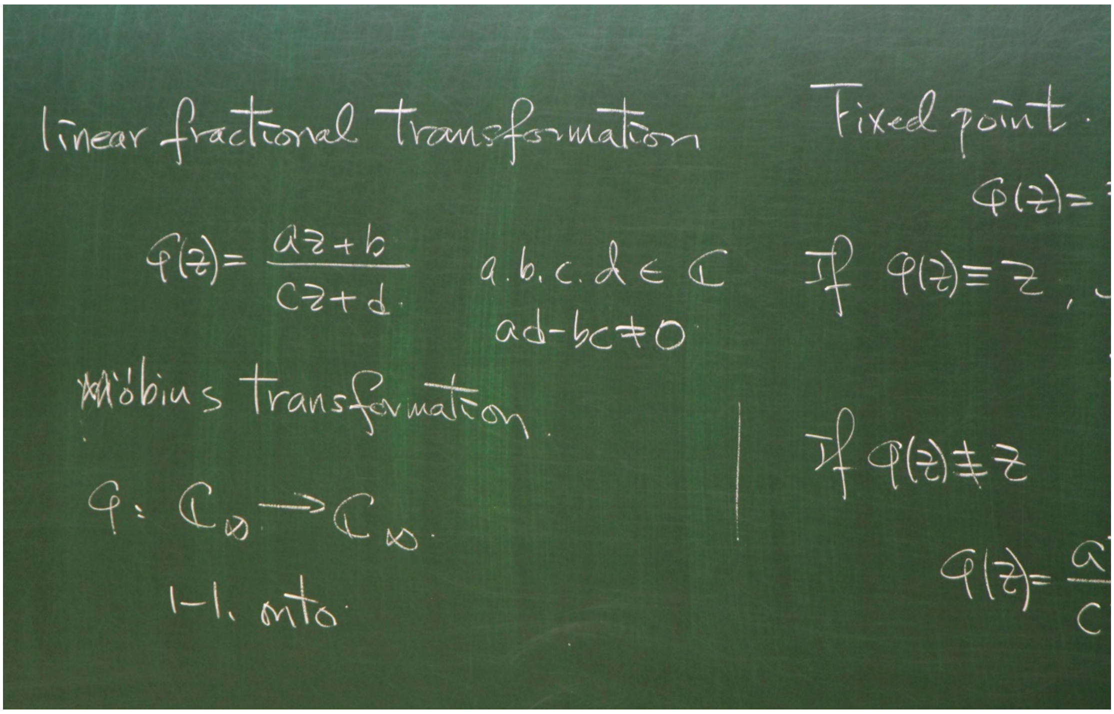
【10920 程守慶教授複變數函數論 / 第 25 堂版書】



...sformation

Fixed point.

$$\varphi(z) = z$$

$a, b, c, d \in \mathbb{C}$

$ad - bc \neq 0$

...ion.

If $\varphi(z) \equiv z$, every point of $\mathbb{C}_\infty$ is a fixed point

If $\varphi(z) \not\equiv z$

$$\varphi(z) = \frac{az+b}{cz+d} = z.$$

(i) $c = 0$

$\varphi(z) = \ldots$

ⓐ $a = \ldots$

ⓑ $a \neq \ldots$

(i) $c = 0$ $\quad a \neq 0,\ d \neq 0$

$$\varphi(z) = \frac{a}{d} z + \frac{b}{d}$$

ⓐ $a = d$. $\quad \varphi(z) = z + \frac{b}{d} \quad b \neq 0$.

Fixed point. ∞.

ⓑ $a \neq d$. $\quad \varphi(z) = \frac{a}{d} z + \frac{b}{d} \quad \frac{a}{d} \neq 1$.

$$\frac{a}{d} z + \frac{b}{d} = z \Rightarrow \left(\frac{a}{d} - 1\right) z = \frac{-b}{d} \quad z = \frac{b}{d-a},\ \infty \leftarrow \text{Fixed point}$$

(ii) $c \neq 0$

$\varphi(z) = \ldots$

$cz^2 + (d - \ldots$

$c \neq \ldots$

Ex: ...

