國立清華大學

Electromagnetism

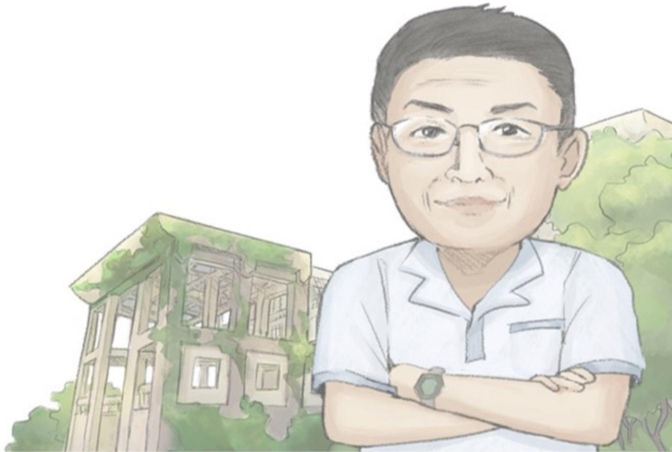
Introduction to Electrodynamics 4th David J. Griffiths

Chap.10 Potentials and Fields

Prof. Tsun Hsu Chang

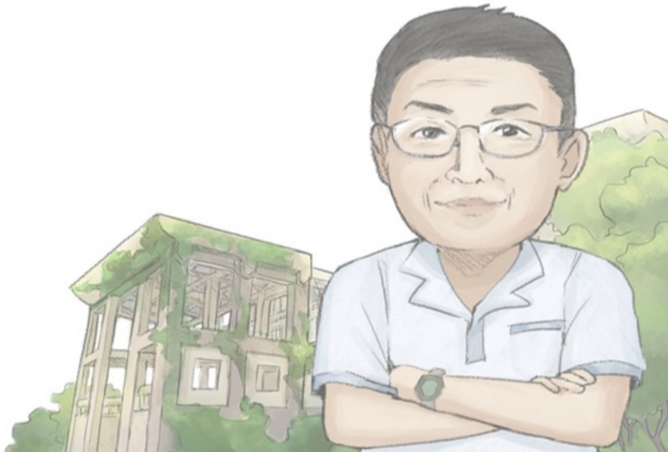
TA: Hung Chun Hsu, Yi Wen Lin, and Tien Fu Yang

2023 Spring



Exercise List

4, 11, 15, 16, 27, 28



Problem 10.4 Suppose $V = 0$ and $\mathbf{A} = A_0 \sin(kx - \omega t)\,\hat{\mathbf{y}}$, where A_0, ω, and k are constants. Find $\mathbf{E}$ and $\mathbf{B}$, and check that they satisfy Maxwell's equations in vacuum. What condition must you impose on ω and k?

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -0 + \omega A_0 \cos(kx - \omega t)\,\hat{\mathbf{y}}$$

$$\mathbf{B} = \nabla \times \mathbf{A} = kA_0 \cos(kx - \omega t)\,\hat{\mathbf{z}}$$

In vacuum $\rho = 0 \;\&\; \mathbf{J} = 0$

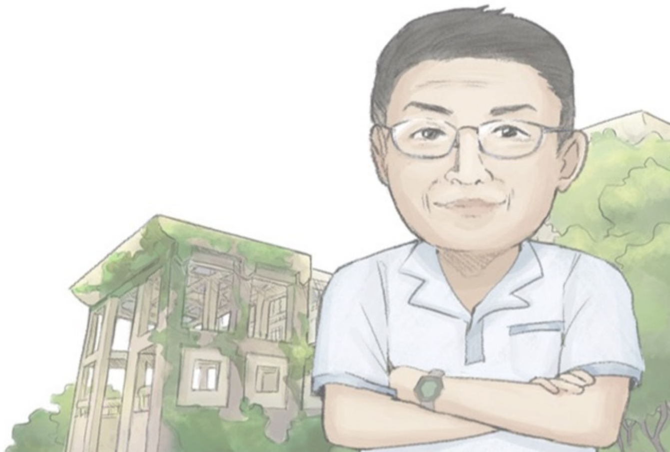
$$\nabla \cdot \mathbf{E} = 0 \Rightarrow \nabla \cdot \mathbf{E} = \partial_y \omega A_0 \cos(kx - \omega t) = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Rightarrow \begin{cases} \nabla \times \mathbf{E} = \partial_x \omega A_0 \cos(kx - \omega t)\,\hat{\mathbf{z}} = -k\omega A_0 \sin(kx - \omega t)\,\hat{\mathbf{z}} \\ \dfrac{\partial \mathbf{B}}{\partial t} = k\omega A_0 \sin(kx - \omega t)\,\hat{\mathbf{z}} \end{cases}$$

$$\nabla \cdot \mathbf{B} = 0 \Rightarrow \nabla \cdot \mathbf{B} = \partial_z kA_0 \cos(kx - \omega t) = 0$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \Rightarrow \begin{cases} \nabla \times \mathbf{B} = -\partial_x kA_0 \cos(kx - \omega t)\,\hat{\mathbf{y}} = k^2 A_0 \sin(kx - \omega t)\,\hat{\mathbf{y}} \\ \dfrac{\partial \mathbf{E}}{\partial t} = \omega^2 A_0 \sin(kx - \omega t)\,\hat{\mathbf{y}} \end{cases}$$

$$k^2 = \mu_0 \varepsilon_0 \omega^2 \Rightarrow \omega = \frac{k}{\sqrt{\mu_0 \varepsilon_0}} = ck$$



Problem 10.11

(a) Suppose the wire in Ex. 10.2 carries a linearly increasing current

$$I(t) = kt,$$

for $t > 0$. Find the electric and magnetic fields generated.

