國立清華大學

Electromagnetism

Introduction to Electrodynamics 4th David J. Griffiths

Chap.1

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Exercise List

1. $\nabla\cdot\left(r^3\mathbf{r}\right)$

2. $\nabla^2\left(\nabla\cdot\left(\dfrac{\mathbf{r}}{r^2}\right)\right)$

3. $\sum_{mn}\varepsilon_{imn}\varepsilon_{jmn}=2\delta_{ij}$

4. $\sum_{ijk}\varepsilon_{ijk}\varepsilon_{ijk}=6$

5. Prove the product rules in page 23

6. If u, v are two scalar functions, prove that

$$\oint u\vec{\nabla}v\cdot d\vec{\ell}=\int_S\left(\vec{\nabla}u\right)\times\left(\vec{\nabla}v\right)\cdot d\mathbf{a}$$

7. If $\mathbf{A}$ is a 3 by 3 matrix and its i-th row j-th column component is denoted as A_{ij}, show that $\det\mathbf{A}=\sum_{ijk}\varepsilon_{ijk}A_{1i}A_{2j}A_{3k}$

8. More problems in Griffiths:
1.5*, 1.7*, 1.12, 1.13*, 1.16*, 1.25, 1.26, 1.33*, 1.36, 1.38*, 1.40*, 1.43*, 1.46*, 1.47, 1.49*, 1.64

1.

$$\nabla\cdot\left(r^3\mathbf{r}\right)$$
$$=\sum_i \partial_i\left(r^3 r_i\right)$$
$$=\sum_i\left[r_i\partial_i\left(r^3\right)+r^3\partial_i\left(r_i\right)\right]$$
$$=\sum_i\left\{r_i\left[\partial_i\left(\sum_j r_j^2\right)^{\frac{3}{2}}\right]+r^3\right\}$$
$$=3r^3+\sum_i\left\{r_i\left[\frac{3}{2}\left(\sum_{j'} r_{j'}^2\right)^{\frac{1}{2}}\sum_j 2r_j\frac{\partial r_j}{\partial r_i}\right]\right\}$$
$$=3r^3+\sum_i\left\{r_i\left[3r\sum_j r_j\delta_{ij}\right]\right\}$$
$$=3r^3+\sum_i\left[r_i\left(3rr_i\right)\right]$$
$$=3r^3+3r^3=6r^3$$

$$\nabla\cdot\left(r^3\mathbf{r}\right)$$
$$=\nabla\cdot\left(r^4\hat{\mathbf{r}}\right)$$
$$=\frac{1}{r^2}\partial_r\left(r^2r^4\right)$$
$$=6r^3$$

2.

$$\nabla^2\left(\nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right)\right)$$
$$=\nabla^2\left(\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r}\right)\right)$$
$$=\nabla^2\left[\frac{1}{r^2}\partial_r\left(r^2\frac{1}{r}\right)\right]$$
$$=\nabla^2\left(\frac{1}{r^2}\right)$$
$$=\frac{1}{r^2}\partial_r\left[r^2\partial_r\left(\frac{1}{r^2}\right)\right]$$
$$=\frac{2}{r^4}$$

3.

$$\sum_{mn}\varepsilon_{imn}\varepsilon_{jmn}$$
$$=\sum_{mn}\left(\delta_{ij}\delta_{mm}-\delta_{im}\delta_{jm}\right)$$
$$=3\delta_{ij}-\delta_{ij}$$
$$=2\delta_{ij}$$

4.

$$\sum_{ijk}\varepsilon_{ijk}\varepsilon_{ijk}$$
$$=\sum_{ijk}2\delta_{ii}$$
$$=6$$

5.

$$\left[\nabla\cdot(\mathbf{A}\times\mathbf{B})\right]_i$$
$$=\partial_i\left(\varepsilon_{ijk}A_jB_k\right)$$
$$=\varepsilon_{ijk}\partial_iA_jB_k+\varepsilon_{ijk}A_j\partial_iB_k$$
$$=B_k\varepsilon_{ijk}\partial_iA_j-A_j\varepsilon_{jik}\partial_iB_k$$
$$=B_i\varepsilon_{ijk}\partial_jA_k-A_i\varepsilon_{ikj}\partial_kB_j$$
$$=\left[\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})\right]_i$$

$$\left[\nabla\times(\mathbf{A}\times\mathbf{B})\right]_i$$
$$=\varepsilon_{ijk}\partial_j\left(\varepsilon_{kmn}A_mB_n\right)$$
$$=\varepsilon_{kij}\varepsilon_{kmn}\left(B_n\partial_jA_m+A_m\partial_jB_n\right)$$
$$=\left(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}\right)\left(B_n\partial_jA_m+A_m\partial_jB_n\right)$$
$$=\left(B_j\partial_jA_i+A_i\partial_jB_j\right)-\left(B_i\partial_jA_j+A_j\partial_jB_i\right)$$
$$=\left[(\mathbf{B}\cdot\nabla)\mathbf{A}+\mathbf{A}(\nabla\cdot\mathbf{B})-\mathbf{B}(\nabla\cdot\mathbf{A})-(\mathbf{A}\cdot\nabla)\mathbf{B}\right]_i$$

