國立清華大學

Electromagnetism

Introduction to Electrodynamics 4th David J. Griffiths

Chap.2

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2023 Fall

Exercise List

2.5, 2.6, *2.9, 2.10, 2.14, 2.16, 2.18, *2.20, 2.21, 2.22, 2.25, 2.27, 2.29, 2.30, 2.33, 2.34, *2.36, 2.38, *2.39, 2.42, *2.43, 2.60

Problem 2.5 Find the electric field a distance z above the center of a circular loop of radius r (Fig. 2.9) that carries a uniform line charge λ.

$$\mathbf{E}(z)=\frac{1}{4\pi\varepsilon_0}\frac{\lambda}{z^2+r^2}\frac{z}{\sqrt{z^2+r^2}}\times 2\pi r\hat{\mathbf{z}}=\frac{1}{2\varepsilon_0}\frac{\lambda z r}{\left(z^2+r^2\right)^{\frac{3}{2}}}\hat{\mathbf{z}}$$

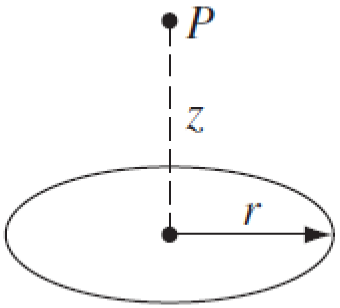


FIGURE 2.9

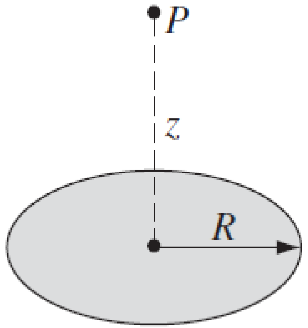


FIGURE 2.10

Problem 2.6 Find the electric field a distance z above the center of a flat circular disk of radius R (Fig. 2.10) that carries a uniform surface charge σ. What does your formula give in the limit $R \to \infty$? Also check the case $z \gg R$.

$$\frac{1}{2\varepsilon_0}\frac{\lambda z r}{\left(z^2+r^2\right)^{\frac{3}{2}}}\hat{\mathbf{z}}\to\frac{1}{2\varepsilon_0}\frac{\sigma z r}{\left(z^2+r^2\right)^{\frac{3}{2}}}\hat{\mathbf{z}}$$

$$\mathbf{E}(z)=\int_0^R\frac{1}{2\varepsilon_0}\frac{\sigma z r}{\left(z^2+r^2\right)^{\frac{3}{2}}}\hat{\mathbf{z}}dr=-\frac{1}{2\varepsilon_0}\frac{\sigma z}{\sqrt{z^2+r^2}}\hat{\mathbf{z}}\Bigg|_0^R=\frac{\sigma z}{2\varepsilon_0}\left[\frac{1}{z}-\frac{1}{\sqrt{z^2+R^2}}\right]\hat{\mathbf{z}}$$

$$\begin{cases}\xrightarrow{R\to\infty}\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}\\[2ex]\xrightarrow[\left(1+\frac{R^2}{z^2}\right)^{-\frac{1}{2}}\approx 1-\frac{1}{2}\frac{R^2}{z^2}]{z\gg R}\dfrac{\sigma z}{2\varepsilon_0}\left[\dfrac{1}{z}-\dfrac{1}{z}\left(1-\dfrac{1}{2}\dfrac{R^2}{z^2}\right)\right]\hat{\mathbf{z}}=\dfrac{\sigma}{4\varepsilon_0}\dfrac{R^2}{z^2}\hat{\mathbf{z}}=\dfrac{1}{4\pi\varepsilon_0}\dfrac{Q=\sigma\pi R^2}{z^2}\hat{\mathbf{z}}\end{cases}$$

*__Problem 2.9__ Suppose the electric field in some region is found to be $\mathbf{E} = kr^3\hat{\mathbf{r}}$, in spherical coordinates (k is some constant).

(a) Find the charge density ρ.

(b) Find the total charge contained in a sphere of radius R, centered at the origin. (Do it two different ways.)