(ii) $c \neq 0$

$$\varphi(z) = \frac{az+b}{cz+d} = z \qquad \varphi(z) = \frac{a+\frac{b}{z}}{c+\frac{d}{z}}$$

$$cz^2 + (d-a)z - b = 0 \qquad \varphi(\infty) = \frac{a}{c}$$

$c \neq 0$

Ex: $c=1.\ d=a=1.\ b=0$

$$\varphi(z) = \frac{z}{z+1} \qquad \frac{z}{z+1} = z$$

$\neq 1$

Fixed points

$$z = \frac{b}{d-a},\ \infty$$

$$z = z^2 + z \Rightarrow z^2 = 0$$

Cross ratio 交比. (交比)

If z_1, z_2, z_3, z_4 are four distinct points in $\mathbb{C}_\infty$.

Define the cross ratio of z_1, z_2, z_3, z_4 by

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

distinct

z_1, z_2, z_3, z_4

$$\frac{(z_3 - z_4)}{(z_3 - z_2)}$$

Ex. $[z_1, z_2, \infty, z_4]$

$$= \lim_{z \to \infty} \frac{(z_1 - z_2)(z - z_4)}{(z_1 - z_4)(z - z_2)}$$

$$= \frac{z_1 - z_2}{z_1 - z_4}$$

Define. let $z_2,$

points in $\mathbb{C}$

$T(z) = [z,$

$= (z -$

Define. let z_2, z_3, z_4 be three distinct points in $\mathbb{C}_\infty$. Define

$$T(z) = [z, z_2, z_3, z_4]$$

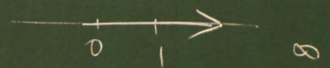
$$= \frac{(z - z_2)(z_3 - z_4)}{(z - z_4)(z_3 - z_2)}$$

$T: z_2 \longmapsto 0$

$z_3 \longmapsto 1$

$z_4 \longmapsto \infty$

$(z_2, z_3, z_4) \longmapsto (0, 1, \infty)$



thm: Cross ratio of any four distinct points z_1, z_2, z_3, z_4 is invariant under every linear fractional transformation φ. i.e.,

$$[z_1, z_2, z_3, z_4] = [\varphi(z_1), \varphi(z_2), \varphi(z_3), \varphi(z_4)]$$

proof. Set

$$[z, \varphi(z_2), \varphi(z_3), \varphi(z_4)] = T\circ\varphi^{-1}(z)$$

$T(z) = [z, z_2, z_3, z_4]$

$$T\circ\varphi^{-1} : \begin{aligned} \varphi(z_2) &\mapsto 0 \\ \varphi(z_3) &\mapsto 1 \\ \varphi(z_4) &\mapsto \infty \end{aligned}$$

Substitute

$z = \varphi_1(z)$

$$\begin{array}{c} [z_1, z_2, z_3, z_4] \\ \parallel \\ T(z) \\ \parallel \end{array}$$

$$[\varphi_1(z), \varphi_2(z), \varphi_3(z), \varphi_4(z)] = T\circ\varphi^{-1}\circ\varphi(z)$$

Ex $\varphi : 0, 1, i \longmapsto i, 1, \infty$

$$\varphi(z) = \frac{az+b}{cz+d} = \frac{az+b}{cz-ib} = \frac{az+b}{bz-ib} = \frac{-ibz+b}{bz-ib} = \frac{-iz+1}{z-i}$$

$\varphi(0) = i = \frac{b}{d}$ $\quad d = \frac{b}{i} = -ib$ $\quad b \neq 0$

$\varphi(i) = \infty \Rightarrow ci = ib \Rightarrow c = b.$

$\varphi(1) = 1 \Rightarrow a + \cancel{b} = \cancel{b} - ib$

$w = \varphi(z)$

$\tilde{a} \to \alpha$

$\tilde{b} \to \beta$

$\tilde{c} \to \gamma$

1

$[z, \tilde{a}, \tilde{b}, \tilde{c}] = [w, \alpha, \beta, \gamma]$

$$\frac{(w-\alpha)(\beta-\gamma)}{(w-\gamma)(\beta-\alpha)}$$

$\varphi(z_4)]$

Ex $\varphi : 0$

$\varphi(z) =$

$\varphi(0) = i$

$\varphi(i) =$

$\varphi(1) =$

Ex $\varphi : 0, 1, i \longmapsto i, 1, \infty$

$$\varphi(z) = \frac{az+b}{cz+d} = \frac{az+b}{cz-ib} = \frac{az+b}{bz-ib} = \frac{-ibz+b}{bz-ib} = \frac{-iz+1}{z-i}$$