It's electically neutral $\Rightarrow V(\mathbf{r},t) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \nabla \frac{\rho(\mathbf{r}', t_t)}{\mathscr{r}}\, d\tau = 0$

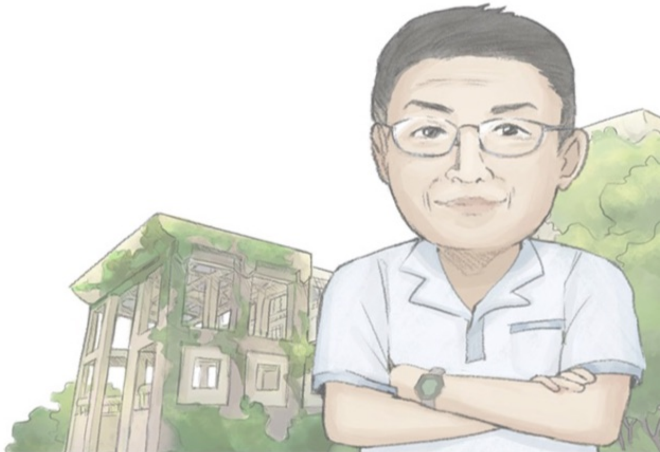
For $t < s/c$, $\mathbf{A} = 0$ For $t > s/c$:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{\mathscr{r}}\, d\tau' = \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\int_{-\sqrt{(ct)^2-s^2}}^{\sqrt{(ct)^2-s^2}} \frac{kt_r}{\sqrt{s^2+z'^2}}\, dz' = \hat{\mathbf{z}}\left(\frac{\mu_0 k}{4\pi}\right)2\int_0^{\sqrt{(ct)^2-s^2}} \frac{t-\sqrt{s^2+z'^2}/c}{\sqrt{s^2+z'^2}}\, dz'$$

$$= \hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left[t\int_0^{\sqrt{(ct)^2-s^2}} \frac{1}{\sqrt{s^2+z'^2}}\, dz' - \frac{1}{c}\int_0^{\sqrt{(ct)^2-s^2}} dz'\right]$$

$$= \hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left[t\ln\left(\sqrt{s^2+z'^2}+z'\right)\Big|_0^{\sqrt{(ct)^2-s^2}} - \frac{\sqrt{(ct)^2-s^2}}{c}\right]$$

$$= \hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left(t\ln\frac{ct+\sqrt{(ct)^2-s^2}}{s} - \frac{\sqrt{(ct)^2-s^2}}{c}\right)$$



Problem 10.11

(a) Suppose the wire in Ex. 10.2 carries a linearly increasing current

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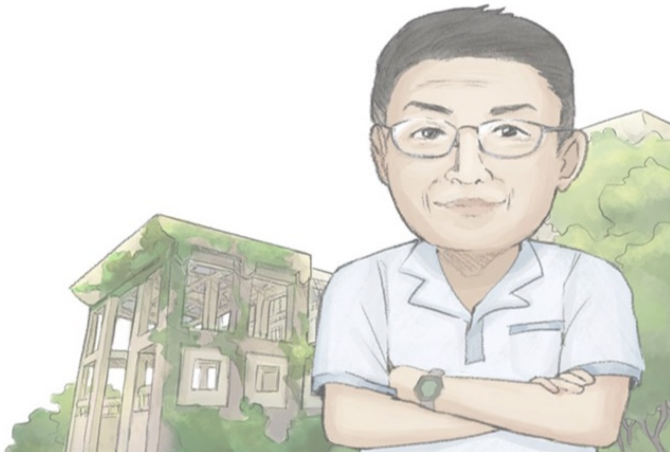
for $t > 0$. Find the electric and magnetic fields generated.

$$\mathbf{A} = \hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left(t\ln\frac{ct+\sqrt{(ct)^2-s^2}}{s} - \frac{\sqrt{(ct)^2-s^2}}{c} \right)$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left[\ln\frac{ct+\sqrt{(ct)^2-s^2}}{s} + \frac{st}{ct+\sqrt{(ct)^2-s^2}}\frac{1}{s}\left(c + \frac{c^2 t}{\sqrt{(ct)^2-s^2}} \right) - \frac{c^2 t}{c\sqrt{(ct)^2-s^2}} \right]$$

$$= -\hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left[\ln\frac{ct+\sqrt{(ct)^2-s^2}}{s} + \frac{ct}{ct+\sqrt{(ct)^2-s^2}}\left(\frac{\sqrt{(ct)^2-s^2}+ct}{\sqrt{(ct)^2-s^2}} \right) - \frac{ct}{\sqrt{(ct)^2-s^2}} \right]$$

$$= -\hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\ln\frac{ct+\sqrt{(ct)^2-s^2}}{s} \quad \text{or} \quad = 0\left(t < s/c\right)$$



Problem 10.11

(a) Suppose the wire in Ex. 10.2 carries a linearly increasing current

$$I(t) = kt,$$

for $t > 0$. Find the electric and magnetic fields generated.