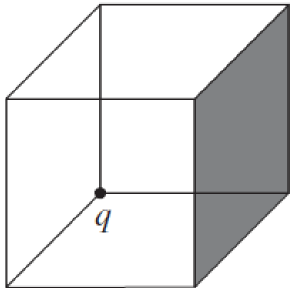


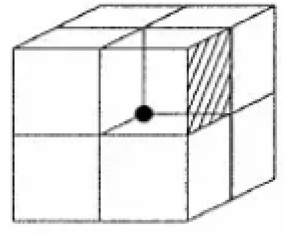
FIGURE 2.17

$$(a)\nabla\cdot\mathbf{E}(r)=\nabla\cdot\left(kr^3\hat{\mathbf{r}}\right)=\frac{1}{r^2}\partial_r\left(r^2E_r\right)=5kr^2=\frac{1}{\varepsilon_0}\rho(r)\to\rho(r)=5\varepsilon_0kr^2$$

$$(b)\int\rho(r)d\tau=\int\rho(r)r^2\sin\theta drd\theta d\phi=\frac{5}{5}\varepsilon_0kr^5(4\pi)\Big|_0^R=4\pi\varepsilon_0kR^5$$

Gauss Law: $Q_{enc}=\varepsilon_0\oint\mathbf{E}\cdot d\mathbf{a}=\varepsilon_0\oint kR^3R^2\sin\theta d\theta d\phi=4\pi\varepsilon_0kR^5$

Problem 2.10 A charge q sits at the back corner of a cube, as shown in Fig. 2.17. What is the flux of $\mathbf{E}$ through the shaded side?



Gauss Law: $Q_{enc}=\varepsilon_0\oint\mathbf{E}\cdot d\mathbf{a}\Rightarrow\int_{\text{shaded face}}E\times da=\frac{1}{6}\times\frac{1}{4}\times\frac{q}{\varepsilon_0}$

Problem 2.12 Use Gauss's law to find the electric field inside a uniformly charged solid sphere (charge density ρ). Compare your answer to Prob. 2.8.

$$\text{Gauss Law: } Q_{enc}(r) = \frac{4\pi r^3 \rho}{3} = \varepsilon_0 \oint \mathbf{E}\cdot d\mathbf{a} = \varepsilon_0 E_r \left(4\pi r^2\right) \Rightarrow \mathbf{E}(r) = \frac{4\pi r^3 \rho}{3\varepsilon_0 4\pi r^2}\hat{\mathbf{r}} = \frac{\rho r}{3\varepsilon_0}\hat{\mathbf{r}}$$

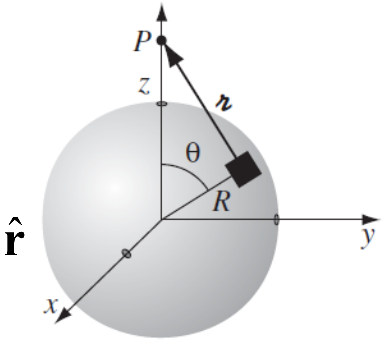


FIGURE 2.11

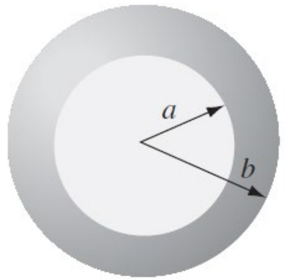


FIGURE 2.25

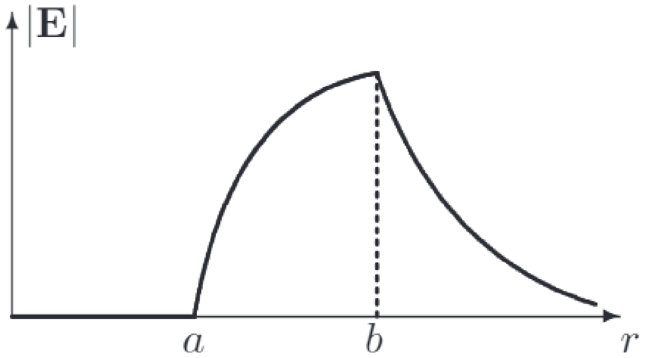
Problem 2.14 Find the electric field inside a sphere that carries a charge density proportional to the distance from the origin, $\rho = kr$, for some constant k. [*Hint:* This charge density is *not* uniform, and you must *integrate* to get the enclosed charge.]

$$\text{Gauss Law: } Q_{enc} = \int kr' r'^2 \sin\theta' dr' d\theta' d\phi' = \pi k r^4 = \varepsilon_0 \oint \mathbf{E}\cdot d\mathbf{a} = \varepsilon_0 E_r \left(4\pi r^2\right) \Rightarrow \mathbf{E}(r) = \frac{\pi k r^4}{\varepsilon_0 4\pi r^2}\hat{\mathbf{r}} = \frac{kr^2}{4\varepsilon_0}\hat{\mathbf{r}}$$

Problem 2.15 A thick spherical shell carries charge density

$$\rho = \frac{k}{r^2} \quad (a \le r \le b)$$

(Fig. 2.25). Find the electric field in the three regions: (i) $r < a$, (ii) $a < r < b$, (iii) $r > b$. Plot $|\mathbf{E}|$ as a function of r, for the case $b = 2a$.



