國立清華大學

Electromagnetism

Introduction to Electrodynamics 4th David J. Griffiths

Chap.10 Potentials and Fields

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Exercise List

4, 11, 15, 16, 27, 28

Problem 10.4 Suppose $V = 0$ and $\mathbf{A} = A_0 \sin(kx - \omega t)\,\hat{\mathbf{y}}$, where A_0, ω, and k are constants. Find $\mathbf{E}$ and $\mathbf{B}$, and check that they satisfy Maxwell's equations in vacuum. What condition must you impose on ω and k?

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -0 + \omega A_0 \cos(kx - \omega t)\,\hat{\mathbf{y}}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = kA_0 \cos(kx - \omega t)\,\hat{\mathbf{z}}$$

In vacuum $\rho = 0$ & $\mathbf{J} = 0$

$$\nabla \cdot \mathbf{E} = 0 \Rightarrow \nabla \cdot \mathbf{E} = \partial_y \omega A_0 \cos(kx - \omega t) = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \begin{cases} \nabla \times \mathbf{E} = \partial_x \omega A_0 \cos(kx - \omega t)\,\hat{\mathbf{z}} = -k\omega A_0 \sin(kx - \omega t)\,\hat{\mathbf{z}} \\ \dfrac{\partial \mathbf{B}}{\partial t} = k\omega A_0 \sin(kx - \omega t)\,\hat{\mathbf{z}} \end{cases}$$

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \nabla \cdot \mathbf{B} = \partial_z kA_0 \cos(kx - \omega t) = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \Rightarrow \begin{cases} \nabla \times \mathbf{B} = -\partial_x kA_0 \cos(kx - \omega t)\,\hat{\mathbf{y}} = k^2 A_0 \sin(kx - \omega t)\,\hat{\mathbf{y}} \\ \dfrac{\partial \mathbf{E}}{\partial t} = \omega^2 A_0 \sin(kx - \omega t)\,\hat{\mathbf{y}} \end{cases}$$

$$k^2 = \mu_0 \varepsilon_0 \omega^2 \Rightarrow \omega = \frac{k}{\sqrt{\mu_0 \varepsilon_0}} = ck$$

Problem 10.11

(a) Suppose the wire in Ex. 10.2 carries a linearly increasing current

$$I(t) = kt,$$

for $t > 0$. Find the electric and magnetic fields generated.

It's electically neutral $\Rightarrow V(\mathbf{r},t) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \nabla \frac{\rho(\mathbf{r}',t_t)}{\eta} d\tau = 0$

For $t < s/c$, $\mathbf{A} = 0$ For $t > s/c$:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{\eta} d\tau' = \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\int_{-\sqrt{(ct)^2-s^2}}^{\sqrt{(ct)^2-s^2}} \frac{kt_r}{\sqrt{s^2+z'^2}} dz' = \hat{\mathbf{z}}\left(\frac{\mu_0 k}{4\pi}\right) 2\int_0^{\sqrt{(ct)^2-s^2}} \frac{t-\sqrt{s^2+z'^2}/c}{\sqrt{s^2+z'^2}} dz'$$

$$= \hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left[t\int_0^{\sqrt{(ct)^2-s^2}} \frac{1}{\sqrt{s^2+z'^2}} dz' - \frac{1}{c}\int_0^{\sqrt{(ct)^2-s^2}} dz'\right]$$

$$= \hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left[t\ln\left(\sqrt{s^2+z'^2}+z'\right)_0^{\sqrt{(ct)^2-s^2}} - \frac{\sqrt{(ct)^2-s^2}}{c}\right]$$

$$= \hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left(t\ln\frac{ct+\sqrt{(ct)^2-s^2}}{s} - \frac{\sqrt{(ct)^2-s^2}}{c}\right)$$

Problem 10.11

$$\mathbf{A}=\hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left(t\ln\frac{ct+\sqrt{(ct)^2-s^2}}{s}-\frac{\sqrt{(ct)^2-s^2}}{c}\right)$$

(a) Suppose the wire in Ex. 10.2 carries a linearly increasing current

$$I(t)=kt,$$

for $t>0$. Find the electric and magnetic fields generated.