$$\left[\nabla(\mathbf{A}\cdot\mathbf{B})\right]_i$$
$$=\partial_i\left(A_jB_j\right)$$
$$=B_j\partial_iA_j+A_j\partial_iB_j$$
$$=B_j\partial_iA_j+A_j\partial_iB_j+\left(B_j\partial_jA_i-B_j\partial_jA_i\right)+\left(A_j\partial_jB_i-A_j\partial_jB_i\right)$$
$$=\left(B_j\partial_iA_j+A_j\partial_iB_j-B_j\partial_jA_i-A_j\partial_jB_i\right)+A_j\partial_jB_i+B_j\partial_jA_i$$
$$=\left[B_j\left(\partial_iA_j-\partial_jA_i\right)+A_j\left(\partial_iB_j-\partial_jB_i\right)\right]+A_j\partial_jB_i+B_j\partial_jA_i$$
$$=\left\{\begin{array}{l}\left[B_j\partial_mA_n\left(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}\right)+A_j\partial_mB_n\left(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}\right)\right]\\+A_j\partial_jB_i+B_j\partial_jA_i\end{array}\right\}$$
$$=\left(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}\right)\left(A_j\partial_mB_n+B_j\partial_mA_n\right)+A_j\partial_jB_i+B_j\partial_jA_i$$
$$=\varepsilon_{kij}\varepsilon_{kmn}\left(A_j\partial_mB_n+B_j\partial_mA_n\right)+A_j\partial_jB_i+B_j\partial_jA_i$$
$$=\varepsilon_{ijk}A_j\varepsilon_{kmn}\partial_mB_n+\varepsilon_{kij}B_j\varepsilon_{kmn}\partial_mA_n+A_j\partial_jB_i+B_j\partial_jA_i$$
$$=\left[\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A})+(\mathbf{A}\cdot\nabla)\mathbf{B}+(\mathbf{B}\cdot\nabla)\mathbf{A}\right]_i$$

6.

$$\oint u\vec{\nabla}v\cdot d\vec{\ell}$$

$$=\int_S \vec{\nabla}\times\left(u\vec{\nabla}v\right)\cdot d\mathbf{a}$$

$$=\int_S \sum_{ijk}\varepsilon_{ijk}\partial_j\left(u\partial_k v\right)da_i$$

$$=\int_S \sum_{ijk}\left[\varepsilon_{ijk}\left(\partial_j u\right)\left(\partial_k v\right)+u\underbrace{\left(\varepsilon_{ijk}\partial_j\partial_k v\right)}_{0}\right]da_i$$

$$=\int_S \left(\vec{\nabla}u\right)\times\left(\vec{\nabla}v\right)\cdot d\mathbf{a}$$

7.

If $\mathbf{A}$ is a 3 by 3 matrix and its i-th row j-th column component is denoted as A_{ij}, show that:

$$\det\mathbf{A}=\sum_{ijk}\varepsilon_{ijk}A_{1i}A_{2j}A_{3k}$$

$$\left|\mathbf{A}\right|=A_{11}\begin{vmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{vmatrix}-A_{12}\begin{vmatrix}A_{21} & A_{23}\\ A_{31} & A_{33}\end{vmatrix}+A_{13}\begin{vmatrix}A_{21} & A_{22}\\ A_{31} & A_{32}\end{vmatrix}$$

$$=\sum_{ijk}\varepsilon_{ijk}A_{1i}A_{2j}A_{3k}$$

👍**Problem 1.5** Prove the **BAC-CAB** rule by writing out both sides in component form.

$$\left[\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\right]_i = \sum_{jkmn} \varepsilon_{ijk} A_j \varepsilon_{kmn} B_m C_n = \sum_{jkmn}\left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right) A_j B_m C_n = \sum_{jk}\left(A_j B_i C_j - A_j B_j C_i\right) = \left[\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})\right]_i$$

👍**Problem 1.7** Find the separation vector $\boldsymbol{\mathscr{r}}$ from the source point (2,8,7) to the field point (4,6,8). Determine its magnitude ($\mathscr{r}$), and construct the unit vector $\hat{\boldsymbol{\mathscr{r}}}$.

$$\vec{\mathscr{r}} = (4,6,8) - (2,8,7) = (2,-2,1) = 3\underbrace{\left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right)}_{\hat{\mathscr{r}}}$$

Problem 1.12 The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^2 - 4y^2 - 18x + 28y + 12),$$

where y is the distance (in miles) north, x the distance east of South Hadley.

(a) Where is the top of the hill located?

(b) How high is the hill?

(c) How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In what direction is the slope steepest, at that point?

$$\nabla h(x,y) = \nabla\left[10\left(2xy - 3x^2 - 4y^2 - 18x + 28y + 12\right)\right]$$

$$= 10(2y - 6x - 18, 2x - 8y + 28)$$

$$= 0 \text{ when } (x,y) = (-2,3) \ldots\ldots (a)$$

$$\Rightarrow h(-2,3) = 10\left[2(-2)3 - 3(-2)^2 - 4(3)^2 - 18(-2) + 28(3) + 12\right] = 720 \text{ ft.} \ldots\ldots (b)$$

$$\nabla h(1,1) = 10(-22,22) = 220\sqrt{2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ ft. per mile} \ldots\ldots (c)$$

Problem 1.13 Let $\boldsymbol{\mathscr{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathscr{r}$ be its length. Show that

(a) $\nabla(\mathscr{r}^2) = 2\boldsymbol{\mathscr{r}}$.

(b) $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$.