$$Q_{enc}(r) = \int_a^r \frac{k}{r'^2} r'^2 \sin\theta' dr' d\theta' d\phi' = \int_a^r k \sin\theta' dr' d\theta' d\phi' = 4\pi k(r-a)$$

$$\text{(ii) } \mathbf{E}_{r<a} = 0 \qquad \text{(ii) } \mathbf{E}_{a<r<b} = \frac{4\pi k(r-a)}{\varepsilon_0 4\pi r^2}\hat{\mathbf{r}} = \frac{k(r-a)}{\varepsilon_0 r^2}\hat{\mathbf{r}} \qquad \text{(iii) } \mathbf{E}_{b<r} = \frac{4\pi k(b-a)}{\varepsilon_0 4\pi r^2}\hat{\mathbf{r}} = \frac{k(b-a)}{\varepsilon_0 r^2}\hat{\mathbf{r}}$$

Problem 2.16 A long coaxial cable (Fig. 2.26) carries a uniform *volume* charge density ρ on the inner cylinder (radius a), and a uniform *surface* charge density on the outer cylindrical shell (radius b). This surface charge is negative and is of just the right magnitude that the cable as a whole is electrically neutral. Find the electric field in each of the three regions: (i) inside the inner cylinder ($s < a$), (ii) between the cylinders ($a < s < b$), (iii) outside the cable ($s > b$). Plot $|\mathbf{E}|$ as a function of s.

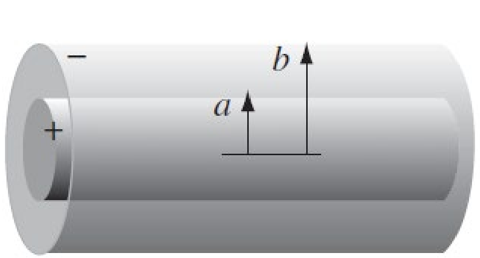


FIGURE 2.26

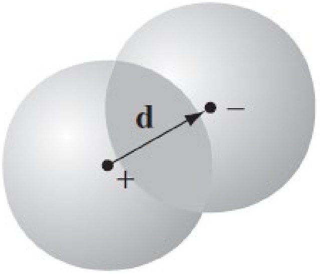
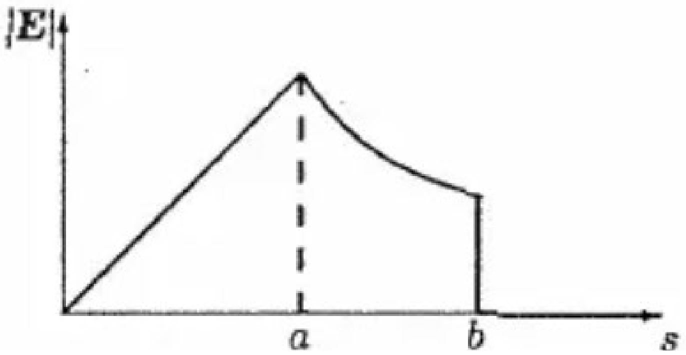


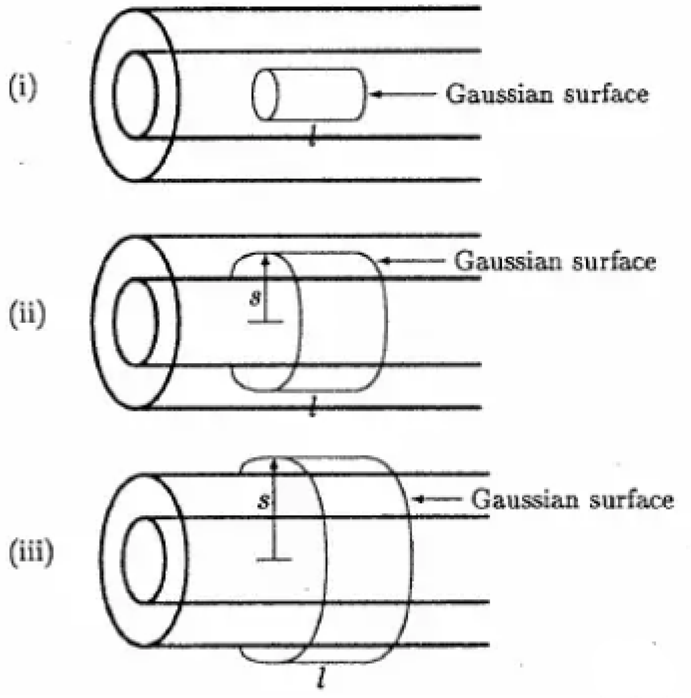
FIGURE 2.28

$$(\mathrm{i})\ \pi s^2 l\rho = \varepsilon_0 E_s 2\pi s l \Rightarrow \mathbf{E} = \frac{\rho s}{2\varepsilon_0}\hat{\mathbf{s}}$$

