國立清華大學

Electromagnetism

Introduction to Electrodynamics 4th David J. Griffiths

Chap.12 Electrodynamics and Relativity

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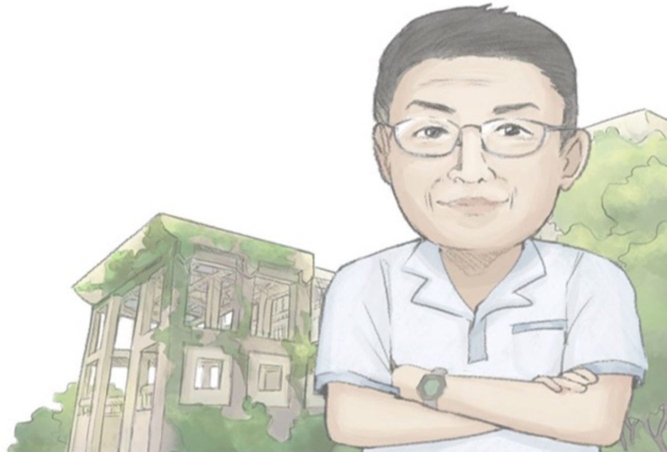
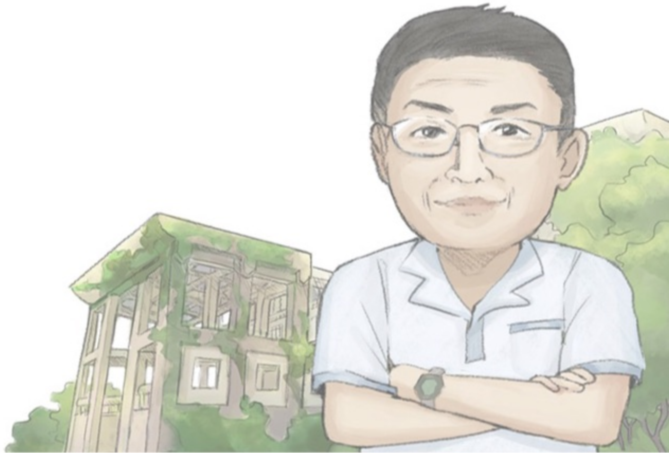


Exercise List

3, 4, 6, 25, 31, 34, 39, 47



Problem 12.3

(a) What's the percent error introduced when you use Galileo's rule, instead of Einstein's, with $v_{AB} = 5$ mi/h and $v_{BC} = 60$ mi/h?

$$v_{Galileo} = v_G = v_{AC} = v_{AB} + v_{BC} \qquad v_{Einstein} = v_E = v_{AC} = \frac{v_{AB} + v_{BC}}{1 + \left(v_{AB} v_{BC} / c^2\right)} \qquad c = 6.7 \times 10^8 \text{ mi/h}$$

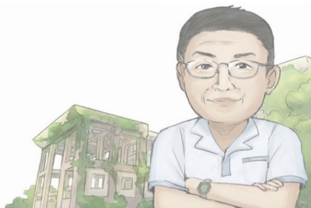
$$\frac{v_G - v_E}{v_G} = 1 - \frac{v_E}{v_G} = \frac{\left(v_{AB} v_{BC} / c^2\right)}{1 + \left(v_{AB} v_{BC} / c^2\right)} = \frac{v_{AB} v_{BC}}{c^2 + v_{AB} v_{BC}} = \frac{5 \times 60}{\left(6.7 \times 10^8\right)^2 + 5 \times 60} \approx \frac{5 \times 60}{\left(6.7 \times 10^8\right)^2} \approx 10^{-16}$$

(b) Suppose you could run at half the speed of light down the corridor of a train going three-quarters the speed of light. What would your speed be relative to the ground?

$$v_E = v_{AC} = \frac{v_{AB} + v_{BC}}{1 + \left(v_{AB} v_{BC} / c^2\right)} = \frac{c/2 + 3c/4}{1 + \left[(c/2)(3c/4)/c^2\right]} = \frac{10}{11} c$$

(c) Prove, using Eq. 12.3, that if $v_{AB} < c$ and $v_{BC} < c$ then $v_{AC} < c$. Interpret this result.