(c) What is the *general* formula for $\nabla(\mathscr{r}^n)$?

$$\vec{\mathscr{r}} = (x, y, z) - (x', y', z') = (\Delta x, \Delta y, \Delta z) \qquad \mathscr{r} = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

$$(a)\nabla\left(\mathscr{r}^2\right) = 2\mathscr{r}\nabla\mathscr{r} = 2\mathscr{r}\left(\frac{2\Delta x}{2\mathscr{r}}, \frac{2\Delta y}{2\mathscr{r}}, \frac{2\Delta z}{2\mathscr{r}}\right) = 2\left(\Delta x, \Delta y, \Delta z\right) = 2\vec{\mathscr{r}}$$

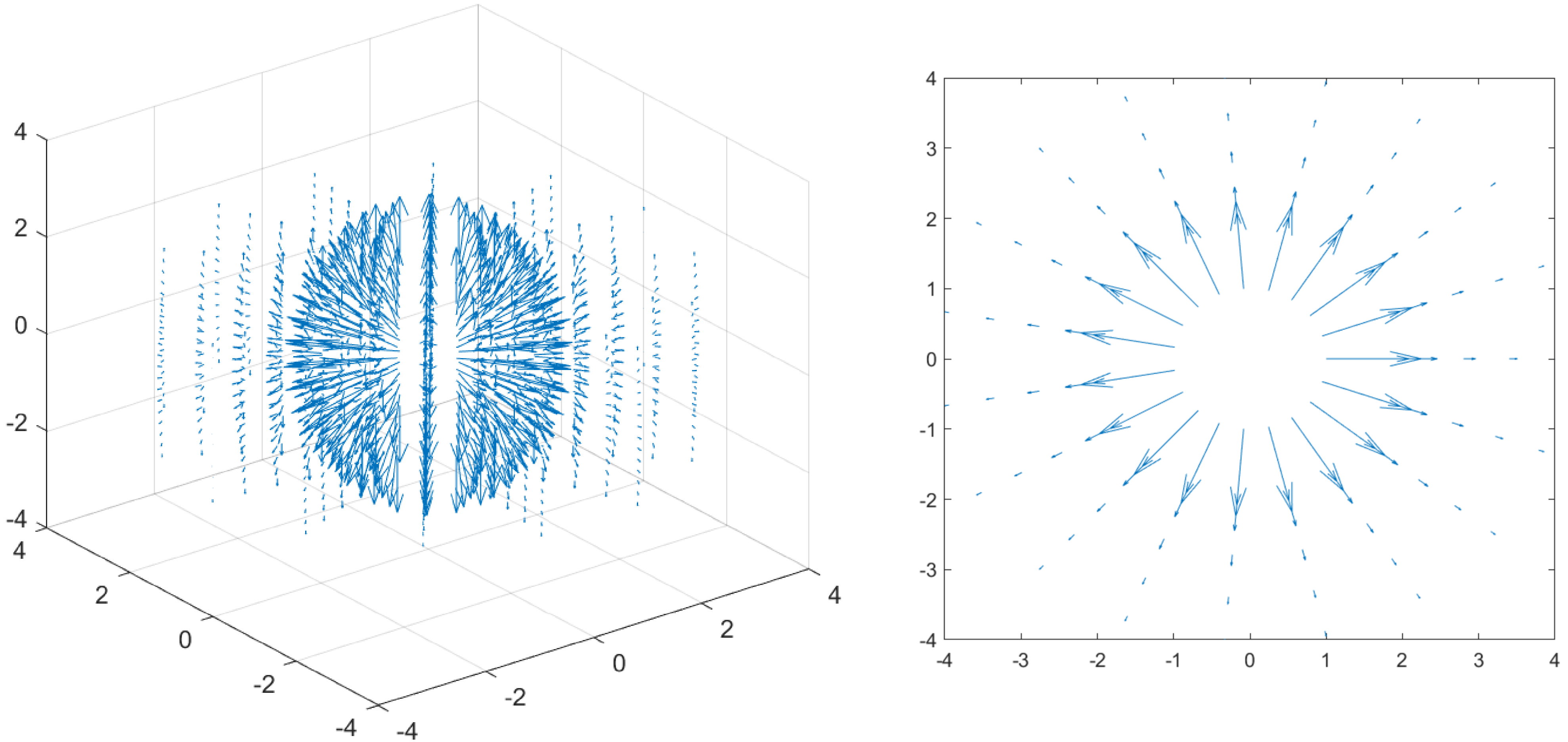
$$(b)\nabla\left(\frac{1}{\mathscr{r}}\right) = -\frac{1}{\mathscr{r}^2}\nabla\mathscr{r} = -\frac{1}{\mathscr{r}^2}\frac{1}{\mathscr{r}}\vec{\mathscr{r}} = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$(c)\nabla\left(\mathscr{r}^n\right) = n\mathscr{r}^{n-1}\nabla\mathscr{r} = n\mathscr{r}^{n-1}\hat{\mathscr{r}}$$

Problem 1.16 Sketch the vector function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2},$$

and compute its divergence. The answer may surprise you. . . can you explain it?



Problem 1.25

(a) Check product rule (iv) (by calculating each term separately) for the functions

$$\mathbf{A} = x\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}} + 3z\,\hat{\mathbf{z}}; \qquad \mathbf{B} = 3y\,\hat{\mathbf{x}} - 2x\,\hat{\mathbf{y}}.$$

(b) Do the same for product rule (ii).