$$(\mathrm{ii})\ \pi a^2 l\rho = \varepsilon_0 E_s 2\pi s l \Rightarrow \mathbf{E} = \frac{\rho a^2}{2\varepsilon_0 s}\hat{\mathbf{s}}$$

$$(\mathrm{iii})\ 0 = \varepsilon_0 E_s 2\pi s l \Rightarrow \mathbf{E} = 0$$





Problem 2.18 Two spheres, each of radius R and carrying uniform volume charge densities $+\rho$ and $-\rho$, respectively, are placed so that they partially overlap (Fig. 2.28). Call the vector from the positive center to the negative center $\mathbf{d}$. Show that the field in the region of overlap is constant, and find its value. [*Hint:* Use the answer to Prob. 2.12.]

$$\mathbf{E}(r) = \frac{\rho r}{3\varepsilon_0}\hat{\mathbf{r}} \Rightarrow \mathbf{E}_+ = \frac{\rho}{3\varepsilon_0}\mathbf{d}_+, \qquad \mathbf{E}_- = \frac{\rho}{3\varepsilon_0}\mathbf{d}_-$$

$$\mathbf{E}_+ - \mathbf{E}_- = \frac{\rho}{3\varepsilon_0}(\mathbf{d}_+ - \mathbf{d}_-) = \frac{\rho}{3\varepsilon_0}\mathbf{d}$$

*__Problem 2.20__ One of these is an impossible electrostatic field. Which one?

(a) $\mathbf{E} = k[xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3xz\,\hat{\mathbf{z}}]$;

(b) $\mathbf{E} = k[y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}]$.

Here k is a constant with the appropriate units. For the *possible* one, find the potential, using the *origin* as your reference point. Check your answer by computing ∇V. [*Hint:* You must select a specific path to integrate along. It doesn't matter *what* path you choose, since the answer is path-independent, but you simply cannot integrate unless you have a definite path in mind.]

$$\mathbf{E} = -\nabla V \Rightarrow \nabla \times (-\nabla V) = 0 = \nabla \times \mathbf{E}$$

$$(a)\nabla \times \mathbf{E} = k[-2y, -3z, -x] \neq 0 \Rightarrow \text{impossible}$$

$$(b)\nabla \times \mathbf{E} = k[2z - 2z, 0, 2\,\mathrm{y} - 2\,\mathrm{y}] = 0 \Rightarrow \text{possible} \Rightarrow V = xy^2 + yz^2 + Const.$$

Problem 2.21 Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose total charge is q. Use infinity as your reference point. Compute the gradient of V in each region, and check that it yields the correct field. Sketch $V(r)$.

$$\mathbf{E}_{out}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

$$\mathbf{E}_{in}(\mathbf{r}) = q \times \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} \Big/ \varepsilon_0 4\pi r^2\,\hat{\mathbf{r}} = \frac{qr}{4\pi\varepsilon_0 R^3}\hat{\mathbf{r}}$$

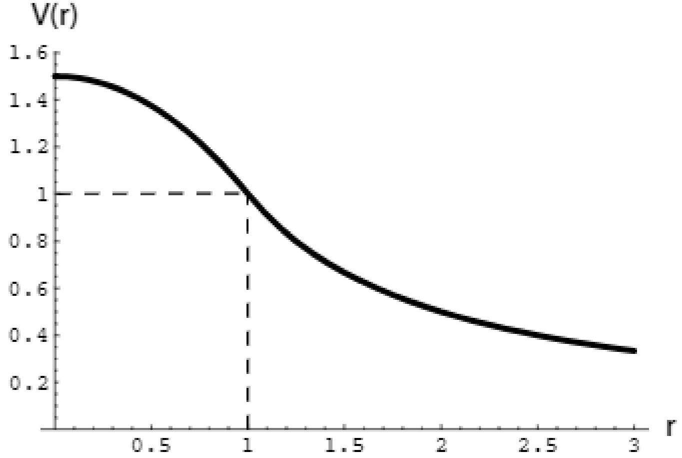
$$V_{out}(\mathbf{r}) = -\int_\infty^r \mathbf{E}_{out}\cdot d\mathbf{l} = \left.\frac{q}{4\pi\varepsilon_0 r'}\right|_\infty^r = \frac{q}{4\pi\varepsilon_0 r}$$