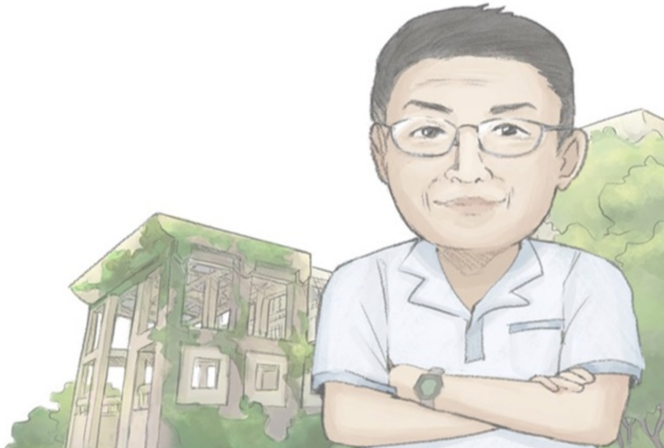
$$\mathbf{A} = \hat{\mathbf{z}}\frac{\mu_0 k}{2\pi}\left(t\ln\frac{ct+\sqrt{(ct)^2-s^2}}{s} - \frac{\sqrt{(ct)^2-s^2}}{c}\right)$$

$$\mathbf{B} = \nabla\times\mathbf{A} = -\partial_s A_z\hat{\phi} = -\hat{\phi}\frac{\mu_0 k}{2\pi}\left[t\frac{s}{ct+\sqrt{(ct)^2-s^2}}\left(-\frac{ct+\sqrt{(ct)^2-s^2}}{s^2} - \frac{s}{s\sqrt{(ct)^2-s^2}}\right) + \frac{s}{c\sqrt{(ct)^2-s^2}}\right]$$

$$= -\hat{\phi}\frac{\mu_0 k}{2\pi}\left[t\frac{s}{ct+\sqrt{(ct)^2-s^2}}\left(-\frac{ct\sqrt{(ct)^2-s^2}+(ct)^2-s^2+s^2}{s^2\sqrt{(ct)^2-s^2}}\right) + \frac{s}{c\sqrt{(ct)^2-s^2}}\right]$$

$$= -\hat{\phi}\frac{\mu_0 k}{2\pi}\left[t\frac{-ct}{s\sqrt{(ct)^2-s^2}} + \frac{s}{c\sqrt{(ct)^2-s^2}}\right]$$

$$= -\hat{\phi}\frac{\mu_0 k}{2\pi}\frac{-(ct)^2+s^2}{sc\sqrt{(ct)^2-s^2}} = \frac{\mu_0 k}{2\pi sc}\sqrt{(ct)^2-s^2}\,\hat{\phi} \quad \text{or} \quad = 0\,(t < s/c)$$



Problem 10.11

(b) Do the same for the case of a sudden burst of current:

$$I(t) = q_0 \delta(t).$$

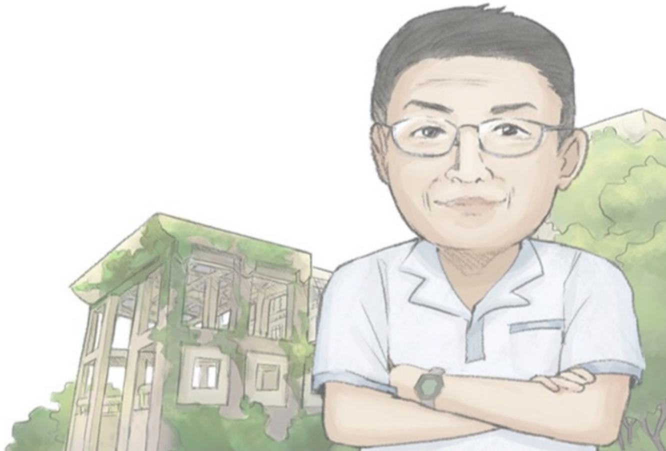
$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}',t_r)}{\mathscr{r}}\,d\tau' = \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}\frac{q_0\delta(t_r)}{\sqrt{s^2+z'^2}}\,dz' = \hat{\mathbf{z}}\frac{\mu_0 q_0}{4\pi}2\int_0^{\infty}\frac{\delta\left(t-\sqrt{s^2+z'^2}/c\right)}{\sqrt{s^2+z'^2}}\,dz'$$

$$= \hat{\mathbf{z}}\frac{\mu_0 q_0}{2\pi}\int_s^{\infty'}\frac{\delta\left(t-\sqrt{s^2+z'^2}/c\right)}{\sqrt{s^2+z'^2}}\left(\frac{c\sqrt{s^2+z'^2}}{z'}\,d\sqrt{s^2+z'^2}/c\right)$$

$$= \frac{\mu_0 q_0}{2\pi}\frac{c}{\sqrt{(ct)^2-s^2}}\hat{\mathbf{z}}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = -\partial_s A_z\hat{\phi} = -\frac{\mu_0 q_0}{2\pi}\frac{cs}{\left[(ct)^2-s^2\right]^{3/2}}\hat{\phi}\ \text{ or } = 0\,(t<s/c)$$

$$\mathbf{E} = -\nabla V - \frac{\partial\mathbf{A}}{\partial t} = \frac{\mu_0 q_0}{2\pi}\frac{c^3 t}{\left[(ct)^2-s^2\right]^{3/2}}\hat{\mathbf{z}}\ \text{ or } = 0\,(t<s/c)$$



Problem 10.15 A particle of charge q moves in a circle of radius a at constant angular velocity ω. (Assume that the circle lies in the xy plane, centered at the origin, and at time $t = 0$ the charge is at $(a, 0)$, on the positive x axis.) Find the Liénard-Wiechert potentials for points on the z axis.