(c) Do the same for rule (vi).

$\mathbf{A} = (x, 2y, 3z) \qquad \mathbf{B} = (3y, -2x, 0)$

$$(a)\quad \nabla\cdot(\mathbf{A}\times\mathbf{B}) = \nabla\cdot\left(\begin{vmatrix} 2y & 3z \\ -2x & 0 \end{vmatrix}, -\begin{vmatrix} x & 3z \\ 3y & 0 \end{vmatrix}, \begin{vmatrix} x & 2y \\ 3y & -2x \end{vmatrix}\right)$$

$$= \nabla\cdot\left(6xz, 9yz, -2x^2 - 6y^2\right) = 6z + 9z + 0 = 15z$$

$$\mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B}) = (3y, -2x, 0)\cdot(0,0,0) - (x, 2y, 3z)\cdot(0,0,-5) = 15z$$

$$(b)\quad \nabla(\mathbf{A}\cdot\mathbf{B}) = \nabla(3xy - 4xy) = (-y, -x)$$

$$\mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times\underbrace{(\nabla\times\mathbf{A})}_{0} + (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A}$$

$$= (x, 2y, 3z)\times(0,0,-5) + 0 + \left(x\partial_x + 2y\partial_y + 3z\partial_z\right)(3y, -2x, 0) + \left(3y\partial_x - 2x\partial_y + 0\right)(x, 2y, 3z)$$

$$= (-10y, 5x, 0) + (6y, -2x, 0) + (3y, -4x, 0) = (-y, -x)$$

Problem 1.25

(a) Check product rule (iv) (by calculating each term separately) for the functions

$$\mathbf{A} = x\,\hat{\mathbf{x}} + 2y\,\hat{\mathbf{y}} + 3z\,\hat{\mathbf{z}}; \qquad \mathbf{B} = 3y\,\hat{\mathbf{x}} - 2x\,\hat{\mathbf{y}}.$$

(b) Do the same for product rule (ii).

(c) Do the same for rule (vi).

$$(c)\quad \nabla\times(\mathbf{A}\times\mathbf{B}) = \nabla\times\left(6xz, 9yz, -2x^2-6y^2\right) = (-12y-9y, 4x+6x, 0) = (-21y, 10x, 0)$$

$$(\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A})$$

$$= (3y, -4x, 0) - (6y, -2x, 0) + (x, 2y, 3z)(0+0+0) - (3y, -2x, 0)(1+2+3)$$

$$= (-y, -x)$$

Problem 1.26 Calculate the Laplacian of the following functions:

(a) $T_a = x^2 + 2xy + 3z + 4$.

(b) $T_b = \sin x \sin y \sin z$.

(c) $T_c = e^{-5x} \sin 4y \cos 3z$.

(d) $\mathbf{v} = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

$(a)\nabla^2 T_a = 2$

$(b)\nabla^2 T_b = -3T_b$

$(c)\nabla^2 T_c = (25-16-9)T_c = 0$

$(d)\nabla^2 \mathbf{v} = \left(\nabla^2 v_x, \nabla^2 v_y, \nabla^2 v_z\right) = (2,6,0)$

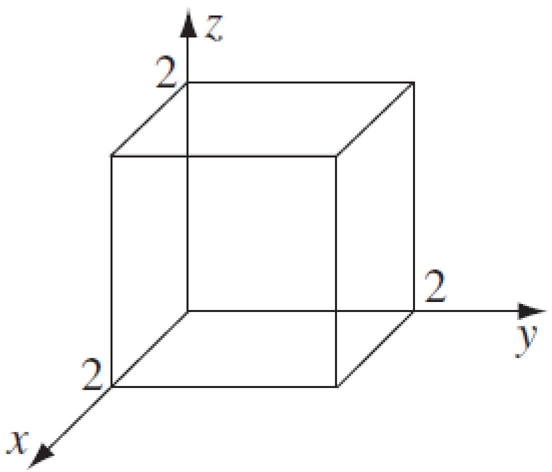


FIGURE 1.30

Problem 1.33 Test the divergence theorem for the function $\mathbf{v} = (xy)\,\hat{\mathbf{x}} + (2yz)\,\hat{\mathbf{y}} + (3zx)\,\hat{\mathbf{z}}$. Take as your volume the cube shown in Fig. 1.30, with sides of length 2.

$\mathbf{v} = (xy)\hat{\mathbf{x}} + (2yz)\hat{\mathbf{y}} + (3zx)\hat{\mathbf{z}} \qquad \nabla\cdot\mathbf{v} = y + 2z + 3x$

$$\int \nabla\cdot\mathbf{v}\,d\tau = \int_0^2\int_0^2\int_0^2 (y+2z+3x)\,dxdydz = \int_0^2\int_0^2\left(y\times2 + 2z\times2 + \frac{3}{2}\times4\right)dydz = \left(\frac{1}{2}+\frac{2}{2}+\frac{3}{2}\right)\times2\times2\times4 = 48$$

$$\oint \mathbf{v}\cdot d\mathbf{a} = \underbrace{\int_0^2\int_0^2 (0)\,ydydz}_{x=0} + \underbrace{\int_0^2\int_0^2 (2)\,ydydz}_{x=2} + \underbrace{\int_0^2\int_0^2 2(0)\,zdxdz}_{y=0} + \underbrace{\int_0^2\int_0^2 2(2)\,zdxdz}_{y=2} + \underbrace{\int_0^2\int_0^2 3(0)\,xdxdy}_{z=0} + \underbrace{\int_0^2\int_0^2 3(2)\,xdxdy}_{z=2}$$

$$=0+8+0+16+0+24=48$$

Problem 1.36

(a) Show that

$$\int_{\mathcal{S}} f(\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int_{\mathcal{S}} [\mathbf{A} \times (\nabla f)] \cdot d\mathbf{a} + \oint_{\mathcal{P}} f\mathbf{A} \cdot d\mathbf{l}. \qquad (1.60)$$