$$\mathbf{E}=-\nabla V-\frac{\partial\mathbf{A}}{\partial t}=-\hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left[\ln\frac{ct+\sqrt{(ct)^2-s^2}}{s}+\frac{st}{ct+\sqrt{(ct)^2-s^2}}\frac{1}{s}\left(c+\frac{c^2 t}{\sqrt{(ct)^2-s^2}}\right)-\frac{c^2 t}{c\sqrt{(ct)^2-s^2}}\right]$$

$$=-\hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left[\ln\frac{ct+\sqrt{(ct)^2-s^2}}{s}+\frac{ct}{ct+\sqrt{(ct)^2-s^2}}\left(\frac{\sqrt{(ct)^2-s^2}+ct}{\sqrt{(ct)^2-s^2}}\right)-\frac{ct}{\sqrt{(ct)^2-s^2}}\right]$$

$$=-\hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\ln\frac{ct+\sqrt{(ct)^2-s^2}}{s}\quad \text{or}\quad =0\left(t<s/c\right)$$

Problem 10.11

(a) Suppose the wire in Ex. 10.2 carries a linearly increasing current

$$I(t) = kt,$$

for $t > 0$. Find the electric and magnetic fields generated.

$$\mathbf{A} = \hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left(t\ln\frac{ct+\sqrt{(ct)^2-s^2}}{s} - \frac{\sqrt{(ct)^2-s^2}}{c}\right)$$

$$\mathbf{B} = \nabla\times\mathbf{A} = -\partial_s A_z\hat{\phi} = -\hat{\phi}\frac{\mu_0 k}{2\pi}\left[t\frac{s}{ct+\sqrt{(ct)^2-s^2}}\left(-\frac{ct+\sqrt{(ct)^2-s^2}}{s^2} - \frac{s}{s\sqrt{(ct)^2-s^2}}\right) + \frac{s}{c\sqrt{(ct)^2-s^2}}\right]$$

$$= -\hat{\phi}\frac{\mu_0 k}{2\pi}\left[t\frac{s}{ct+\sqrt{(ct)^2-s^2}}\left(-\frac{ct\sqrt{(ct)^2-s^2}+(ct)^2-s^2+s^2}{s^2\sqrt{(ct)^2-s^2}}\right) + \frac{s}{c\sqrt{(ct)^2-s^2}}\right]$$

$$= -\hat{\phi}\frac{\mu_0 k}{2\pi}\left[t\frac{-ct}{s\sqrt{(ct)^2-s^2}} + \frac{s}{c\sqrt{(ct)^2-s^2}}\right]$$

$$= -\hat{\phi}\frac{\mu_0 k}{2\pi}\frac{-(ct)^2+s^2}{sc\sqrt{(ct)^2-s^2}} = \frac{\mu_0 k}{2\pi sc}\sqrt{(ct)^2-s^2}\,\hat{\phi} \quad \text{or} \quad = 0\,(t < s/c)$$

Problem 10.11

(b) Do the same for the case of a sudden burst of current:

$$I(t) = q_0\delta(t).$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}d\tau' = \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}\frac{q_0\delta(t_r)}{\sqrt{s^2+z'^2}}dz' = \hat{\mathbf{z}}\frac{\mu_0 q_0}{4\pi}2\int_0^{\infty}\frac{\delta\left(t-\sqrt{s^2+z'^2}/c\right)}{\sqrt{s^2+z'^2}}dz'$$

$$= \hat{\mathbf{z}}\frac{\mu_0 q_0}{2\pi}\int_s^{\infty'}\frac{\delta\left(t-\sqrt{s^2+z'^2}/c\right)}{\sqrt{s^2+z'^2}}\left(\frac{c\sqrt{s^2+z'^2}}{z'}d\sqrt{s^2+z'^2}/c\right)$$

$$= \frac{\mu_0 q_0}{2\pi}\frac{c}{\sqrt{(ct)^2-s^2}}\hat{\mathbf{z}}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = -\partial_s A_z\hat{\phi} = -\frac{\mu_0 q_0}{2\pi}\frac{cs}{\left[(ct)^2-s^2\right]^{3/2}}\hat{\phi} \text{ or } = 0\,(t<s/c)$$