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})}, \tag{10.46}$$

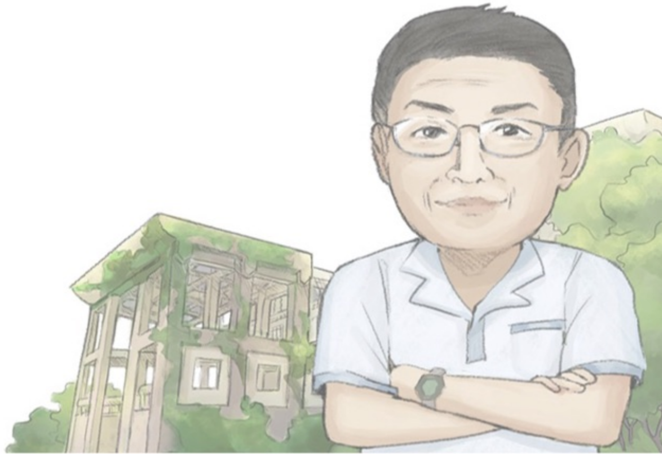
$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} = \frac{\mathbf{v}}{c^2} V(\mathbf{r}, t). \tag{10.47}$$

$$\mathbf{w}(t) = a\left[\cos(\omega t)\hat{\mathbf{x}} + \sin(\omega t)\hat{\mathbf{y}}\right] \Rightarrow \mathbf{v}(t) = \omega a\left[-\sin(\omega t)\hat{\mathbf{x}} + \cos(\omega t)\hat{\mathbf{y}}\right]$$

$$\boldsymbol{\mathscr{r}} = z\hat{\mathbf{z}} - \mathbf{w}(t_r) = z\hat{\mathbf{z}} - a\left[\cos(\omega t_r)\hat{\mathbf{x}} + \sin(\omega t_r)\hat{\mathbf{y}}\right], \quad \mathscr{r} = \sqrt{z^2 + a^2} = c(t - t_r) \Rightarrow t_r = t - \frac{\sqrt{z^2 + a^2}}{c}$$

$$\boldsymbol{\mathscr{r}} \cdot \mathbf{v}(t_r) = \omega a^2 \sin(\omega t_r)\cos(\omega t_r)(1-1) = 0$$

$$\begin{cases} V(\mathbf{r}, t) = \dfrac{1}{4\pi\varepsilon_0} \dfrac{qc}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q}{\sqrt{z^2 + a^2}} \\ \mathbf{A}(\mathbf{r}, t) = \dfrac{\mu_0}{4\pi} \dfrac{qc\mathbf{v}}{(\mathscr{r}c - \boldsymbol{\mathscr{r}} \cdot \mathbf{v})} = \dfrac{\mu_0}{4\pi} \dfrac{q\omega a}{\sqrt{z^2 + a^2}}\left[-\sin(\omega t_r)\hat{\mathbf{x}} + \cos(\omega t_r)\hat{\mathbf{y}}\right] = \dfrac{1}{4\pi\varepsilon_0} \dfrac{q\omega a}{c^2\sqrt{z^2 + a^2}}\left[-\sin(\omega t_r)\hat{\mathbf{x}} + \cos(\omega t_r)\hat{\mathbf{y}}\right] \\ t_r = t - \dfrac{\sqrt{z^2 + a^2}}{c} \end{cases}$$



$$V(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}}, \qquad (10.49)$$

Problem 10.16 Show that the scalar potential of a point charge moving with constant velocity (Eq. 10.49) can be written more simply as

$$V(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\frac{q}{R\sqrt{1-v^2\sin^2\theta/c^2}}, \qquad (10.51)$$

where $\mathbf{R}\equiv\mathbf{r}-\mathbf{v}t$ is the vector from the *present (!)* position of the particle to the field point $\mathbf{r}$, and θ is the angle between $\mathbf{R}$ and $\mathbf{v}$ (Fig. 10.9). Note that for nonrelativistic velocities ($v^2 \ll c^2$),

$$V(\mathbf{r},t)\approx\frac{1}{4\pi\epsilon_0}\frac{q}{R}.$$

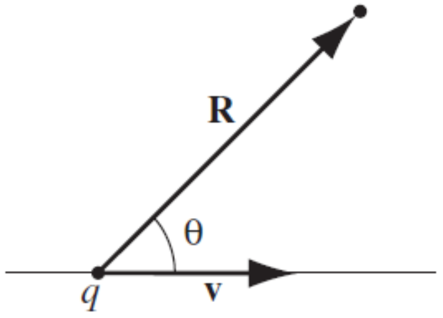
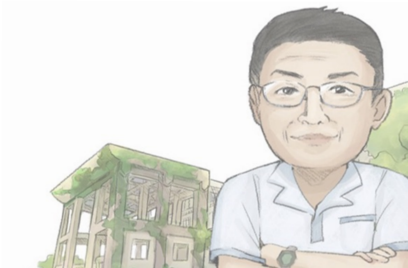