$\varphi(0) = i = \frac{b}{d}$ $\quad d = \frac{b}{i} = -ib$ $\quad b \neq 0$

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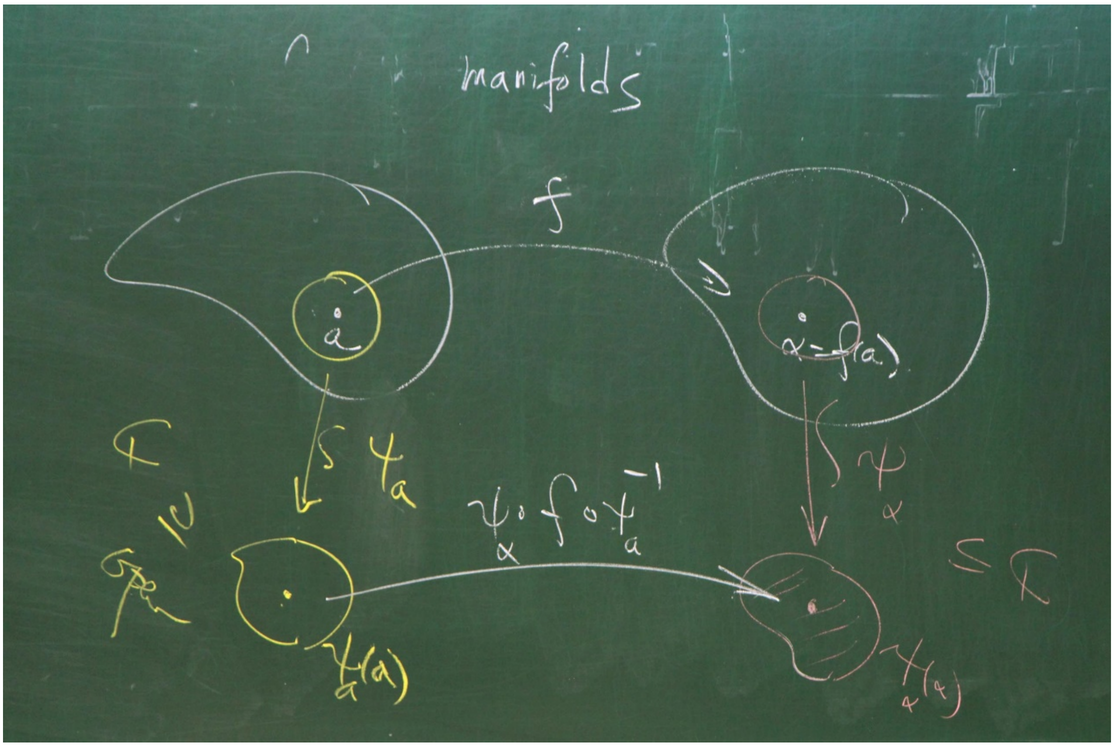
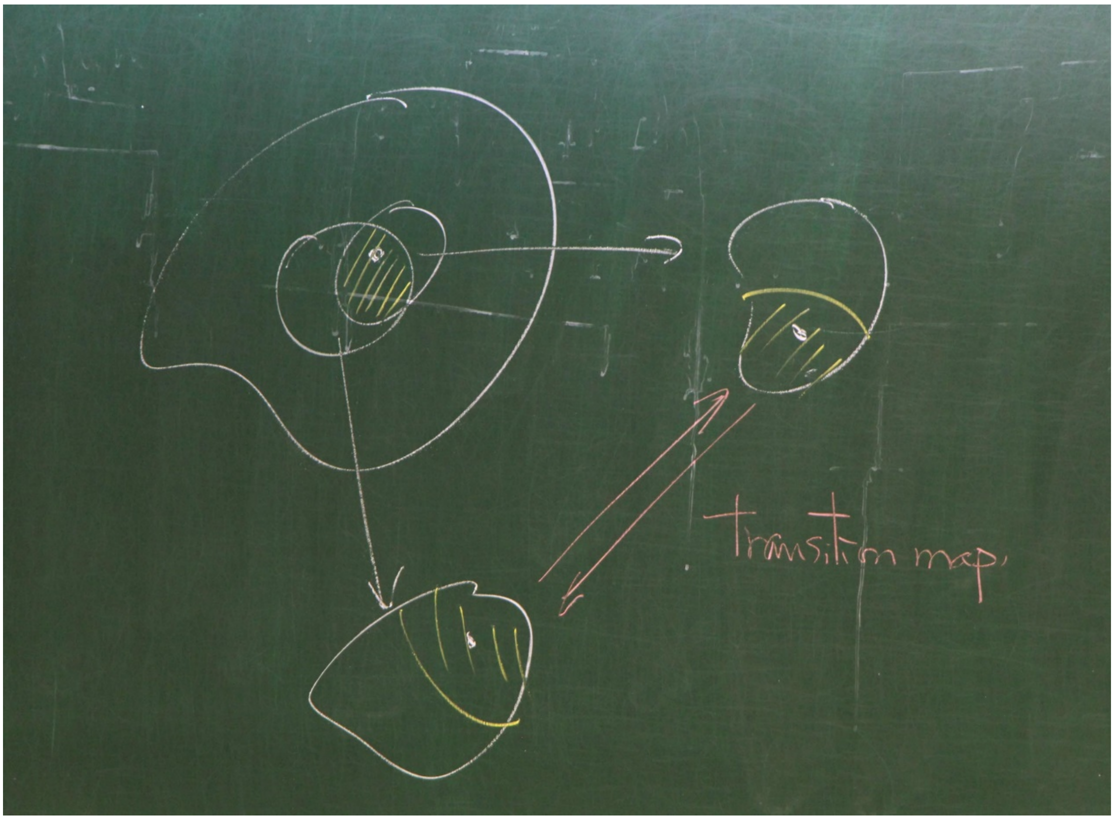
$\varphi(1) = 1 \Rightarrow a + \cancel{b} = \cancel{b} - ib$