(b) Show that

$$\int_{\mathcal{V}} \mathbf{B} \cdot (\nabla \times \mathbf{A})\, d\tau = \int_{\mathcal{V}} \mathbf{A} \cdot (\nabla \times \mathbf{B})\, d\tau + \oint_{\mathcal{S}} (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a}. \qquad (1.61)$$

$$(a)\int_S f\left(\nabla\times\mathbf{A}\right)\cdot d\mathbf{a} = \int_S \sum_{ijk} f\,\varepsilon_{ijk}\left(\partial_j A_k\right)da_i = \int_S \sum_{ijk}\left[\varepsilon_{ijk}\partial_j\left(fA_k\right) - \varepsilon_{ijk}\left(\partial_j f\right)A_k\right]da_i$$

$$= \int_S\left[\nabla\times\left(f\mathbf{A}\right) - \left(\nabla f\right)\times\mathbf{A}\right]\cdot d\mathbf{a} = \oint_P f\mathbf{A}\cdot d\mathbf{l} + \int_S\left[\mathbf{A}\times\left(\nabla f\right)\right]\cdot d\mathbf{a}$$

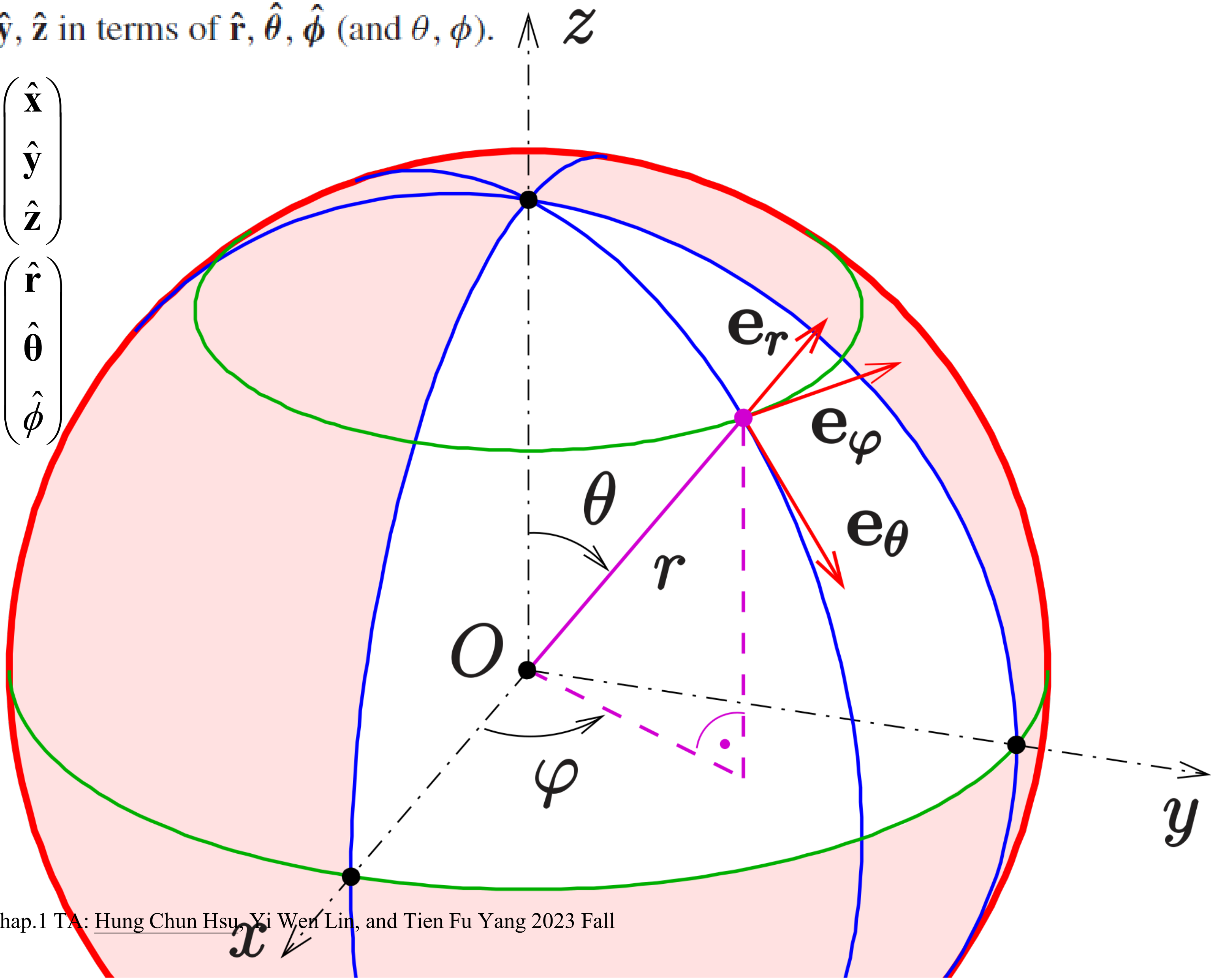
$$(b)\int_V \mathbf{B}\cdot\left(\nabla\times\mathbf{A}\right)d\tau = \int_V \sum_{ijk} B_i\varepsilon_{ijk}\left(\partial_j A_k\right)d\tau = \int_V \sum_{ijk}\left[\varepsilon_{ijk}\partial_j\left(B_i A_k\right) - \varepsilon_{ijk}\left(\partial_j B_i\right)A_k\right]d\tau$$