$$\frac{v_{AC}}{c} = \frac{v_{AB}/c + v_{BC}/c}{1 + (v_{AB}/c)(v_{BC}/c)} = \frac{\alpha_{AB} + \alpha_{BC}}{1 + \alpha_{AB}\alpha_{BC}} = 1 - \frac{1 + \alpha_{AB}\alpha_{BC} - \left[\alpha_{AB} + \alpha_{BC}\right]}{1 + \alpha_{AB}\alpha_{BC}} = 1 - \underbrace{\frac{(1 - \alpha_{AB})(1 - \alpha_{BC})}{1 + \alpha_{AB}\alpha_{BC}}}_{positive} \Rightarrow v_{AC} < c$$



Problem 12.4 As the outlaws escape in their getaway car, which goes $\frac{3}{4}c$, the police officer fires a bullet from the pursuit car, which only goes $\frac{1}{2}c$ (Fig. 12.3). The muzzle velocity of the bullet (relative to the gun) is $\frac{1}{3}c$. Does the bullet reach its target (a) according to Galileo, (b) according to Einstein?

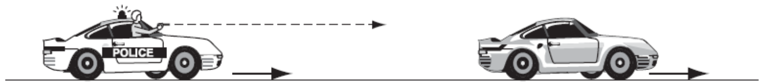
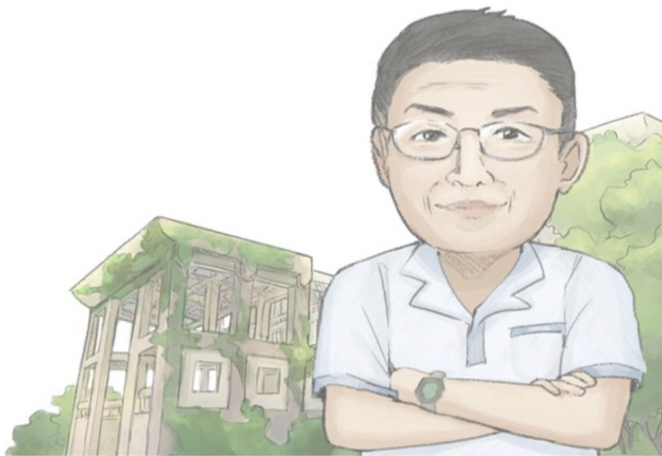


FIGURE 12.3

$$v_G = v_{AC} = v_{AB} + v_{BC} = \frac{1}{2}c + \frac{1}{3}c = \frac{5}{6}c > \frac{3}{4}c$$

$$v_E = v_{AC} = \frac{v_{AB} + v_{BC}}{1 + \left(v_{AB}v_{BC} / c^2\right)} = \frac{c/2 + c/3}{1 + \left[(c/2)(c/3)/c^2\right]} = \frac{5}{7}c < \frac{3}{4}c$$



Problem 12.6 Every 2 years, more or less, *The New York Times* publishes an article in which some astronomer claims to have found an object traveling faster than the speed of light. Many of these reports result from a failure to distinguish what is *seen* from what is *observed*—that is, from a failure to account for light travel time. Here's an example: A star is traveling with speed v at an angle θ to the line of sight (Fig. 12.6). What is its apparent speed across the sky? (Suppose the light signal from b reaches the earth at a time Δt after the signal from a, and the star has meanwhile advanced a distance Δs across the celestial sphere; by "apparent speed," I mean $\Delta s/\Delta t$.) What angle θ gives the maximum apparent speed? Show that the apparent speed can be much greater than c, even if v itself is less than c.

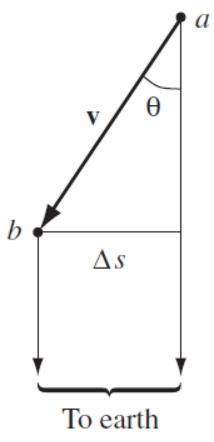


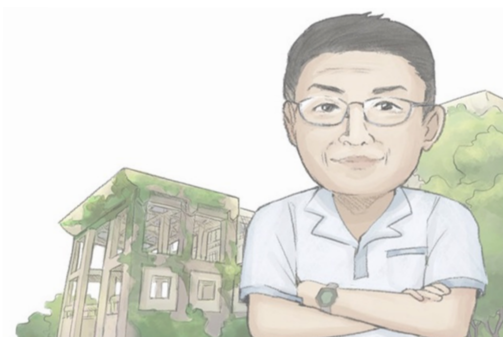
FIGURE 12.6

Emitting light from $\begin{Bmatrix} a \\ b \end{Bmatrix}$ at time $\begin{Bmatrix} t_a \\ t_b \end{Bmatrix}$; arrives at earth at time $\begin{Bmatrix} t_a' = t_a + d_a/c \\ t_b' = t_b + d_b/c \end{Bmatrix}$ $\quad t_b - t_a = \dfrac{\Delta s}{v\sin\theta}; \; d_a - d_b = \dfrac{\Delta s\cos\theta}{\sin\theta}$