thm: $\varphi \in \mathrm{Aut}(\mathbb{C}_\infty)$

$\Longleftrightarrow$ φ is a linear fractional transformation.


manifolds
f
a
$\alpha = f(a)$
ψ_a
ψ_α
$\psi_\alpha \circ f \circ \psi_a^{-1}$
open
$\psi_a(a)$
$\psi_\alpha(\alpha)$


transition map

thm: $\varphi \in \mathrm{Aut}(\mathbb{C}_\infty)$.

$\Longleftrightarrow$ φ is a linear fractional transformation.

proof: "$\Longleftarrow$" φ is a linear fractional transformation

$$\varphi(z) = \frac{az+b}{cz+d} \qquad ad-bc \neq 0.$$

$\varphi: \mathbb{C}_\infty \to \mathbb{C}_\infty$. 1-1. onto.

$\varphi: \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$

$\infty \longmapsto \infty$.

$$\varphi(z) = \frac{az+b}{cz+d} =$$

① $\infty \longmapsto \infty$.

② $\infty \longmapsto$ finite point

finite point $\longmapsto \infty$

If $c \neq 0$. $\varphi(\infty) =$

$$\varphi(z) = \frac{a}{d} z + \frac{b}{d}$$

$$\psi(z) = \frac{1}{z}$$

$$\psi \circ \varphi \circ \psi(z) =$$

e point

$\to \infty$

$(z) = \frac{1}{z}$

$\infty \mapsto \infty.$

$\varphi(z) = \frac{az+b}{cz+d} \Rightarrow c = 0. \therefore a \neq 0$

$d \neq 0$

If $c \neq 0$. $\varphi(\infty) = \frac{a}{c}.$

$\varphi(z) = \frac{a}{d} z + \frac{b}{d}.$

$\psi \circ \varphi \circ \psi(z) = \frac{1}{\frac{a}{d}\frac{1}{z} + \frac{b}{d}} = \frac{dz}{a+bz}.$ $0 \mapsto 0$

$\therefore a \neq 0$

$d \neq 0$

$\infty \mapsto$ finite point

$\therefore c \neq 0$

$\varphi(z) = \frac{az+b}{cz+d}$

$\varphi \circ \psi = \frac{a\frac{1}{z}+b}{c\frac{1}{z}+d} = \frac{a+bz}{c+dz}.$

$\frac{dz}{a+bz}.$ $0 \mapsto 0$

finite point

$c \neq 0$

When $z = -\frac{d}{c}$

az

ψ

finite point $\mapsto \infty$

$c \neq 0$

When $z = -\frac{d}{c}$ $\quad \varphi\left(-\frac{d}{c}\right) = \infty$.

$$az+b = a\left(-\frac{d}{c}\right)+b = \frac{-ad+bc}{c} \neq 0$$

$$\psi \circ \varphi(z) = \frac{1}{\varphi(z)} = \frac{cz+d}{az+b}.$$

$z = -\frac{d}{c} \mapsto 0.$

$\Rightarrow \varphi \in \mathrm{Aut}(\mathbb{C}_\infty)$.