FIGURE 10.9

$$\begin{aligned}
&(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)\\
&=\left[c^2t-(\mathbf{R}+\mathbf{v}t)\cdot\mathbf{v}\right]^2+(c^2-v^2)\left[(\mathbf{R}+\mathbf{v}t)\cdot(\mathbf{R}+\mathbf{v}t)-c^2t^2\right]\\
&=(c^2t-Rv\cos\theta-v^2t)^2+(c^2-v^2)(R^2+2Rvt\cos\theta+v^2t^2-c^2t^2)\\
&=\left[(c^2-v^2)t-Rv\cos\theta\right]^2+(c^2-v^2)(R^2+2Rvt\cos\theta)-\left[(c^2-v^2)t\right]^2\\
&=\left[(c^2-v^2)t\right]^2-2(c^2-v^2)tRv\cos\theta+(Rv\cos\theta)^2\\
&\quad+(c^2-v^2)(R^2+2Rvt\cos\theta)-\left[(c^2-v^2)t\right]^2\\
&=R^2\left[(v\cos\theta)^2+c^2-v^2\right]
\end{aligned}$$

$$\begin{aligned}
V(\mathbf{r},t)&=\frac{1}{4\pi\varepsilon_0}\frac{qc}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}}\\
&=\frac{1}{4\pi\varepsilon_0}\frac{qc}{\sqrt{R^2\left[(v\cos\theta)^2+c^2-v^2\right]}}\\
&=\frac{1}{4\pi\varepsilon_0}\frac{q}{R\sqrt{\left[1+(v\cos\theta/c)^2-(v/c)^2\right]}}\\
&=\frac{1}{4\pi\varepsilon_0}\frac{q}{R\sqrt{\left[1-(v\sin\theta/c)^2\right]}}
\end{aligned}$$



$$V(\mathbf{r},t)=\frac{1}{4\pi\epsilon_0}\frac{qc}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}}, \tag{10.49}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\frac{qc\mathbf{v}}{\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{v})^2+(c^2-v^2)(r^2-c^2t^2)}}. \tag{10.50}$$

Problem 10.27 Check that the potentials of a point charge moving at constant velocity (Eqs. 10.49 and 10.50) satisfy the Lorenz gauge condition (Eq. 10.12).

$$\boxed{\nabla\cdot\mathbf{A}=-\mu_0\epsilon_0\frac{\partial V}{\partial t}.} \tag{10.12}$$

$$\nabla\cdot\mathbf{A}=\frac{\mu_0}{4\pi}qc\left\{\left[\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)\right]^{-1/2}\underbrace{\nabla\cdot\mathbf{v}}_{=0}+\mathbf{v}\cdot\nabla\left[\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)\right]^{-1/2}\right\}$$

$$=\frac{\mu_0}{4\pi}qc\mathbf{v}\cdot\left\{\left(-\frac{1}{2}\right)\left[\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)\right]^{-3/2}\left[2\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)\nabla\left(-\mathbf{r}\cdot\mathbf{v}\right)+\left(c^2-v^2\right)\nabla\left(r^2\right)\right]\right\}$$

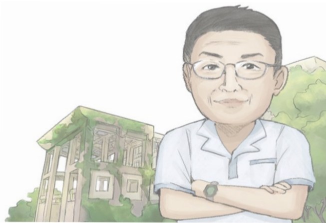
$$=\frac{\mu_0}{8\pi}\frac{qc}{\left(\sqrt{\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)}\right)^3}\mathbf{v}\cdot\left[2\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)\nabla\left(\mathbf{r}\cdot\mathbf{v}\right)-\left(c^2-v^2\right)\nabla\left(r^2\right)\right]$$

$$\nabla\cdot\mathbf{A}=\frac{\mu_0}{8\pi}\frac{qc}{\left(\sqrt{\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)}\right)^3}\mathbf{v}\cdot\left[2\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)\mathbf{v}-\left(c^2-v^2\right)2\mathbf{r}\right]$$

$$\boxed{\begin{aligned}&\nabla\left(\mathbf{r}\cdot\mathbf{v}\right)=\underbrace{\left(\mathbf{r}\cdot\nabla\right)\mathbf{v}}_{=0}+\left(\mathbf{v}\cdot\nabla\right)\mathbf{r}+\mathbf{r}\times\underbrace{\left(\nabla\times\mathbf{v}\right)}_{=0}+\mathbf{v}\times\underbrace{\left(\nabla\times\mathbf{r}\right)}_{=0}=\mathbf{v}\\&\nabla\left(r^2\right)=\nabla\left(\mathbf{r}\cdot\mathbf{r}\right)=2\left(\mathbf{r}\cdot\nabla\right)\mathbf{r}+2\mathbf{r}\times\underbrace{\left(\nabla\times\mathbf{r}\right)}_{=0}=2\mathbf{r}\end{aligned}}$$

$$=\frac{\mu_0 qc}{4\pi}\frac{\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)v^2-\left(c^2-v^2\right)\mathbf{v}\cdot\mathbf{r}}{\left(\sqrt{\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)}\right)^3}$$

$$=\frac{\mu_0 qc^3}{4\pi}\frac{v^2t-\mathbf{v}\cdot\mathbf{r}}{\left(\sqrt{\left(c^2t-\mathbf{r}\cdot\mathbf{v}\right)^2+\left(c^2-v^2\right)\left(r^2-c^2t^2\right)}\right)^3}$$



$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{qc}{\sqrt{(c^2 t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}, \quad (10.49)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\mathbf{v}}{\sqrt{(c^2 t - \mathbf{r}\cdot\mathbf{v})^2 + (c^2 - v^2)(r^2 - c^2 t^2)}}. \quad (10.50)$$

$$\boxed{\nabla \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}.} \quad (10.12)$$

Problem 10.27 Check that the potentials of a point charge moving at constant velocity (Eqs. 10.49 and 10.50) satisfy the Lorenz gauge condition (Eq. 10.12).