$$\text{Apparent speed } v' = \frac{\Delta s}{t_b' - t_a'} = \frac{\Delta s}{(t_b + d_b/c) - (t_a + d_a/c)} = \frac{\Delta s}{(t_b - t_a) - (d_a - d_b)/c} = \frac{\Delta s}{\Delta s/v\sin\theta - \Delta s\cos\theta/c\sin\theta} = \frac{v\sin\theta}{1 - v\cos\theta/c}$$

$$\frac{\partial v'}{\partial\theta} = \frac{(1 - v\cos\theta/c)v\cos\theta - v\sin\theta(v\sin\theta/c)}{(1 - v\cos\theta/c)^2} = \frac{v\cos\theta - v^2\cos^2\theta/c - v^2\sin^2\theta/c}{(1 - v\cos\theta/c)^2} = \frac{v(\cos\theta - v/c)}{(1 - v\cos\theta/c)^2} \Rightarrow \theta_{\max} = \cos^{-1}\left(\frac{v}{c}\right)$$

$$v'_{\max} = \frac{v\sqrt{1-(v/c)^2}}{1-(v/c)^2} = \frac{v}{\sqrt{1-(v/c)^2}} \quad \text{as } v \to c \Rightarrow v'_{\max} \to \infty \text{ even though } v < c$$



Problem 12.25 A car is traveling along the 45° line in $\mathcal{S}$ (Fig. 12.25), at (ordinary) speed $(2/\sqrt{5})c$.

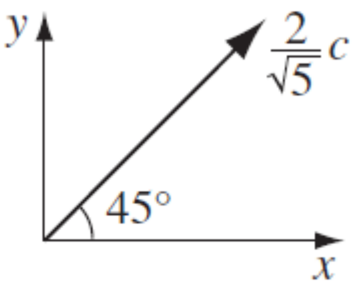


FIGURE 12.25

(a) Find the components u_x and u_y of the (ordinary) velocity.

(b) Find the components η_x and η_y of the proper velocity.

(c) Find the zeroth component of the 4-velocity, η^0.

System $\bar{\mathcal{S}}$ is moving in the x direction with (ordinary) speed $\sqrt{2/5}\,c$, relative to $\mathcal{S}$. By using the appropriate transformation laws:

(d) Find the (ordinary) velocity components $\bar{u}_x$ and $\bar{u}_y$ in $\bar{\mathcal{S}}$.

$$\eta = \frac{1}{\sqrt{1-u^2/c^2}}\mathbf{u}. \tag{12.40}$$

$$\left.\begin{aligned} \bar{u}_x &= \frac{d\bar{x}}{d\bar{t}} = \frac{u_x - v}{(1 - vu_x/c^2)}, \\ \bar{u}_y &= \frac{d\bar{y}}{d\bar{t}} = \frac{u_y}{\gamma(1 - vu_x/c^2)}, \\ \bar{u}_z &= \frac{d\bar{z}}{d\bar{t}} = \frac{u_z}{\gamma(1 - vu_x/c^2)}. \end{aligned}\right\} \tag{12.45}$$

(a) $u_x = u_y = u\cos 45^\circ = \frac{1}{\sqrt{2}}\frac{2}{\sqrt{5}}c = \sqrt{\frac{2}{5}}c$

(b) $\gamma = \frac{1}{\sqrt{1-(u/c)^2}} = \frac{1}{\sqrt{1-4/5}} = \sqrt{5}$

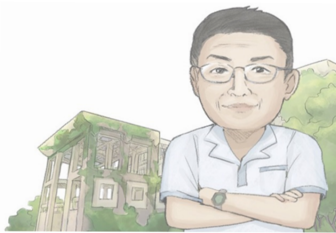
$$\eta = \frac{u}{\sqrt{1-(u/c)^2}} \Rightarrow \eta_x = \eta_y = \sqrt{2}c$$

(c) $\eta^0 = \gamma c = \sqrt{5}c$

(d)

$$\bar{u}_x = \frac{u_x - v_{\bar{S}}}{1-u_x v_{\bar{S}}/c^2} = \frac{c\sqrt{2/5} - c\sqrt{2/5}}{1-\left(c\sqrt{2/5}\right)\left(c\sqrt{2/5}\right)/c^2} = 0$$

$$\bar{u}_y = \frac{1}{\gamma}\frac{u_x}{1-u_x v_{\bar{S}}/c^2} = \sqrt{1-\frac{2}{5}}\frac{c\sqrt{2/5}}{1-\left(c\sqrt{2/5}\right)\left(c\sqrt{2/5}\right)/c^2} = \frac{2}{3}c$$



Problem 12.25 A car is traveling along the 45° line in $\mathcal{S}$ (Fig. 12.25), at (ordinary) speed $(2/\sqrt{5})c$.