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} = \frac{\mu_0 q_0}{2\pi}\frac{c^3 t}{\left[(ct)^2-s^2\right]^{3/2}}\hat{\mathbf{z}} \text{ or } = 0\,(t<s/c)$$

Problem 10.15 A particle of charge q moves in a circle of radius a at constant angular velocity ω. (Assume that the circle lies in the xy plane, centered at the origin, and at time $t=0$ the charge is at $(a,0)$, on the positive x axis.) Find the Liénard-Wiechert potentials for points on the z axis.

$$V(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\frac{qc}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})}, \tag{10.46}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})}=\frac{\mathbf{v}}{c^2}V(\mathbf{r},t). \tag{10.47}$$

$$\mathbf{w}(t)=a\left[\cos(\omega t)\hat{\mathbf{x}}+\sin(\omega t)\hat{\mathbf{y}}\right]\Rightarrow\mathbf{v}(t)=\omega a\left[-\sin(\omega t)\hat{\mathbf{x}}+\cos(\omega t)\hat{\mathbf{y}}\right]$$

$$\boldsymbol{\mathscr{r}}=z\hat{\mathbf{z}}-\mathbf{w}(t_r)=z\hat{\mathbf{z}}-a\left[\cos(\omega t_r)\hat{\mathbf{x}}+\sin(\omega t_r)\hat{\mathbf{y}}\right],\ \mathscr{r}=\sqrt{z^2+a^2}=c(t-t_r)\Rightarrow t_r=t-\frac{\sqrt{z^2+a^2}}{c}$$

$$\boldsymbol{\mathscr{r}}\cdot\mathbf{v}(t_r)=\omega a^2\sin(\omega t_r)\cos(\omega t_r)(1-1)=0$$

$$\begin{cases}V(\mathbf{r},t)=\dfrac{1}{4\pi\varepsilon_0}\dfrac{qc}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})}=\dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{\sqrt{z^2+a^2}}\\[2ex]\mathbf{A}(\mathbf{r},t)=\dfrac{\mu_0}{4\pi}\dfrac{qc\mathbf{v}}{(\mathscr{r}c-\boldsymbol{\mathscr{r}}\cdot\mathbf{v})}=\dfrac{\mu_0}{4\pi}\dfrac{q\omega a}{\sqrt{z^2+a^2}}\left[-\sin(\omega t_r)\hat{\mathbf{x}}+\cos(\omega t_r)\hat{\mathbf{y}}\right]=\dfrac{1}{4\pi\varepsilon_0}\dfrac{q\omega a}{c^2\sqrt{z^2+a^2}}\left[-\sin(\omega t_r)\hat{\mathbf{x}}+\cos(\omega t_r)\hat{\mathbf{y}}\right]\\[2ex]t_r=t-\dfrac{\sqrt{z^2+a^2}}{c}\end{cases}$$

Problem 10.16 Show that the scalar potential of a point charge moving with constant velocity (Eq. 10.49) can be written more simply as

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{R\sqrt{1 - v^2 \sin^2\theta/c^2}}, \qquad (10.51)$$

where $\mathbf{R} \equiv \mathbf{r} - \mathbf{v}t$ is the vector from the *present (!)* position of the particle to the field point $\mathbf{r}$, and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$ (Fig. 10.9). Note that for nonrelativistic velocities ($v^2 \ll c^2$),

$$V(\mathbf{r}, t) \approx \frac{1}{4\pi\epsilon_0} \frac{q}{R}.$$

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{\sqrt{(c^2 t - \mathbf{r} \cdot \mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}, \qquad (10.49)$$