$$= \int_V\left[\nabla\cdot\left(\mathbf{A}\times\mathbf{B}\right) + \mathbf{A}\cdot\left(\nabla\times\mathbf{B}\right)\right]d\tau = \oint_S\left(\mathbf{A}\times\mathbf{B}\right)\cdot d\mathbf{a} + \int_V \mathbf{A}\cdot\left(\nabla\times\mathbf{B}\right)d\tau$$

Problem 1.38 Express the unit vectors $\hat{\mathbf{r}}, \boldsymbol{\theta}, \boldsymbol{\phi}$ in terms of $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ (that is, derive Eq. 1.64). Check your answers several ways ($\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \stackrel{?}{=} 1,\ \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} \stackrel{?}{=} 0,\ \hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \stackrel{?}{=} \hat{\boldsymbol{\phi}}, \ldots$). Also work out the inverse formulas, giving $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ in terms of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ (and θ, ϕ).

$$\left(\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\phi}\right) = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix}$$

$$\left(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\right) = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{\mathbf{r}} \\ \hat{\boldsymbol{\theta}} \\ \hat{\phi} \end{pmatrix}$$



Problem 1.40 Compute the divergence of the function

$$\mathbf{v} = (r\cos\theta)\,\hat{\mathbf{r}} + (r\sin\theta)\,\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\,\hat{\boldsymbol{\phi}}.$$

Check the divergence theorem for this function, using as your volume the inverted hemispherical bowl of radius R, resting on the xy plane and centered at the origin (Fig. 1.40).

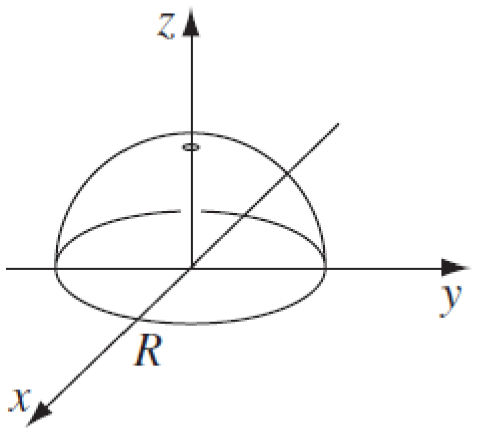


FIGURE 1.40

$$\mathbf{v} = (r\cos\theta)\hat{\mathbf{r}} + (r\sin\theta)\hat{\boldsymbol{\theta}} + (r\sin\theta\cos\phi)\hat{\phi}$$

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\partial_r\left(r^2 v_r\right) + \frac{1}{r\sin\theta}\partial_\theta\left(\sin\theta v_\theta\right) + \frac{1}{r\sin\theta}\partial_\phi\left(v_\phi\right)$$

$$= \frac{1}{r^2}\partial_r\left(r^3\cos\theta\right) + \frac{1}{r\sin\theta}\partial_\theta\left(r\sin^2\theta\right) + \frac{1}{r\sin\theta}\partial_\phi\left(r\sin\theta\cos\phi\right)$$

$$= 3\cos\theta + 2\cos\theta - \sin\phi = 5\cos\theta - \sin\phi$$

$$\int \nabla\cdot\mathbf{v}\,d\tau = \int_{\frac{\pi}{2}}^{0}\int_0^{2\pi}\int_0^R (5\cos\theta - \sin\phi)r^2 dr d\phi d(-\cos\theta) = \frac{1}{3}R^3(10\pi)\left(\frac{1}{2}\right) = \frac{5\pi R^3}{3}$$

$$\oint \mathbf{v}\cdot d\mathbf{a} = \int_0^{2\pi}\int_{\frac{\pi}{2}}^{0} r\cos\theta r^2\sin\theta d\theta d\phi\bigg|_{r=R} + \int_0^{2\pi}\int_0^R r\sin\theta r dr d\phi\bigg|_{\theta=\frac{\pi}{2}} = \pi R^3 + \frac{2\pi R^3}{3} = \frac{5\pi R^3}{3}$$

Problem 1.43

(a) Find the divergence of the function

$$\mathbf{v} = s(2+\sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}.$$

(b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in Fig. 1.43.

(c) Find the curl of **v**.