System $\bar{\mathcal{S}}$ is moving in the x direction with (ordinary) speed $\sqrt{2/5}\,c$, relative to $\mathcal{S}$. By using the appropriate transformation laws:

(e) Find the proper velocity components $\bar{\eta}_x$ and $\bar{\eta}_y$ in $\bar{\mathcal{S}}$.

(f) As a consistency check, verify that

$$\bar{\eta} = \frac{\bar{\mathbf{u}}}{\sqrt{1-\bar{u}^2/c^2}}.$$

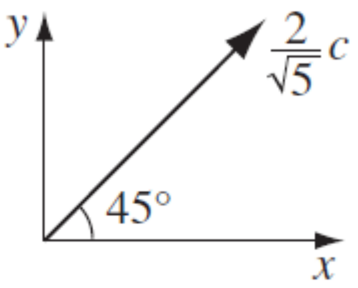


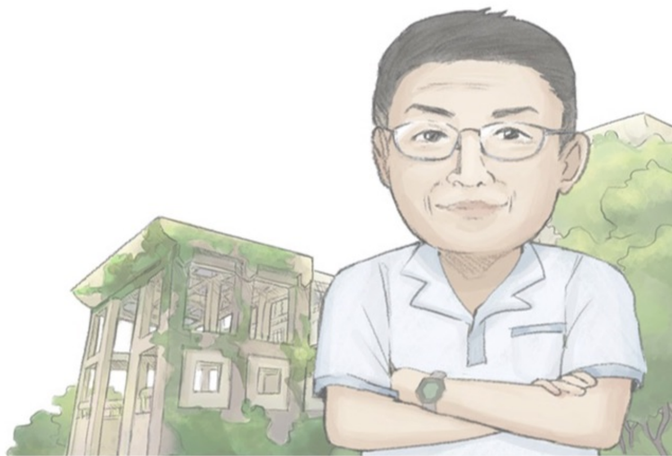
FIGURE 12.25

$$\left.\begin{aligned} \bar{\eta}^0 &= \gamma(\eta^0 - \beta\eta^1), \\ \bar{\eta}^1 &= \gamma(\eta^1 - \beta\eta^0), \\ \bar{\eta}^2 &= \eta^2, \\ \bar{\eta}^3 &= \eta^3. \end{aligned}\right\} \qquad (12.43)$$

(e) $\bar{\eta}_x = \gamma\left(\eta_x - \beta\eta^0\right) = \sqrt{\frac{3}{5}}\left(\sqrt{2}c - \sqrt{\frac{2}{5}}\sqrt{5}c\right) = 0$

$\bar{\eta}_y = \eta_y = \sqrt{2}c$

(f) $\dfrac{1}{\sqrt{1-(\bar{u}/c)^2}} = \dfrac{1}{\sqrt{1-2/3}} = \sqrt{3} \Rightarrow \bar{\eta} = \sqrt{3}\bar{u} \Rightarrow \begin{cases} \bar{\eta}_x = \sqrt{3}\bar{u}_x = 0 \\ \bar{\eta}_y = \sqrt{3}\bar{u}_y = \sqrt{2}c \end{cases}$



Problem 12.31 Suppose you have a collection of particles, all moving in the x direction, with energies E_1, E_2, E_3, …and momenta p_1, p_2, p_3, …. Find the velocity of the **center of momentum** frame, in which the total momentum is zero.