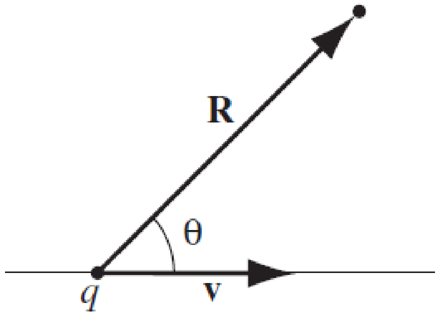


FIGURE 10.9

$$\begin{aligned}
&(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\\
&=\left[c^2t-(\mathbf{R}+\mathbf{v}t)\cdot\mathbf{v}\right]^2+(c^2-v^2)\left[(\mathbf{R}+\mathbf{v}t)\cdot(\mathbf{R}+\mathbf{v}t)-c^2t^2\right]\\
&=(c^2t-Rv\cos\theta-v^2t)^2+(c^2-v^2)(R^2+2Rvt\cos\theta+v^2t^2-c^2t^2)\\
&=\left[(c^2-v^2)t-Rv\cos\theta\right]^2+(c^2-v^2)(R^2+2Rvt\cos\theta)-\left[(c^2-v^2)t\right]^2\\
&=\left[(c^2-v^2)t\right]^2-2(c^2-v^2)tRv\cos\theta+(Rv\cos\theta)^2\\
&\quad+(c^2-v^2)(R^2+2Rvt\cos\theta)-\left[(c^2-v^2)t\right]^2\\
&=R^2\left[(v\cos\theta)^2+c^2-v^2\right]
\end{aligned}$$

$$\begin{aligned}
V(\mathbf{r},t)&=\frac{1}{4\pi\varepsilon_0}\frac{qc}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}}\\
&=\frac{1}{4\pi\varepsilon_0}\frac{qc}{\sqrt{R^2\left[(v\cos\theta)^2+c^2-v^2\right]}}\\
&=\frac{1}{4\pi\varepsilon_0}\frac{q}{R\sqrt{\left[1+(v\cos\theta/c)^2-(v/c)^2\right]}}\\
&=\frac{1}{4\pi\varepsilon_0}\frac{q}{R\sqrt{\left[1-(v\sin\theta/c)^2\right]}}
\end{aligned}$$

$$V(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}}, \quad (10.49)$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}}. \quad (10.50)$$

$$\nabla\cdot\mathbf{A}=-\mu_0\epsilon_0\frac{\partial V}{\partial t}. \quad (10.12)$$

Problem 10.27 Check that the potentials of a point charge moving at constant velocity (Eqs. 10.49 and 10.50) satisfy the Lorenz gauge condition (Eq. 10.12).

$$\nabla\cdot\mathbf{A}=\frac{\mu_0}{4\pi}qc\left\{\left[\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)\right]^{-1/2}\underbrace{\nabla\cdot\mathbf{v}}_{=0}+\mathbf{v}\cdot\nabla\left[\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)\right]^{-1/2}\right\}$$

$$=\frac{\mu_0}{4\pi}qc\mathbf{v}\cdot\left\{\left(-\frac{1}{2}\right)\left[\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)\right]^{-3/2}\left[2\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)\nabla\left(-\mathbf{r}\cdot\mathbf{v}\right)+\left(c^2-v^2\right)\nabla\left(r^2\right)\right]\right\}$$

$$=\frac{\mu_0}{8\pi}\frac{qc}{\left(\sqrt{\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)}\right)^3}\mathbf{v}\cdot\left[2\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)\nabla\left(\mathbf{r}\cdot\mathbf{v}\right)-\left(c^2-v^2\right)\nabla\left(r^2\right)\right]$$

$$\nabla\cdot\mathbf{A}=\frac{\mu_0}{8\pi}\frac{qc}{\left(\sqrt{\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)}\right)^3}\mathbf{v}\cdot\left[2\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)\mathbf{v}-\left(c^2-v^2\right)2\mathbf{r}\right]$$

$$\nabla\left(\mathbf{r}\cdot\mathbf{v}\right)=\underbrace{\left(\mathbf{r}\cdot\nabla\right)\mathbf{v}}_{=0}+\left(\mathbf{v}\cdot\nabla\right)\mathbf{r}+\mathbf{r}\times\underbrace{\left(\nabla\times\mathbf{v}\right)}_{=0}+\mathbf{v}\times\underbrace{\left(\nabla\times\mathbf{r}\right)}_{=0}=\mathbf{v}$$