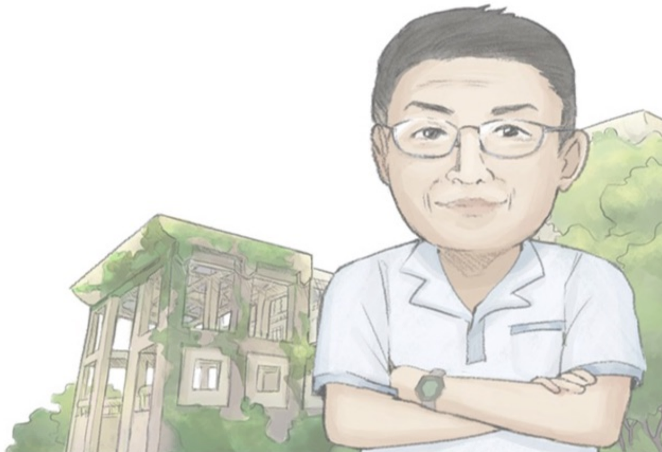
$$\frac{\partial V}{\partial t} = \frac{qc}{4\pi\varepsilon_0} \frac{\partial}{\partial t} \left[\left(c^2 t - \mathbf{r}\cdot\mathbf{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2 t^2\right) \right]^{-1/2}$$

$$= \frac{qc}{4\pi\varepsilon_0} \left(-\frac{1}{2}\right) \left[\left(c^2 t - \mathbf{r}\cdot\mathbf{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2 t^2\right) \right]^{-3/2} \left[2\left(c^2 t - \mathbf{r}\cdot\mathbf{v}\right)\left(c^2 - \underbrace{\frac{\partial \mathbf{r}}{\partial t}}_{=0}\cdot\mathbf{v}\right) + \left(c^2 - v^2\right)\left(2\mathbf{r}\cdot\underbrace{\frac{\partial \mathbf{r}}{\partial t}}_{=0} - 2c^2 t\right) \right]$$

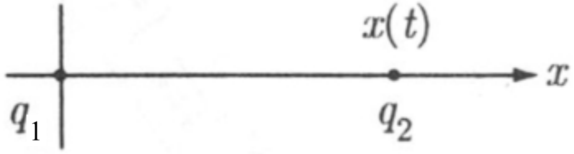
$$= -\frac{qc^3}{4\pi\varepsilon_0} \frac{v^2 t - \mathbf{r}\cdot\mathbf{v}}{\left(\sqrt{\left(c^2 t - \mathbf{r}\cdot\mathbf{v}\right)^2 + \left(c^2 + v^2\right)\left(r^2 - c^2 t^2\right)}\right)^3}$$

$$\nabla \cdot \mathbf{A} = \frac{\mu_0 q c^3}{4\pi} \frac{v^2 t - \mathbf{v}\cdot\mathbf{r}}{\left(\sqrt{\left(c^2 t - \mathbf{r}\cdot\mathbf{v}\right)^2 + \left(c^2 - v^2\right)\left(r^2 - c^2 t^2\right)}\right)^3}$$

$$\Rightarrow -\mu_0 \varepsilon_0 \frac{\partial V}{\partial t} = \nabla \cdot \mathbf{A}$$



Problem 10.28 One particle, of charge q_1, is held at rest at the origin. Another particle, of charge q_2, approaches along the x axis, in hyperbolic motion:



$$x(t) = \sqrt{b^2 + (ct)^2};$$

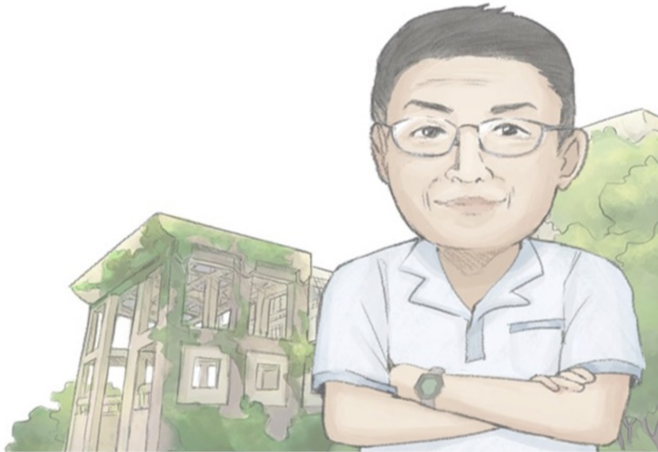
it reaches the closest point, b, at time $t = 0$, and then returns out to infinity.