$$E_{total} = E_1 + E_2 + \cdots, \quad p_{total} = p_1 + p_2 + \cdots$$

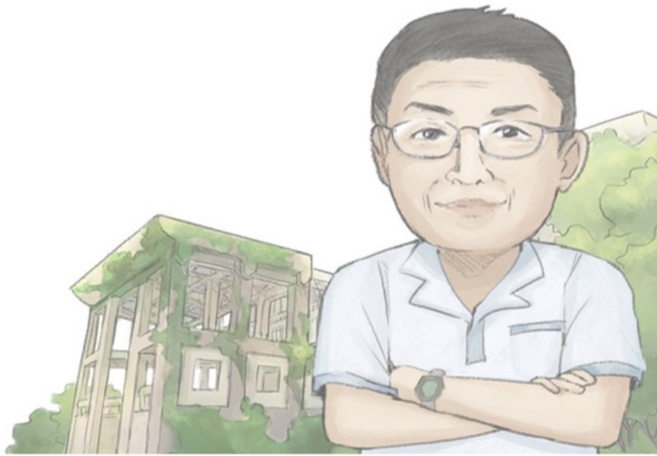
$$\overline{p}_{total} = \gamma\left(p_{total} - \frac{\beta E_{total}}{c} \right) = 0 \Rightarrow \beta = \frac{v}{c} = \frac{p_{total}c}{E_{total}}$$

$$\Rightarrow v = \frac{p_{total}c^2}{E_{total}} = \frac{c^2\left(p_1 + p_2 + \cdots\right)}{E_1 + E_2 + \cdots}$$

$$\left.\begin{aligned} \bar{\eta}^0 &= \gamma(\eta^0 - \beta\eta^1), \\ \bar{\eta}^1 &= \gamma(\eta^1 - \beta\eta^0), \\ \bar{\eta}^2 &= \eta^2, \\ \bar{\eta}^3 &= \eta^3. \end{aligned}\right\} \quad (12.43)$$

Einstein identified p^0c as **relativistic energy**:

$$\boxed{E \equiv \frac{mc^2}{\sqrt{1 - u^2/c^2}};} \quad (12.49)$$



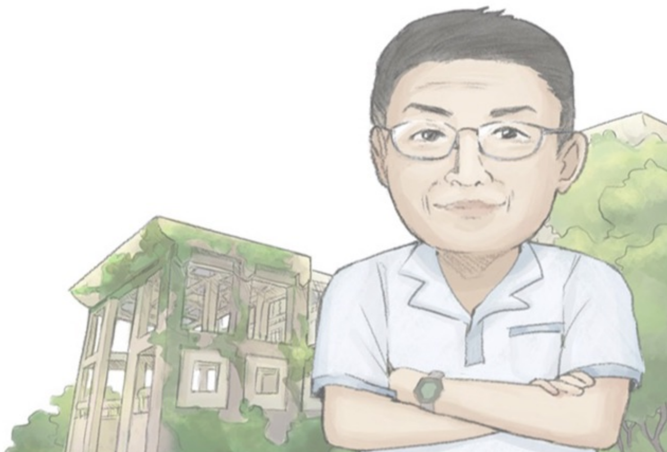
Problem 12.34 A neutral pion of (rest) mass m and (relativistic) momentum $p = \frac{3}{4}mc$ decays into two photons. One of the photons is emitted in the same direction as the original pion, and the other in the opposite direction. Find the (relativistic) energy of each photon.

$$E^2 = p^2c^2 + m^2c^4 = \left(\frac{3}{4}mc\right)^2 c^2 + m^2c^4 = \frac{25}{16}m^2c^4 \Rightarrow E = \frac{5}{4}mc^2$$

Conservation of energy: $\frac{5}{4}mc^2 = E_A + E_B$

Conservation of momentum: $p = \frac{3}{4}mc = p_A - p_B = \frac{E_A}{c} - \frac{E_B}{c}$

$$\Rightarrow \begin{cases} E_A = mc^2 \\ E_B = \frac{1}{4}mc^2 \end{cases}$$



Problem 12.39 Define **proper acceleration** in the obvious way:

$$d\tau = \sqrt{1 - u^2/c^2}\, dt. \tag{12.37}$$

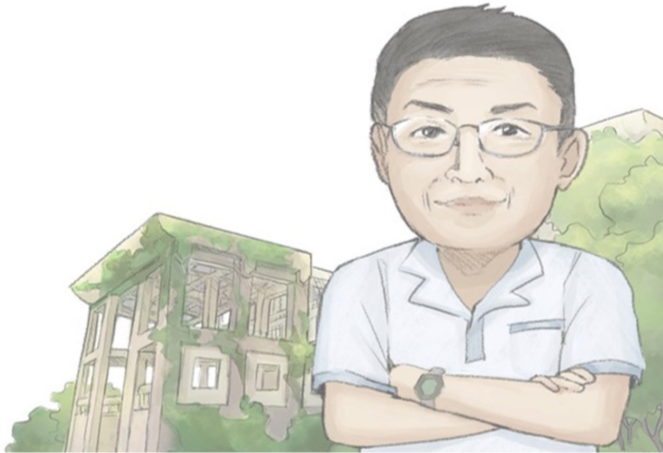
$$\alpha^\mu \equiv \frac{d\eta^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2}. \tag{12.75}$$