$$\nabla\left(r^2\right)=\nabla\left(\mathbf{r}\cdot\mathbf{r}\right)=2\left(\mathbf{r}\cdot\nabla\right)\mathbf{r}+2\mathbf{r}\times\underbrace{\left(\nabla\times\mathbf{r}\right)}_{=0}=2\mathbf{r}$$

$$=\frac{\mu_0 qc}{4\pi}\frac{\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)v^2-\left(c^2-v^2\right)\mathbf{v}\cdot\mathbf{r}}{\left(\sqrt{\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)}\right)^3}$$

$$=\frac{\mu_0 qc^3}{4\pi}\frac{v^2t-\mathbf{v}\cdot\mathbf{r}}{\left(\sqrt{\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)}\right)^3}$$

$$V(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}}, \qquad (10.49)$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}}. \qquad (10.50)$$

$$\boxed{\nabla\cdot\mathbf{A}=-\mu_0\epsilon_0\frac{\partial V}{\partial t}.} \qquad (10.12)$$

Problem 10.27 Check that the potentials of a point charge moving at constant velocity (Eqs. 10.49 and 10.50) satisfy the Lorenz gauge condition (Eq. 10.12).

$$\frac{\partial V}{\partial t}=\frac{qc}{4\pi\varepsilon_0}\frac{\partial}{\partial t}\left[\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)\right]^{-1/2}$$

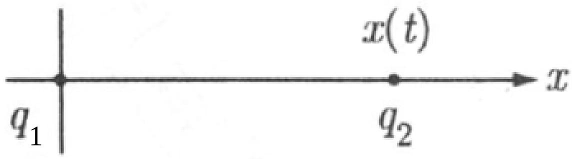
$$=\frac{qc}{4\pi\varepsilon_0}\left(-\frac{1}{2}\right)\left[\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)\right]^{-3/2}\left[2\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)\left(c^2-\underbrace{\frac{\partial\mathbf{r}}{\partial t}}_{=0}\cdot\mathbf{v}\right)+\left(c^2-v^2\right)\left(2\mathbf{r}\cdot\underbrace{\frac{\partial\mathbf{r}}{\partial t}}_{=0}-2c^2t\right)\right]$$

$$=-\frac{qc^3}{4\pi\varepsilon_0}\frac{v^2t-\mathbf{r}\cdot\mathbf{v}}{\left(\sqrt{\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2+v^2\right)\left(r^2-c^2t^2\right)}\right)^3}$$

$$\nabla\cdot\mathbf{A}=\frac{\mu_0qc^3}{4\pi}\frac{v^2t-\mathbf{v}\cdot\mathbf{r}}{\left(\sqrt{\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)}\right)^3}$$

$$\Rightarrow -\mu_0\varepsilon_0\frac{\partial V}{\partial t}=\nabla\cdot\mathbf{A}$$

Problem 10.28 One particle, of charge q_1, is held at rest at the origin. Another particle, of charge q_2, approaches along the x axis, in hyperbolic motion:



$$x(t) = \sqrt{b^2 + (ct)^2};$$

it reaches the closest point, b, at time $t = 0$, and then returns out to infinity.

(a) What is the force F_2 on q_2 (due to q_1) at time t?

$$\mathbf{F}_2 = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{1}{b^2 + c^2 t^2} \hat{\mathbf{x}}$$

(b) What total impulse $\left(I_2 = \int_{-\infty}^{\infty} F_2 dt\right)$ is delivered to q_2 by q_1?

$$I_2 = \int_{-\infty}^{\infty} F_2 dt = \int_{-\infty}^{\infty} \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{1}{b^2 + c^2 t^2} dt = \frac{q_1 q_2}{4\pi\varepsilon_0} \left[\frac{1}{bc} \tan^{-1} \frac{ct}{b} \right]_{-\infty}^{\infty} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{1}{bc} \left[\frac{\pi}{2} - \left(\frac{\pi}{2} \right) \right] = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\pi}{bc}$$

Problem 10.28

(c) What is the force F_1 on q_1 (due to q_2) at time t?