(a) What is the force F_2 on q_2 (due to q_1) at time t?

$$\mathbf{F}_2 = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{1}{b^2 + c^2 t^2} \hat{\mathbf{x}}$$

(b) What total impulse $\left(I_2 = \int_{-\infty}^{\infty} F_2 dt\right)$ is delivered to q_2 by q_1?

$$I_2 = \int_{-\infty}^{\infty} F_2 dt = \int_{-\infty}^{\infty} \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{1}{b^2 + c^2 t^2} dt = \frac{q_1 q_2}{4\pi\varepsilon_0} \left[\frac{1}{bc} \tan^{-1} \frac{ct}{b}\right]_{-\infty}^{\infty} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{1}{bc} \left[\frac{\pi}{2} - \left(\frac{\pi}{2}\right)\right] = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\pi}{bc}$$



Problem 10.28

(c) What is the force F_1 on q_1 (due to q_2) at time t?

$$c(t-t_r)=\left|\mathbf{r}-\mathbf{w}(t_r)\right|=x(t_r)=\sqrt{b^2+c^2t_r^2}\Rightarrow t_r=\frac{c^2t^2-b^2}{2c^2t}$$

$$\begin{cases} x(t)=\sqrt{b^2+c^2t^2} \\ v(t)=\dfrac{c^2t}{\sqrt{b^2+c^2t^2}}=\dfrac{c^2t}{x} \end{cases}\Rightarrow\begin{cases} x(t_r)=c(t-t_r)=\dfrac{c^2t^2+b^2}{2ct} \\ v(t_r)=\dfrac{c^2t_r}{x(t_r)}=\dfrac{c^2t^2-b^2}{2t}\dfrac{2ct}{c^2t^2+b^2}=c\left(\dfrac{c^2t^2-b^2}{c^2t^2+b^2}\right) \end{cases}$$

$$\mathbf{E}=-\frac{q_2}{4\pi\varepsilon_0}\frac{1}{x(t_r)^2}\left(\frac{c-v}{c+v}\right)\hat{\mathbf{x}}=-\frac{q_2}{4\pi\varepsilon_0}\frac{1}{x(t_r)^2}\left(1-\frac{c^2t^2-b^2}{c^2t^2+b^2}\right)\left(1+\frac{c^2t^2-b^2}{c^2t^2+b^2}\right)^{-1}\hat{\mathbf{x}}$$

$$=-\frac{q_2}{4\pi\varepsilon_0}\left(\frac{c^2t^2+b^2}{2ct}\right)^{-2}\frac{2b^2}{c^2t^2+b^2}\frac{c^2t^2+b^2}{2c^2t^2}\hat{\mathbf{x}}$$

$$=-\frac{q_2}{4\pi\varepsilon_0}\frac{4c^2t^2}{\left(c^2t^2+b^2\right)^2}\frac{b^2}{c^2t^2}\hat{\mathbf{x}}$$

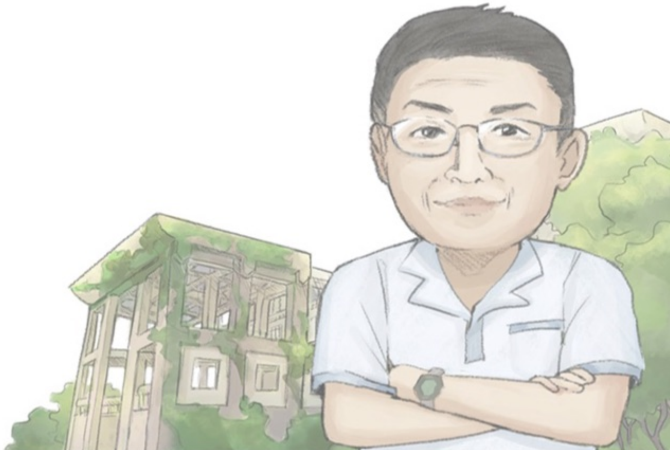
$$=-\frac{q_2}{\pi\varepsilon_0}\frac{b^2}{\left(c^2t^2+b^2\right)^2}\hat{\mathbf{x}}$$

Problem 10.20 Suppose a point charge q is constrained to move along the x axis. Show that the fields at points on the axis to the *right* of the charge are given by

$$\mathbf{E}=\frac{q}{4\pi\epsilon_0}\frac{1}{\mathscr{r}^2}\left(\frac{c+v}{c-v}\right)\hat{\mathbf{x}},\quad \mathbf{B}=\mathbf{0}.$$

(Do not assume v is constant!) What are the fields on the axis to the *left* of the charge?

$$\mathbf{E}=\begin{cases} \dfrac{q_2}{4\pi\varepsilon_0}\dfrac{1}{\mathscr{r}^2}\left(\dfrac{c+v}{c-v}\right)\hat{\mathbf{x}} & \text{, to the right of the charge} \\ -\dfrac{q_2}{4\pi\varepsilon_0}\dfrac{1}{\mathscr{r}^2}\left(\dfrac{c-v}{c+v}\right)\hat{\mathbf{x}} & \text{, to the left of the charge} \end{cases}$$