(a) Find α^0 and $\boldsymbol{\alpha}$ in terms of $\mathbf{u}$ and $\mathbf{a}$ (the ordinary acceleration).

$$\alpha^0 = \frac{d\eta_0}{d\tau} = \frac{d\eta_0}{dt}\frac{dt}{d\tau} = \left[\frac{d}{dt}\left(\frac{c}{\sqrt{1-u^2/c^2}}\right)\right]\frac{1}{\sqrt{1-u^2/c^2}} = \left[\left(-\frac{1}{2}\right)\frac{c\left(-2\mathbf{u}\cdot\mathbf{a}/c^2\right)}{\left(1-u^2/c^2\right)^{3/2}}\right]\frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{c}\frac{\mathbf{u}\cdot\mathbf{a}}{\left(1-u^2/c^2\right)^2}$$

$$\boldsymbol{\alpha} = \frac{d\boldsymbol{\eta}}{d\tau} = \frac{dt}{d\tau}\frac{d\boldsymbol{\eta}}{dt} = \frac{1}{\sqrt{1-u^2/c^2}}\frac{d}{dt}\left(\frac{\mathbf{u}}{\sqrt{1-u^2/c^2}}\right) = \frac{1}{\sqrt{1-u^2/c^2}}\left(\frac{\mathbf{a}}{\sqrt{1-u^2/c^2}} + \mathbf{u}\left(-\frac{1}{2}\right)\frac{c\left(-2\mathbf{u}\cdot\mathbf{a}/c^2\right)}{\left(1-u^2/c^2\right)^{3/2}}\right)$$

$$= \frac{1}{1-u^2/c^2}\left[\mathbf{a} + \frac{u\left(\mathbf{u}\cdot\mathbf{a}\right)}{c^2 - u^2}\right]$$



Problem 12.39

(b) Express $\alpha_\mu \alpha^\mu$ in terms of $\mathbf{u}$ and $\mathbf{a}$.

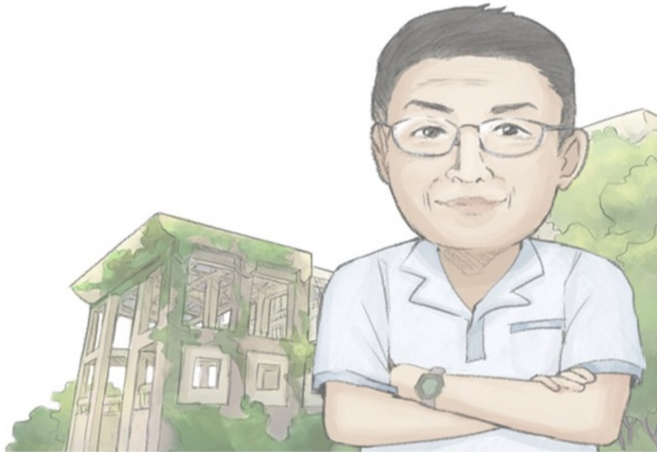
$$\alpha_\mu \alpha^\mu = -\left(\alpha^0\right)^2 + \boldsymbol{\alpha}\cdot\boldsymbol{\alpha} = -\frac{1}{c^2}\frac{(\mathbf{u}\cdot\mathbf{a})^2}{\left(1-u^2/c^2\right)^4} + \frac{1}{\left(1-u^2/c^2\right)^4}\left[\mathbf{a}\left(1-\frac{u^2}{c^2}\right)+\frac{1}{c^2}\mathbf{u}(\mathbf{u}\cdot\mathbf{a})\right]^2$$

$$= \frac{1}{\left(1-u^2/c^2\right)^4}\left\{-\frac{1}{c^2}(\mathbf{u}\cdot\mathbf{a})^2 + a^2\left(1-\frac{u^2}{c^2}\right)^2 + \frac{2}{c^2}\left(1-\frac{u^2}{c^2}\right)(\mathbf{u}\cdot\mathbf{a})^2 + \frac{1}{c^4}u^2(\mathbf{u}\cdot\mathbf{a})^2\right\}$$