Problem 10.20 Suppose a point charge q is constrained to move along the x axis. Show that the fields at points on the axis to the *right* of the charge are given by

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c+v}{c-v}\right)\hat{\mathbf{x}}, \quad \mathbf{B} = \mathbf{0}.$$

(Do not assume v is constant!) What are the fields on the axis to the *left* of the charge?

$$\mathbf{E} = \begin{cases} \frac{q_2}{4\pi\varepsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c+v}{c-v}\right)\hat{\mathbf{x}} & \text{, to the right of the charge} \\ -\frac{q_2}{4\pi\varepsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c-v}{c+v}\right)\hat{\mathbf{x}} & \text{, to the left of the charge} \end{cases}$$

$$c(t-t_r) = \left|\mathbf{r}-\mathbf{w}(t_r)\right| = x(t_r) = \sqrt{b^2+c^2t_r^2} \Rightarrow t_r = \frac{c^2t^2-b^2}{2c^2t}$$

$$\begin{cases} x(t) = \sqrt{b^2+c^2t^2} \\ v(t) = \dfrac{c^2t}{\sqrt{b^2+c^2t^2}} = \dfrac{c^2t}{x} \end{cases} \Rightarrow \begin{cases} x(t_r) = c(t-t_r) = \dfrac{c^2t^2+b^2}{2ct} \\ v(t_r) = \dfrac{c^2t_r}{x(t_r)} = \dfrac{c^2t^2-b^2}{2t}\dfrac{2ct}{c^2t^2+b^2} = c\left(\dfrac{c^2t^2-b^2}{c^2t^2+b^2}\right) \end{cases}$$

$$\mathbf{E} = -\frac{q_2}{4\pi\varepsilon_0}\frac{1}{x(t_r)^2}\left(\frac{c-v}{c+v}\right)\hat{\mathbf{x}} = -\frac{q_2}{4\pi\varepsilon_0}\frac{1}{x(t_r)^2}\left(1-\frac{c^2t^2-b^2}{c^2t^2+b^2}\right)\left(1+\frac{c^2t^2-b^2}{c^2t^2+b^2}\right)^{-1}\hat{\mathbf{x}}$$

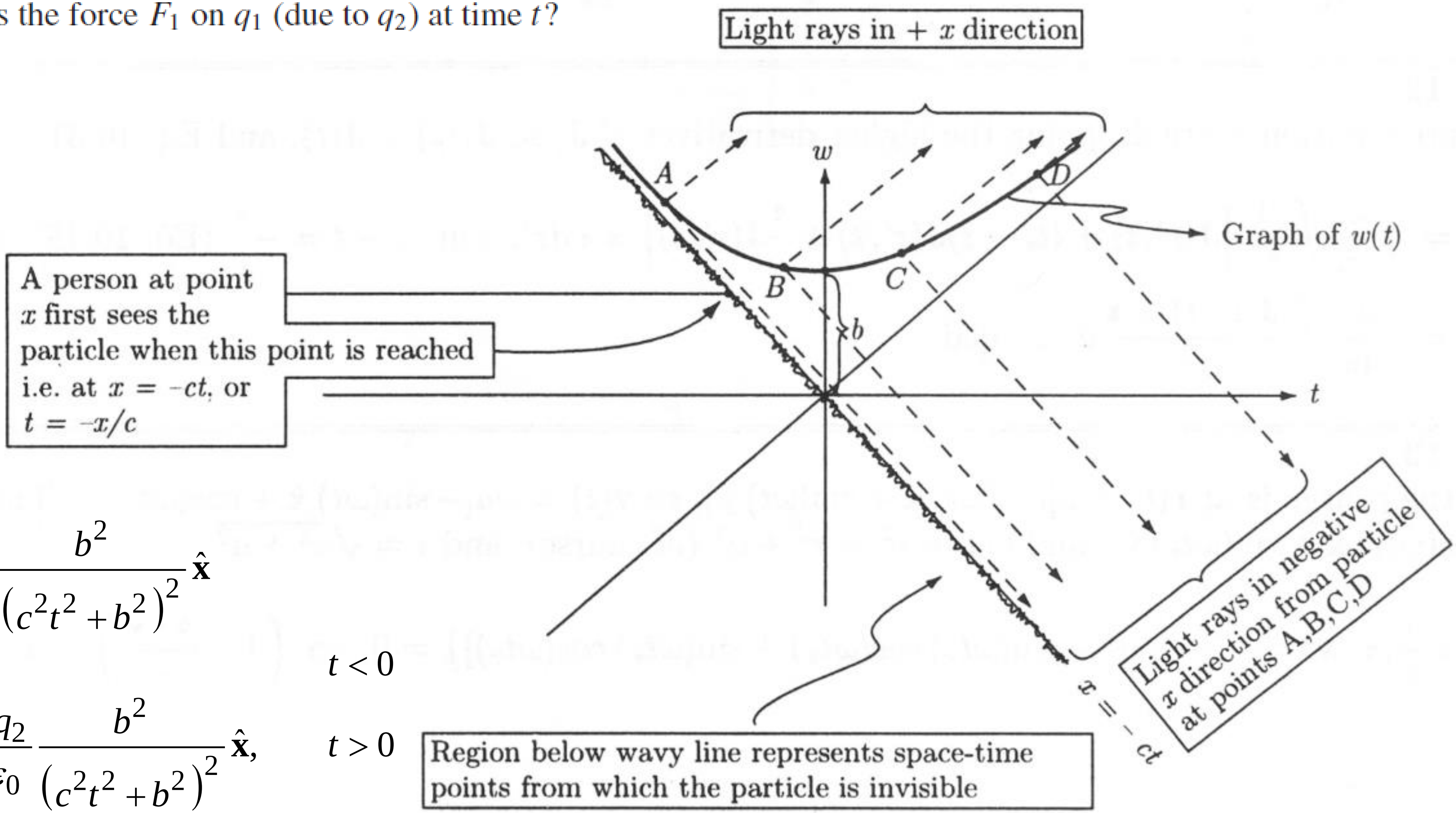
$$= -\frac{q_2}{4\pi\varepsilon_0}\left(\frac{c^2t^2+b^2}{2ct}\right)^{-2}\frac{2b^2}{c^2t^2+b^2}\frac{c^2t^2+b^2}{2c^2t^2}\hat{\mathbf{x}}$$