$$V_{in}(\mathbf{r}) = -\int_\infty^R \mathbf{E}_{out}\cdot d\mathbf{l} - \int_R^r \mathbf{E}_{in}\cdot d\mathbf{l} = \left.\frac{q}{4\pi\varepsilon_0 r'}\right|_\infty^R - \left.\frac{qr'^2}{8\pi\varepsilon_0 R^3}\right|_R^r$$

$$= \frac{3q}{8\pi\varepsilon_0 R} - \frac{qr^2}{8\pi\varepsilon_0 R^3} = \frac{q}{8\pi\varepsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)$$

check:

$$\mathbf{E} = -\nabla V \Rightarrow \begin{cases} -\nabla V_{out} = -\partial_r (V_r) = \dfrac{q}{4\pi\varepsilon_0 r^2} = \mathbf{E}_{out} \\ -\nabla V_{in} = \dfrac{qr}{4\pi\varepsilon_0 R^3} = \mathbf{E}_{in} \end{cases}$$



Problem 2.22 Find the potential a distance s from an infinitely long straight wire that carries a uniform line charge λ. Compute the gradient of your potential, and check that it yields the correct field.

$$\mathbf{E}(s)=\frac{1}{\varepsilon_0}\frac{\lambda L}{2\pi s L}=\frac{1}{2\pi\varepsilon_0}\frac{\lambda}{s}$$

The more charges we take in account, the greater the potential will be, and since the line charge is infinite long, the potential will be infinite. Therefore, the boundaries of integration for potential should be finite.

$$V=\int -\mathbf{E}\cdot d\mathbf{l}=\lim_{a\to\infty}-\int_a^s\frac{1}{2\pi\varepsilon_0}\frac{\lambda}{s'}ds'=\lim_{a\to\infty}\frac{\lambda}{2\pi\varepsilon_0}\ln\left|\frac{a}{s}\right|$$

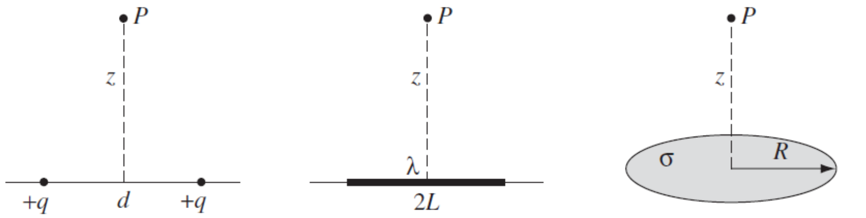
$$-\nabla V=-\partial_s V_s=\lim_{a\to\infty}-\frac{\lambda}{2\pi\varepsilon_0}\frac{s}{a}\left(-\frac{a}{s^2}\right)=\frac{1}{2\pi\varepsilon_0}\frac{\lambda}{s}=\mathbf{E},\ \text{independent of reference point } a.$$

$$V=\int_{V_{-\infty}}^{V_\infty}dV'=\lim_{y_a\to\infty}2\int_0^{y_a}\frac{1}{4\pi\varepsilon_0}\frac{\lambda}{\sqrt{s^2+y'^2}}dy'=\lim_{y_a\to\infty}\frac{\lambda}{2\pi\varepsilon_0}\ln\left|\frac{y_a+\sqrt{y_a^2+s^2}}{s}\right|$$

$$-\nabla V=-\partial_s V_s=\lim_{y_a\to\infty}-\frac{\lambda}{2\pi\varepsilon_0}\frac{s}{y_a+\sqrt{y_a^2+s^2}}\left(-\frac{y_a+\sqrt{y_a^2+s^2}}{s^2}+\frac{1}{\sqrt{y_a^2+s^2}}\right)=\lim_{y_a\to\infty}\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{s\sqrt{1+\frac{s^2}{y_a^2}}}=\frac{1}{2\pi\varepsilon_0}\frac{\lambda}{s}=\mathbf{E}$$

*__Problem 2.25__ Using Eqs. 2.27 and 2.30, find the potential at a distance z above the center of the charge distributions in Fig. 2.34. In each case, compute $\mathbf{E} = -\nabla V$, and compare your answers with Ex. 2.1, Ex. 2.2, and Prob. 2.6, respectively. Suppose that we changed the right-hand charge in Fig. 2.34a to $-q$; what then is the potential at P? What field does that suggest? Compare your answer to Prob. 2.2, and explain carefully any discrepancy.

We have known that $E_s = E_\theta = 0$ at point P.