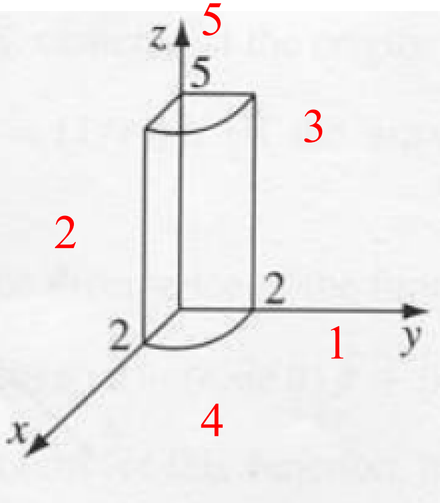


FIGURE 1.43

$$(a)\ \nabla\cdot\mathbf{v} = \frac{1}{s}\frac{\partial}{\partial s}(s v_s) + \frac{1}{s}\frac{\partial v_\phi}{\partial\phi} + \frac{\partial v_z}{\partial z} = 2\left(2+\sin^2\phi\right) + \cos^2\phi - \sin^2\phi + 3 = 8$$

(b)

$$\int_V \nabla\cdot\mathbf{v}\,d\tau = 8\times\text{volumn} = 40\pi$$

$$\oint_A \mathbf{v}\cdot d\mathbf{a} = \oint_A \left[s\left(2+\sin^2\phi\right), s\sin\phi\cos\phi, 3z\right]\cdot\left(sd\phi dz, dsdz, sdsd\phi\right)$$

$$= \oint_A s\left(2+\sin^2\phi\right)sd\phi dz + s\sin\phi\cos\phi dsdz + 3zsdsd\phi = \overbrace{2^2\left(2\times\frac{\pi}{2}+\frac{\pi}{4}\right)\times 5}^{\text{face 1}} + \overbrace{0}^{\text{face 2}} + \overbrace{0}^{\text{face 3}} + \overbrace{0}^{\text{face 4}} + \overbrace{3\times 5\times 2\times\frac{\pi}{2}}^{\text{face 5}} = 40\pi$$

Problem 1.43

(a) Find the divergence of the function

$$\mathbf{v} = s(2 + \sin^2\phi)\,\hat{\mathbf{s}} + s\sin\phi\cos\phi\,\hat{\boldsymbol{\phi}} + 3z\,\hat{\mathbf{z}}.$$

(b) Test the divergence theorem for this function, using the quarter-cylinder (radius 2, height 5) shown in Fig. 1.43.

(c) Find the curl of $\mathbf{v}$.

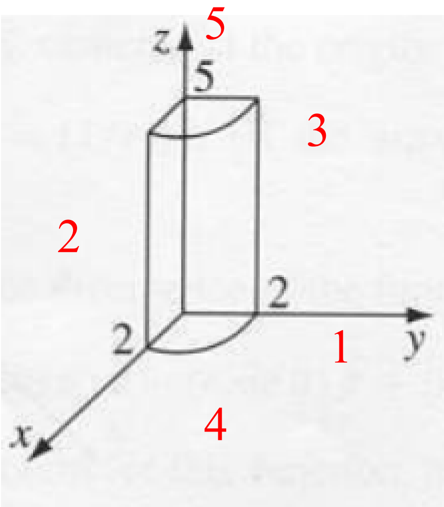


FIGURE 1.43

$$(c)\nabla\times\mathbf{v} = \left[\left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right), \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{s}\right), \frac{1}{s}\left(\frac{\partial}{\partial s}\left(s v_\phi\right) - \frac{\partial v_s}{\partial\phi}\right)\right] = \left[0, 0, \frac{1}{s}\left(2s\sin\phi\cos\phi - 2s\sin\phi\cos\phi\right)\right] = 0$$

Problem 1.46

(a) Show that

$$x \frac{d}{dx}(\delta(x)) = -\delta(x).$$

[*Hint:* Use integration by parts.]

$$\int f(x)\left[x\frac{d}{dx}\delta(x)\right]dx = f(x)x\delta(x)\Big|_{-\infty}^{\infty} - \int \frac{d}{dx}\left[f(x)x\right]\delta(x)dx = -\int \left(f(x)+f'(x)x\right)\delta(x)dx$$

$$= -\int f(x)\delta(x)dx \Rightarrow x\frac{d}{dx}\delta(x) = -\delta(x)$$

Problem 1.46

(b) Let $\theta(x)$ be the **step function**:

$$\theta(x) \equiv \left\{ \begin{array}{ll} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{array} \right\}. \tag{1.95}$$

Show that $d\theta/dx = \delta(x)$.

$$\theta(x) = \left\{ \begin{array}{ll} 1, & x > 0 \\ 0, & x \leq 0 \end{array} \right\}$$

$$\int f(x)\theta'(x)\,dx = f(x)\theta(x)\Big|_{-\infty}^{\infty} - \int f'(x)\theta(x)\,dx = f(\infty) - \int_0^{\infty} f'(x)\,dx = f(\infty) - f(\infty) + f(0) = f(0)$$

$$\Rightarrow \theta'(x) = \delta(x)$$

Problem 1.47

(a) Write an expression for the volume charge density $\rho(\mathbf{r})$ of a point charge q at $\mathbf{r}'$. Make sure that the volume integral of ρ equals q.

(b) What is the volume charge density of an electric dipole, consisting of a point charge $-q$ at the origin and a point charge $+q$ at $\mathbf{a}$?