$$= \frac{1}{\left(1-u^2/c^2\right)^4}\left\{a^2\left(1-\frac{u^2}{c^2}\right)^2 + \frac{(\mathbf{u}\cdot\mathbf{a})^2}{c^2}\left[-1+2\left(1-\frac{u^2}{c^2}\right)+\frac{u^2}{c^2}\right]\right\}$$

$$= \frac{1}{\left(1-u^2/c^2\right)^4}\left\{a^2\left(1-\frac{u^2}{c^2}\right)^2 + \frac{(\mathbf{u}\cdot\mathbf{a})^2}{c^2}\left(1-\frac{u^2}{c^2}\right)\right\}$$

$$= \frac{1}{\left(1-u^2/c^2\right)^2}\left\{a^2 + \frac{(\mathbf{u}\cdot\mathbf{a})^2}{c^2}\left(1-\frac{u^2}{c^2}\right)^{-1}\right\} = \frac{1}{\left(1-u^2/c^2\right)^2}\left[a^2 + \frac{(\mathbf{u}\cdot\mathbf{a})^2}{c^2-u^2}\right]$$



Problem 12.39

(c) Show that $\eta^\mu \alpha_\mu = 0$.

(d) Write the Minkowski version of Newton's second law, Eq. 12.68, in terms of α^μ. Evaluate the invariant product $K^\mu \eta_\mu$.

$$p^\mu \equiv m\eta^\mu, \quad (12.47)$$

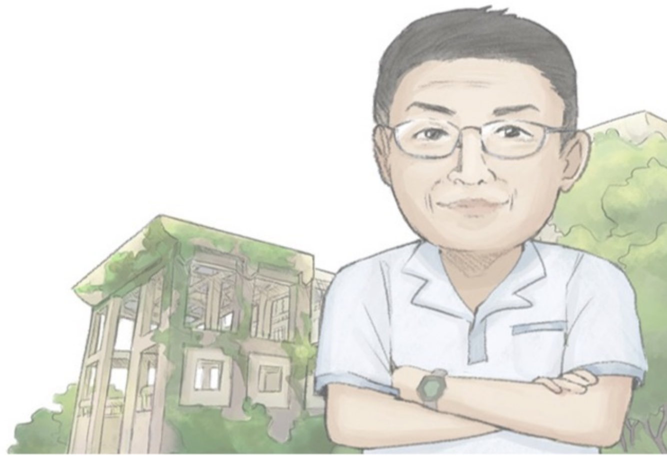
$$p^\mu p_\mu = -(p^0)^2 + (\mathbf{p}\cdot\mathbf{p}) = -m^2c^2, \quad (12.53)$$

(c)

$$\eta^\mu \eta_\mu = -c^2 \Rightarrow \frac{d}{d\tau}\left(\eta^\mu \eta_\mu\right) = \alpha^\mu \eta_\mu + \eta^\mu \alpha_\mu = 2\alpha^\mu \eta_\mu = 0 \Rightarrow \alpha^\mu \eta_\mu = 0$$

(d)

$$K^\mu = \frac{dp^\mu}{d\tau} = \frac{d}{d\tau}\left(m\eta^\mu\right) = m\alpha^\mu \Rightarrow K^\mu \eta_\mu = m\alpha^\mu \eta_\mu = 0$$



Problem 12.47

$$\bar{I} = \frac{1}{\gamma} I. \tag{12.108}$$

(a) Show that $(\mathbf{E} \cdot \mathbf{B})$ is relativistically invariant.

$$\bar{\mathbf{E}} \cdot \bar{\mathbf{B}} = \bar{E}_x \bar{B}_x + \bar{E}_y \bar{B}_y + \bar{E}_z \bar{B}_z = E_x B_x + \gamma^2 \left(E_y - vB_z\right)\left(B_y + \frac{v}{c^2} E_z\right) + \gamma^2 \left(E_z + vB_y\right)\left(B_z - \frac{v}{c^2} E_y\right)$$

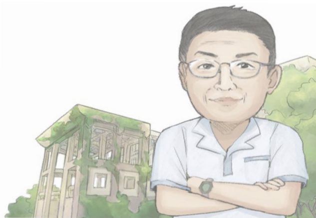
$$= E_x B_x + \gamma^2 \left(E_y B_y \cancel{- vB_y B_z} \cancel{+ \frac{v}{c^2} E_y E_z} - \frac{v^2}{c^2} B_z E_z + E_z B_z \cancel{+ vB_y B_z} \cancel{- \frac{v}{c^2} E_z E_y} - \frac{v^2}{c^2} B_y E_y \right)$$

$$= E_x B_x + \gamma^2 \left(E_y B_y + E_z B_z\right)\left(1 - \frac{v^2}{c^2}\right) = E_x B_x + E_y B_y + E_z B_z = \mathbf{E} \cdot \mathbf{B}$$

Like **E**, the component of **B** *parallel* to the motion is unchanged.