(a) Two point charges (b) Uniform line charge (c) Uniform surface charge

FIGURE 2.34

$$(a)V_P = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{z^2+\left(\frac{d}{2}\right)^2}}+\frac{q}{\sqrt{z^2+\left(\frac{d}{2}\right)^2}}\right) = \frac{1}{4\pi\varepsilon_0}\frac{2q}{\sqrt{z^2+\left(\frac{d}{2}\right)^2}}$$

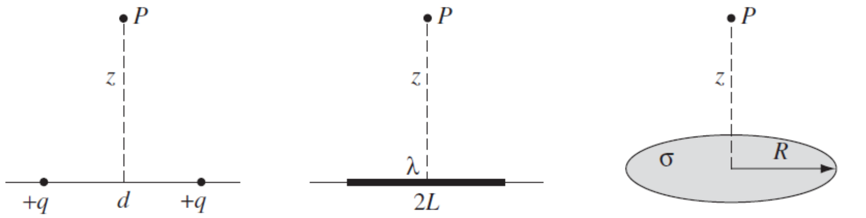
$$\mathbf{E} = -\nabla V\big|_P = -\partial_z V_P\hat{\mathbf{z}} = \frac{1}{4\pi\varepsilon_0}\frac{2qz}{\left[z^2+\left(\frac{d}{2}\right)^2\right]^{\frac{3}{2}}}\hat{\mathbf{z}}$$

$$(b)V_P = \frac{1}{4\pi\varepsilon_0}\int\frac{\lambda(\mathbf{r}')}{r}dl' = \frac{1}{4\pi\varepsilon_0}\int_{-L}^{L}\frac{\lambda}{\sqrt{z^2+l'^2}}dl' = \frac{\lambda}{4\pi\varepsilon_0}\ln\left|l'+\sqrt{z^2+l'^2}\right|_{-L}^{L} = \frac{\lambda}{4\pi\varepsilon_0}\ln\left|\frac{L+\sqrt{z^2+L^2}}{-L+\sqrt{z^2+L^2}}\right|$$

$$\mathbf{E} = -\nabla V\big|_P = -\frac{\lambda}{4\pi\varepsilon_0}\frac{-L+\sqrt{z^2+L^2}}{L+\sqrt{z^2+L^2}}\left(\frac{1}{-L+\sqrt{z^2+L^2}}\frac{z}{\sqrt{z^2+L^2}}-\frac{L+\sqrt{z^2+L^2}}{\left(-L+\sqrt{z^2+L^2}\right)^2}\frac{z}{\sqrt{z^2+L^2}}\right)\hat{\mathbf{z}} = \frac{\lambda}{2\pi\varepsilon_0}\frac{1}{\sqrt{z^2+L^2}}\frac{L}{z}\hat{\mathbf{z}}$$

*Problem 2.25** Using Eqs. 2.27 and 2.30, find the potential at a distance z above the center of the charge distributions in Fig. 2.34. In each case, compute $\mathbf{E} = -\nabla V$, and compare your answers with Ex. 2.1, Ex. 2.2, and Prob. 2.6, respectively. Suppose that we changed the right-hand charge in Fig. 2.34a to $-q$; what then is the potential at P? What field does that suggest? Compare your answer to Prob. 2.2, and explain carefully any discrepancy.

We have known that $E_s = E_\theta = 0$ at point P.



(a) Two point charges (b) Uniform line charge (c) Uniform surface charge

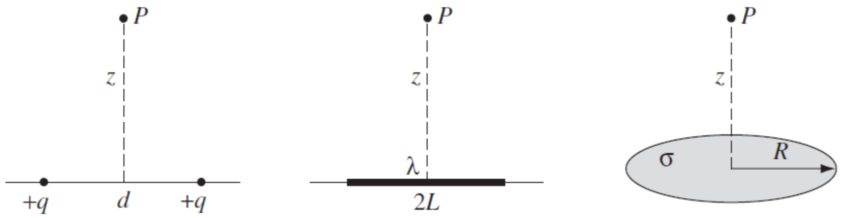
FIGURE 2.34

$$(c)\, V = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma(\mathbf{r}')}{r}\,da' = \frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^{R} \frac{\sigma}{\sqrt{z^2+s'^2}}\,s'ds'd\phi' = \frac{\sigma}{2\varepsilon_0}\sqrt{z^2+s'^2}\Big|_0^R = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2}-z\right)$$

$$\mathbf{E} = -\nabla V\big|_P = -\frac{\sigma}{2\varepsilon_0}\left(\frac{z}{\sqrt{z^2+R^2}}-1\right)\hat{\mathbf{z}} = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)\hat{\mathbf{z}}$$

*__Problem 2.25__ Using Eqs. 2.27 and 2.30, find the potential at a distance z above the center of the charge distributions in Fig. 2.34. In each case, compute $\mathbf{E} = -\nabla V$, and compare your answers with Ex. 2.1, Ex. 2.2, and Prob. 2.6, respectively. Suppose that we changed the right-hand charge in Fig. 2.34a to $-q$; what then is the potential at P? What field does that suggest? Compare your answer to Prob. 2.2, and explain carefully any discrepancy.