Map. $0, 1, \infty \mapsto \alpha, \beta, \gamma$.

Set $T(z) = [z, \alpha, \beta, \gamma]$

Consider.

$$T \circ \varphi \in \mathrm{Aut}(\mathbb{C}_\infty).$$

maps $0, 1, \infty \mapsto 0, 1, \infty$

Consider

$T \circ \varphi$

Consider

$$T\circ\varphi\Big|_{\mathbb{C}} \in \mathrm{Aut}(\mathbb{C}).$$

$\therefore$ $T\circ\varphi(z)\Big|_{\mathbb{C}} = az+b$. $\quad a\neq 0$.

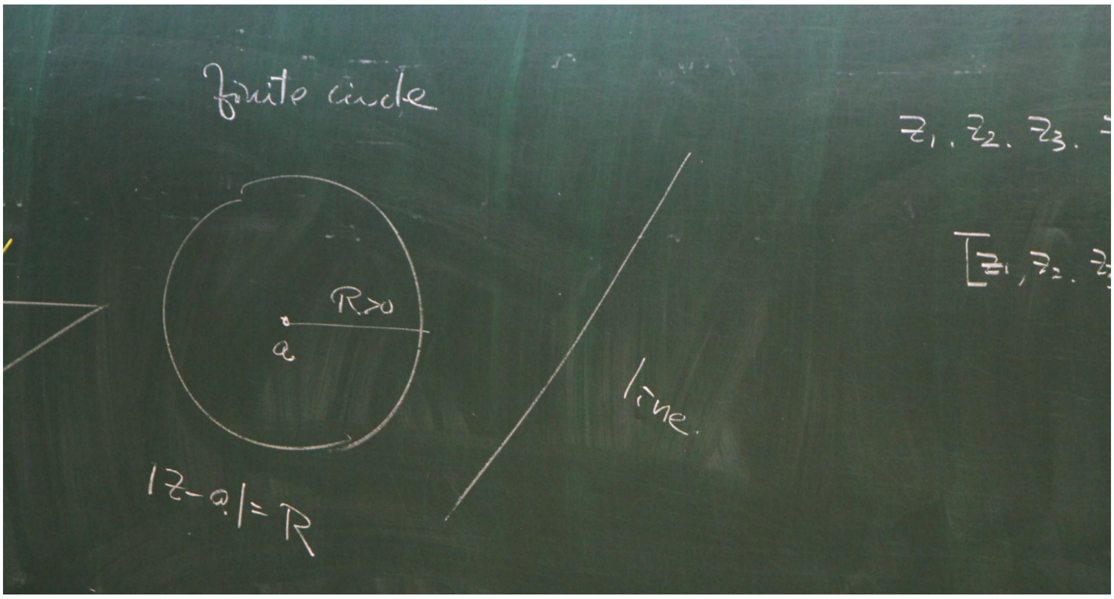
$T\circ\varphi(0)=0$ $\quad\therefore b=0$. $\quad\therefore \varphi(z)=T^{-1}(z)$