Problem 10.28

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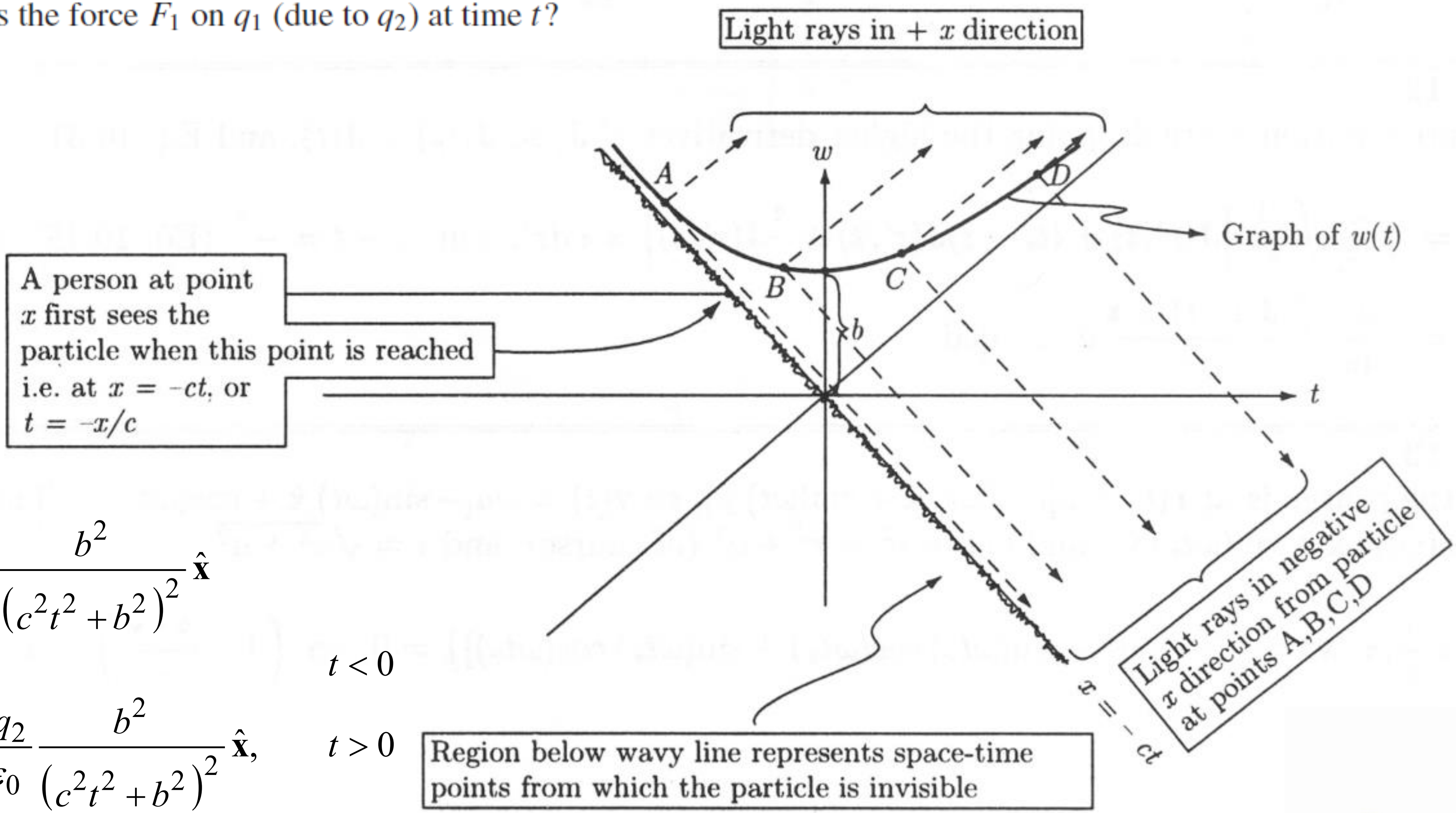
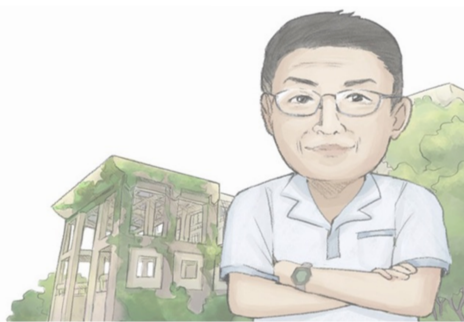


$$\mathbf{E} = -\frac{q_2}{\pi\varepsilon_0}\frac{b^2}{\left(c^2t^2+b^2\right)^2}\hat{\mathbf{x}}$$

$$\mathbf{F}_1 = \begin{cases} 0, & t<0 \\ -\dfrac{q_1 q_2}{\pi\varepsilon_0}\dfrac{b^2}{\left(c^2t^2+b^2\right)^2}\hat{\mathbf{x}}, & t>0 \end{cases}$$



Problem 10.28

(d) What total impulse $\left(I_1 = \int_{-\infty}^{\infty} F_1 dt\right)$ is delivered to q_1 by q_2? [*Hint:* It might help to review Prob. 10.17 before doing this integral. *Answer:* $I_2 = -I_1 = q_1 q_2 / 4\epsilon_0 bc$]

$$I_2 = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\pi}{bc}$$

$$I_1 = -\frac{q_1 q_2}{\pi\varepsilon_0} b^2 \int_0^\infty \frac{1}{\left(c^2t^2 + b^2\right)^2} dt = -\frac{q_1 q_2}{\pi\varepsilon_0} b^2 \frac{1}{2b^2}\left(\frac{t}{b^2 + c^2t^2}\bigg|_0^\infty + \int_0^\infty \frac{1}{b^2 + c^2t^2} dt \right) = -\frac{q_1 q_2}{2\pi\varepsilon_0}\left(\frac{1}{bc}\frac{\pi}{2}\right) = -\frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\pi}{bc}$$

$$\Rightarrow I_2 = -I_1 = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\pi}{bc}$$