We have known that $E_s = E_\theta = 0$ at point P.



(a) Two point charges (b) Uniform line charge (c) Uniform surface charge

FIGURE 2.34

$$(a)\,V_P = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{r_i} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\sqrt{z^2+\left(\frac{d}{2}\right)^2}} + \frac{-q}{\sqrt{z^2+\left(\frac{d}{2}\right)^2}}\right) = 0$$

Basically, the ∇ is a operator, and it should be applied on a certain "distribution", not a single point.

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} + \frac{-q}{r_-}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right) \simeq \frac{q}{4\pi\varepsilon_0}\frac{d\cos\theta}{r^2}\text{(seeing P.151, electric dipole)}$$

$$\mathbf{E} = -\nabla V = -\left(\partial_r, \frac{1}{r}\partial_\theta, \frac{1}{r\sin\theta}\partial_\phi\right)\frac{qd}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2} = -\frac{qd}{4\pi\varepsilon_0}\left(\frac{-2\cos\theta}{r^3}, \frac{-\sin\theta}{r^3}, 0\right) = \frac{qd}{4\pi\varepsilon_0 r^3}(2\cos\theta, \sin\theta, 0)\text{(seeing P.158)}$$

$$\xrightarrow{(\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}})\to(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})} = \frac{qd}{4\pi\varepsilon_0 r^3}\left(3\sin\theta\cos\theta\cos\phi, 3\sin\theta\cos\theta\sin\phi, 2\cos^2\theta - \sin^2\theta\right)$$

$$\xrightarrow{\theta=\frac{\pi}{2},\phi=0} = \frac{qd}{4\pi\varepsilon_0 r^3}(0,0,-1)\text{, here } \hat{\mathbf{z}} \text{ is the same as } \hat{\mathbf{x}}' \text{ in Prob.2.}$$

Problem 2.27 Find the potential on the axis of a uniformly charged solid cylinder, a distance z from the center. The length of the cylinder is L, its radius is R, and the charge density is ρ. Use your result to calculate the electric field at this point. (Assume that $z > L/2$.)

$$V=\frac{1}{4\pi\varepsilon_0}\int\frac{\sigma(\mathbf{r}')}{r}da'=\frac{1}{4\pi\varepsilon_0}\int_0^{2\pi}\int_0^R\frac{\sigma}{\sqrt{z^2+s'^2}}s'ds'd\phi'=\frac{\sigma}{2\varepsilon_0}\sqrt{z^2+s'^2}\Big|_0^R=\frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2}-z\right)$$

$$\mathbf{E}=-\nabla V\big|_P=-\frac{\sigma}{2\varepsilon_0}\left(\frac{z}{\sqrt{z^2+R^2}}-1\right)\hat{\mathbf{z}}=\frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\sqrt{z^2+R^2}}\right)\hat{\mathbf{z}}$$

$$\Rightarrow\mathbf{E}=\int_{-\frac{L}{2}}^{\frac{L}{2}}\frac{\rho}{2\varepsilon_0}\left(1-\frac{z+l}{\sqrt{(z+l)^2+R^2}}\right)\hat{\mathbf{z}}dl=\frac{\rho}{2\varepsilon_0}\left(L-\sqrt{(z+l)^2+R^2}\Big|_{-\frac{L}{2}}^{\frac{L}{2}}\right)\hat{\mathbf{z}}=\frac{\rho}{2\varepsilon_0}\left(L-\sqrt{\left(z+\tfrac{L}{2}\right)^2+R^2}+\sqrt{\left(z-\tfrac{L}{2}\right)^2+R^2}\right)\hat{\mathbf{z}}$$

Problem 2.29 Check that Eq. 2.29 satisfies Poisson's equation, by applying the Laplacian and using Eq. 1.102.

$$\nabla_r^2V(\mathbf{r})=\nabla_r^2\left(\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}'-\mathbf{r}|}d\tau'\right)=\frac{1}{4\pi\varepsilon_0}\int\rho(\mathbf{r}')\left(\nabla_r^2\frac{1}{|\mathbf{r}'-\mathbf{r}|}\right)d\tau'=\frac{1}{4\pi\varepsilon_0}\int\rho(\mathbf{r}')-4\pi\varepsilon_0\delta(\mathbf{r}'-\mathbf{r})d\tau'=-\frac{\rho(\mathbf{r})}{\varepsilon_0}$$

Problem 2.30

(a) Check that the results of Exs. 2.5 and 2.6, and Prob. 2.11, are consistent with Eq. 2.33.