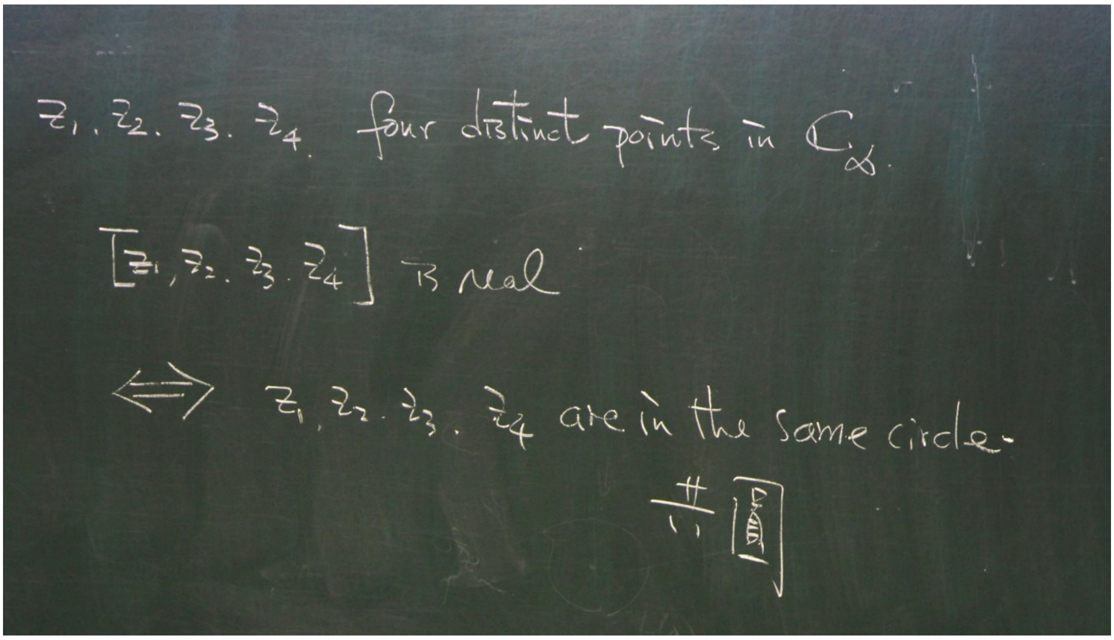
$T\circ\varphi(1)=1$ $\quad a=1$.

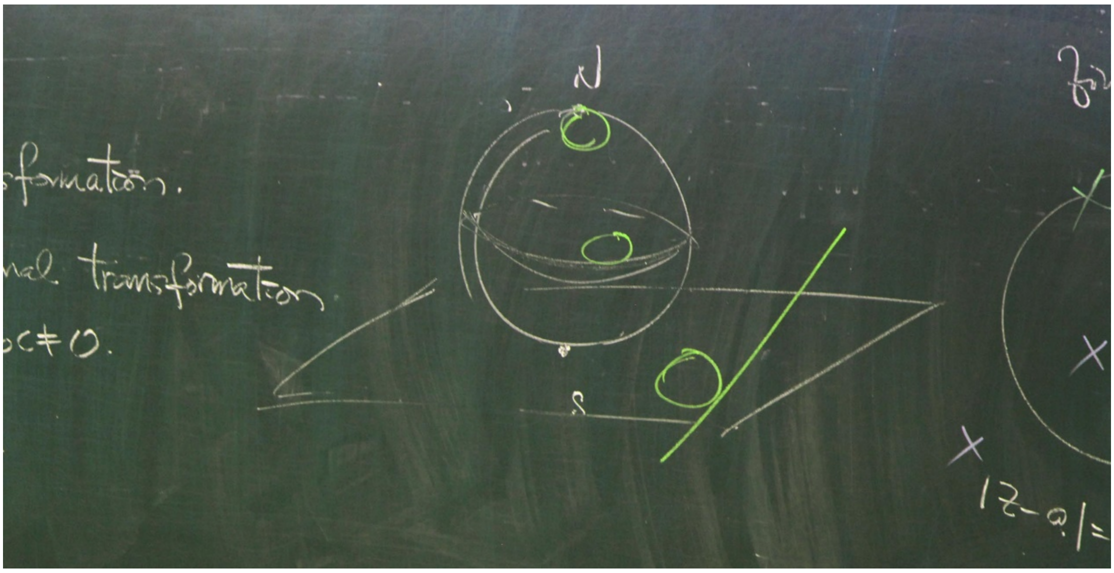
$\therefore T\circ\varphi(z)\equiv z$.

z_1, z_2, z_3, z_4 four distinct points in $\mathbb{C}_\infty$.

$[z_1, z_2, z_3, z_4]$


finite circle
R>0
a
|z-a|=R
line.
z1 . z2 . z3 .
[z1, z2 . z3


z1 . z2 . z3 . z4 four distinct points in C∞.
[z1, z2 . z3 . z4] is real
⟺ z1, z2 . z3 . z4 are in the same circle.
#


sformation.
nal transformation
N
S
|z-a|=