Here, then, is the complete set of transformation rules:

$$\begin{array}{lll} \bar{E}_x = E_x, & \bar{E}_y = \gamma(E_y - vB_z), & \bar{E}_z = \gamma(E_z + vB_y), \\ \bar{B}_x = B_x, & \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2} E_z\right), & \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2} E_y\right). \end{array} \tag{12.109}$$



Problem 12.47

(b) Show that $(E^2 - c^2B^2)$ is relativistically invariant.

$$\bar{E}^2 - c^2\bar{B}^2 = \left[E_x^2 + \gamma^2\left(E_y - vB_z\right)^2 + \gamma^2\left(E_z + vB_y\right)^2\right] - c^2\left[B_x^2 + \gamma^2\left(B_y + \frac{v}{c^2}E_z\right)^2 + \gamma^2\left(B_z - \frac{v}{c^2}E_y\right)^2\right]$$

$$= E_x^2 + \gamma^2\left(E_y^2 - 2vE_yB_z + v^2B_z^2\right) + \gamma^2\left(E_z^2 + 2vE_zB_y + v^2B_y^2\right)$$

$$-c^2B_x^2 - c^2\gamma^2\left(B_y^2 + 2\frac{v}{c^2}B_yE_z + \frac{v^2}{c^4}E_z^2\right) - c^2\gamma^2\left(B_z^2 - 2\frac{v}{c^2}B_zE_y + \frac{v^2}{c^4}E_y^2\right)$$

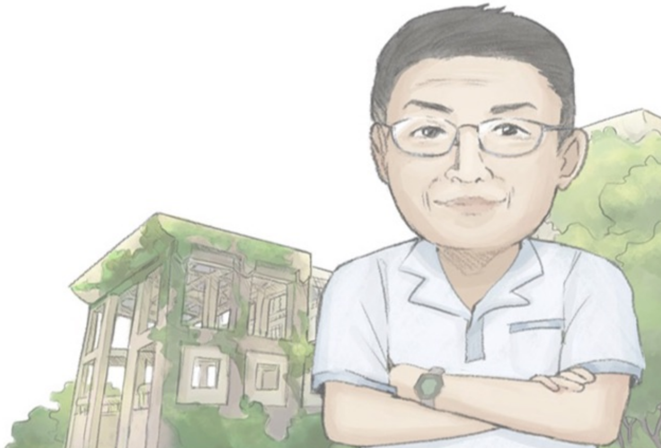
$$= E_x^2 - c^2B_x^2 + \gamma^2\begin{pmatrix} E_y^2 \cancel{-2vE_yB_z} + v^2B_z^2 + E_z^2 \cancel{+2vE_zB_y} + v^2B_y^2 \\ -c^2B_y^2 \cancel{-2vB_yE_z} - \frac{v^2}{c^2}E_z^2 - c^2B_z^2 \cancel{+2vB_zE_y} - \frac{v^2}{c^2}E_y^2 \end{pmatrix}$$

$$= E_x^2 - c^2B_x^2 + \gamma^2\left(E_y^2 - \frac{v^2}{c^2}E_y^2 + E_z^2 - \frac{v^2}{c^2}E_z^2 - c^2B_y^2 + v^2B_y^2 - c^2B_z^2 + v^2B_z^2\right)$$

$$= E_x^2 - c^2B_x^2 + \gamma^2\left(E_y^2 + E_z^2 - c^2B_y^2 - c^2B_z^2\right)\left(1 - \frac{v^2}{c^2}\right)$$

$$= \left(E_x^2 + E_y^2 + E_z^2\right) - c^2\left(B_x^2 + B_y^2 + B_z^2\right)$$

$$= E \quad - B^2c^2$$



Problem 12.47

(c) Suppose that in one inertial system $\mathbf{B} = \mathbf{0}$ but $\mathbf{E} \neq \mathbf{0}$ (at some point P). Is it possible to find another system in which the *electric* field is zero at P?

No.

For $B = 0$ in one system, then $\left(E^2 - c^2B^2\right)$ is **positive**.
Since it is invariant, it must be positive in any system.
Therefore $E \neq 0$ in all system