \text{for } n \geq 0. \end{aligned}$$

3. (a) Let the generating functions for a_n, b_n, and c_n be $A(x)$, $B(x)$, and $C(x)$, respectively. We have

$$\begin{aligned} A(x) - a_0 &= 2xA(x) + 6xB(x) - 3xC(x) \\ B(x) - b_0 &= 4xB(x) - xC(x) \\ C(x) - c_0 &= 2xC(x) \end{aligned}$$

which yield

$$\begin{aligned} &(1-2x)A(x) - 6xB(x) + 3xC(x) = 0 \\ &(1-4x)B(x) + xC(x) = 0 \\ &(1-2x)C(x) = 1. \end{aligned}$$

Therefore,

$$\begin{aligned} C(x) &= \frac{1}{1-2x} \\ B(x) &= \frac{-x}{(1-2x)(1-4x)} \end{aligned}$$

and

$$A(x) = \frac{-3x}{(1-2x)(1-4x)}.$$

(b) From (a), we have

$$(1 - 6x + 8x^2)A(x) = -3x$$

yielding

$$\begin{aligned} a_0 &= 0 \\ a_1 - 6a_0 &= -3 \\ a_n - 6a_{n-1} + 8a_{n-2} &= 0, \text{ for } n \geq 2. \end{aligned}$$

Therefore, the homogeneous recurrence relation that a_n satisfies is

$$a_n - 6a_{n-1} + 8a_{n-2} = 0, \text{ for } n \geq 2$$

with $a_0 = 0$ and $a_1 = -3$.

(c) From (a), we have

$$A(x) = \frac{-3x}{(1-2x)(1-4x)} = \frac{3/2}{(1-2x)} - \frac{3/2}{(1-4x)}.$$

Hence, for $n \geq 0$

$$a_n = \frac{3}{2}(2^n - 4^n).$$

4. (a) $a_1 = 2$ and $a_2 = 4$.

(b) We have

$$A(x) = \underbrace{(1 + x^2 + x^4 + ...)}_{\alpha}\underbrace{(1 + x + x^2 + ...)}_{\beta}\underbrace{(1 + x + x^2 + ...)}_{\gamma}$$
$$= \frac{1}{(1 - x^2)}\frac{1}{(1 - x)}\frac{1}{(1 - x)} = \frac{1}{(1 + x)(1 - x)^3}.$$

(c) We have

$$A(x) = \frac{1}{(1 + x)(1 - x)^3} = \frac{1/8}{1 + x} + \frac{1/8}{1 - x} + \frac{1/4}{(1 - x)^2} + \frac{1/2}{(1 - x)^3}.$$

Note that

$$\frac{1}{(1 - x)^2} = (1 - x)^{-2} = \sum_{n \geq 0} \binom{-2}{n} x^n = \sum_{n \geq 0} \binom{n + 1}{1} x^n$$
$$\frac{1}{(1 - x)^3} = (1 - x)^{-3} = \sum_{n \geq 0} \binom{-3}{n} x^n = \sum_{n \geq 0} \binom{n + 2}{2} x^n.$$

Hence, for $n \geq 0$

$$a_n = \frac{1}{8}(-1)^n + \frac{1}{8} + \frac{1}{4}(n + 1) + \frac{1}{2}\frac{(n + 2)(n + 1)}{2}$$
$$= \frac{7}{8} + \frac{1}{8}(-1)^n + n + \frac{n^2}{4}.$$

5. (a) The generating function for $p(n|\text{only even parts can occur more than once})$ is given by

$$(1 + x)(1 + x^2 + x^4 + \cdots)(1 + x^3)(1 + x^4 + x^8 + \cdots)(1 + x^5)(1 + x^6 + x^{12} + \cdots)\cdots$$
$$= \prod_{i=1}^{\infty} \frac{1 + x^{2i-1}}{1 - x^{2i}}.$$

(b) The generating function for $p(n|\text{each part is a multiple of 3})$ is given by

$$(1 + x^3 + x^6 + \cdots)(1 + x^6 + x^{12} + \cdots)(1 + x^9 + x^{18} + \cdots)\cdots = \prod_{i=1}^{\infty} \frac{1}{1 - x^{3i}}.$$

(c) Consider the two Ferrers graphs for the partition of n in which each part is 1 or 2 and the partition of $n + 3$ which have exactly two distinct parts, shown in Fig. 1. After the three red dots are added/removed, one graph is the transposition of the other graph, and vice versa. Therefore, there is a one-to-one correspondence between the sets of partitions of the two kinds, so they have the same cardinality.

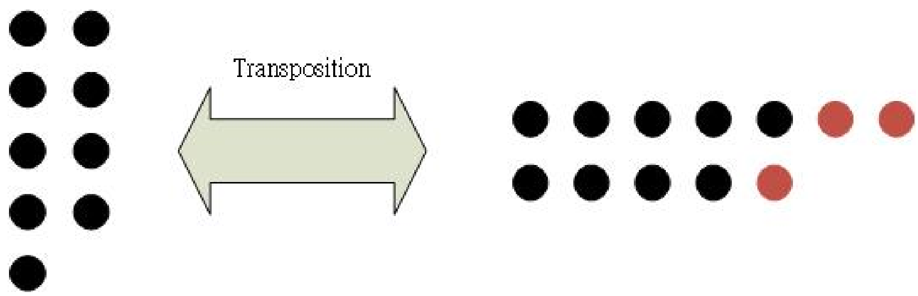

Figure 1: Ferrers graphs for Problem 5.(c).

6. For (a), $f_1(n) = n^2$, which is $O(n^2)$. For (b), $f_2(n) = \lfloor \log_2 n \rfloor + 1$, which is $O(\log_2 n)$. For (c), $f_3(n) = n(\lfloor \log_2 n \rfloor + 1)$, which is $O(n \log_2 n)$. Therefore, the procedure in (b) has the least complexity.

7. (a) The corresponding graph is shown in Fig. 2. So it has 3 components.

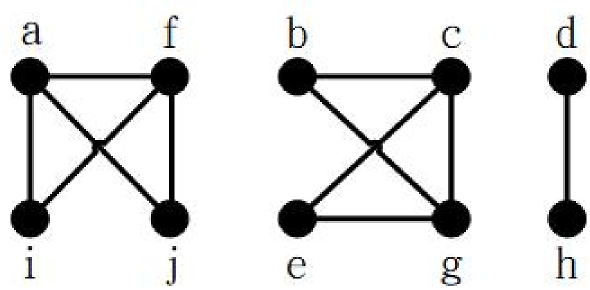

Figure 2: Graph for Problem 7.(a).

(b) No, the two graphs are not isomorphic. Note that there are cycles of length 3 in the graph on the left-hand side but there are no such cycles in the graph on the right-hand side.

(c) i. The corresponding graph is shown in Fig. 3.

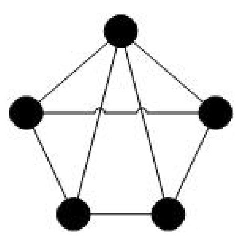

Figure 3: Graph for Problem 7.(c).i.

ii. No, it is not possible. The sum of the degrees must be even.

iii. No, it is not possible. Since there is a vertex with degree 4, it is adjacent to all the other vertices. Hence, there can not be a vertex with degree 0.