$$= -\frac{q_2}{4\pi\varepsilon_0}\frac{4c^2t^2}{(c^2t^2+b^2)^2}\frac{b^2}{c^2t^2}\hat{\mathbf{x}}$$

$$= -\frac{q_2}{\pi\varepsilon_0}\frac{b^2}{(c^2t^2+b^2)^2}\hat{\mathbf{x}}$$

Problem 10.28

(c) What is the force F_1 on q_1 (due to q_2) at time t?



$$\mathbf{E} = -\frac{q_2}{\pi\varepsilon_0}\frac{b^2}{\left(c^2t^2+b^2\right)^2}\hat{\mathbf{x}}$$

$$\mathbf{F}_1 = \begin{cases} 0, & t<0 \\ -\dfrac{q_1q_2}{\pi\varepsilon_0}\dfrac{b^2}{\left(c^2t^2+b^2\right)^2}\hat{\mathbf{x}}, & t>0 \end{cases}$$

Problem 10.28

(d) What total impulse $\left(I_1 = \int_{-\infty}^{\infty} F_1 dt\right)$ is delivered to q_1 by q_2? [*Hint:* It might help to review Prob. 10.17 before doing this integral. *Answer:* $I_2 = -I_1 = q_1 q_2 / 4\epsilon_0 bc$]

$$I_2 = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\pi}{bc}$$

$$I_1 = -\frac{q_1 q_2}{\pi\varepsilon_0} b^2 \int_0^\infty \frac{1}{\left(c^2t^2 + b^2\right)^2} dt = -\frac{q_1 q_2}{\pi\varepsilon_0} b^2 \frac{1}{2b^2} \left(\frac{t}{b^2 + c^2t^2}\bigg|_0^\infty + \int_0^\infty \frac{1}{b^2 + c^2t^2} dt \right) = -\frac{q_1 q_2}{2\pi\varepsilon_0} \left(\frac{1}{bc} \frac{\pi}{2} \right) = -\frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\pi}{bc}$$

$$\Rightarrow I_2 = -I_1 = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\pi}{bc}$$