(c) What is the volume charge density (in spherical coordinates) of a uniform, infinitesimally thin spherical shell of radius R and total charge Q, centered at the origin? [*Beware:* the integral over all space must equal Q.]

$$(a)\ \rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{r}')$$

$$(b)\ \rho(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{a}) - q\delta^3(\mathbf{r})$$

$$(c)\ Q = \int \rho(\mathbf{r}')\,d\tau = \int q(R)\,\delta(r'-R)\,r'^2 \sin\theta' dr' d\theta' d\phi'$$

$$= \int q(R)R^2(2)(2\pi) \Rightarrow q(R) = \frac{Q}{4\pi R^2} \Rightarrow \rho(\mathbf{r}) = \frac{Q}{4\pi R^2}\delta(r'-R)$$

Problem 1.49 Evaluate the integral

$$J = \int_{\mathcal{V}} e^{-r} \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau$$

(where $\mathcal{V}$ is a sphere of radius R, centered at the origin) by two different methods, as in Ex. 1.16.

$$J = \int_V e^{-r} \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau$$

*Sol.*1:

$$J = \int_V e^{-r} 4\pi\delta^3(\mathbf{r})\, d\tau = 4\pi$$

*Sol.*2:

$$J = \int_V e^{-r} \left(\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} \right) d\tau = -\int_V \frac{\hat{\mathbf{r}}}{r^2} \cdot \left(\nabla e^{-r} \right) d\tau + \oint_S e^{-r} \frac{\hat{\mathbf{r}}}{r^2} \cdot d\mathbf{a}$$

$$= \int_V \frac{e^{-r}}{r^2} r^2 \sin\theta\, dr\, d\theta\, d\phi + \oint_S \frac{e^{-r}}{r^2} r^2 \sin\theta\, d\theta\, d\phi = -4\pi e^{-r}\Big|_0^R + 4\pi e^{-R} = 4\pi$$

Problem 1.64 In case you're not persuaded that $\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r})$ (Eq. 1.102 with $\mathbf{r}' = \mathbf{0}$ for simplicity), try replacing r by $\sqrt{r^2+\epsilon^2}$, and watching what happens as $\epsilon \to 0$.[16] Specifically, let

$$D(r,\epsilon) \equiv -\frac{1}{4\pi}\nabla^2\frac{1}{\sqrt{r^2+\epsilon^2}}.$$

$$(a)\,D(r,\varepsilon) = -\frac{1}{4\pi}\nabla^2\left(\frac{1}{\sqrt{r^2+\varepsilon^2}}\right) = -\frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{1}{\sqrt{r^2+\varepsilon^2}}\right)\right] = -\frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-\frac{1}{2}\right)\left(r^2+\varepsilon^2\right)^{-\frac{3}{2}}(2r)\right]$$

$$= -\frac{1}{4\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\left[-r^3\left(r^2+\varepsilon^2\right)^{-\frac{3}{2}}\right] = \frac{1}{4\pi}\frac{1}{r^2}\left[3r^2\left(r^2+\varepsilon^2\right)^{-\frac{3}{2}} - \frac{3}{2}r^3\left(r^2+\varepsilon^2\right)^{-\frac{5}{2}}(2r)\right] = \frac{3}{4\pi}\left(r^2+\varepsilon^2\right)^{-\frac{5}{2}}\left(r^2+\varepsilon^2-r^2\right)$$

$$= \frac{3\varepsilon^2}{4\pi}\left(r^2+\varepsilon^2\right)^{-\frac{5}{2}}$$

$$(b)\lim_{\varepsilon\to 0} D(0,\varepsilon) = \lim_{\varepsilon\to 0}\frac{3}{4\pi\varepsilon^3} = \infty$$

Problem 1.64 In case you're not persuaded that $\nabla^2(1/r) = -4\pi\delta^3(\mathbf{r})$ (Eq. 1.102 with $\mathbf{r}' = \mathbf{0}$ for simplicity), try replacing r by $\sqrt{r^2+\epsilon^2}$, and watching what happens as $\epsilon \to 0$.[16] Specifically, let

$$D(r,\epsilon) \equiv -\frac{1}{4\pi}\nabla^2\frac{1}{\sqrt{r^2+\epsilon^2}}.$$

$$(c)\lim_{\varepsilon\to 0} D(r,\varepsilon) = \lim_{\varepsilon\to 0}\frac{3\varepsilon^2}{4\pi}\left(r^2+\varepsilon^2\right)^{-\frac{5}{2}} = \lim_{\varepsilon\to 0}\frac{3\varepsilon^2}{4\pi r^5}\left(1+\frac{\varepsilon^2}{r^2}\right)^{-\frac{5}{2}} = \lim_{\varepsilon\to 0}\frac{3\varepsilon^2}{4\pi r^5}\left(1-\frac{5}{2}\frac{\varepsilon^2}{r^2}+\frac{25}{4}\frac{\varepsilon^4}{r^4}-\cdots\right) = 0$$

$$(d)\int D(r',\varepsilon)\,d\tau = -\frac{1}{4\pi}\int\nabla\cdot\left[\nabla\left(\frac{1}{\sqrt{r'^2+\varepsilon^2}}\right)\right]d\tau = -\frac{1}{4\pi}\oint_{r'\to\infty}\nabla\left(\frac{1}{\sqrt{r'^2+\varepsilon^2}}\right)\cdot d\mathbf{a}$$

$$= -\frac{1}{4\pi}\oint_{r'\to\infty}(-r')\left(r'^2+\varepsilon^2\right)^{-\frac{3}{2}} r'^2\sin\theta' d\theta' d\phi' = \left(1+\frac{\varepsilon^2}{r'^2}\right)^{-\frac{3}{2}}_{r'\to\infty} = 1$$