(b) Use Gauss's law to find the field inside and outside a long hollow cylindrical tube, which carries a uniform surface charge σ. Check that your result is consistent with Eq. 2.33.

(c) Check that the result of Ex. 2.8 is consistent with boundary conditions 2.34 and 2.36.

(a)

Ex.5

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}\Big|_{\text{interface}} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}} - \left(-\frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}\right) = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

Prob.11

$$\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}\Big|_{\text{interface}} = \frac{R^2\sigma}{\varepsilon_0 r^2}\hat{\mathbf{r}} - 0\Big|_{r=R} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{r}}$$

Ex.6

$$+\sigma : \mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}\Big|_{\text{interface}} = \left(\mathbf{E}_+ - \mathbf{E}_-\right) - \left(-\mathbf{E}_+ - \mathbf{E}_-\right) = 2\mathbf{E}_+ = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}\left(\text{to the left}\right)$$

$$-\sigma : \mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}\Big|_{\text{interface}} = \left(-\mathbf{E}_+ - \mathbf{E}_-\right) - \left(-\mathbf{E}_+ + \mathbf{E}_-\right) - = -2\mathbf{E}_- = -\frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}\left(\text{to the right}\right)$$

Problem 2.30

(a) Check that the results of Exs. 2.5 and 2.6, and Prob. 2.11, are consistent with Eq. 2.33.

(b) Use Gauss's law to find the field inside and outside a long hollow cylindrical tube, which carries a uniform surface charge σ. Check that your result is consistent with Eq. 2.33.

(c) Check that the result of Ex. 2.8 is consistent with boundary conditions 2.34 and 2.36.

(b) $$\mathbf{E}_{\text{outside}} - \mathbf{E}_{\text{inside}}\Big|_{s=R} = \frac{\sigma R}{\varepsilon_0 s}\hat{\mathbf{n}} - 0\Bigg|_{s=R} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}}$$

(c) $$V_{\text{outside}}\Big|_{r=R} = \frac{R^2\sigma}{\varepsilon_0 r}\Bigg|_{r=R} = \frac{R\sigma}{\varepsilon_0} \quad V_{\text{inside}}\Big|_{r=R} = \frac{R\sigma}{\varepsilon_0}\Bigg|_{r=R} = \frac{R\sigma}{\varepsilon_0} = V_{\text{outside}}\Big|_{r=R}$$

$$\frac{\partial V_{\text{outside}}}{\partial r} - \frac{\partial V_{\text{inside}}}{\partial r}\Bigg|_{r=R} = \frac{R\sigma}{\varepsilon_0}\left(-\frac{R}{r^2} - 0\right)\hat{\mathbf{r}}\Bigg|_{r=R} = -\frac{\sigma}{\varepsilon_0}\hat{\mathbf{r}}$$

Problem 2.33 Consider an infinite chain of point charges, $\pm q$ (with alternating signs), strung out along the x axis, each a distance a from its nearest neighbors. Find the work per particle required to assemble this system. [*Partial Answer:* $-\alpha q^2/(4\pi\epsilon_0 a)$, for some dimensionless number α; your problem is to determine α. It is known as the **Madelung constant**. Calculating the Madelung constant for 2- and 3-dimensional arrays is much more subtle and difficult.]

$$W_0 = 0, W_1 = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q^2}{a}\right), W_2 = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q^2}{a}+\frac{q^2}{2a}\right), W_3 = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q^2}{a}+\frac{q^2}{2a}-\frac{q^2}{3a}\right)$$

$$\Rightarrow W_n = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}\sum_{i=1}^{n}\frac{(-1)^i}{i} \Rightarrow \lim_{n\to\infty} W_n = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}\lim_{n\to\infty}\sum_{i=1}^{n}\frac{(-1)^i}{i} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{a}(-\ln 2)$$

Problem 2.34 Find the energy stored in a uniformly charged solid sphere of radius R and charge q. Do it three different ways:

(a) Use Eq. 2.43. You found the potential in Prob. 2.21.

(b) Use Eq. 2.45. Don't forget to integrate over *all space*.

(c) Use Eq. 2.44. Take a spherical volume of radius a. What happens as $a \to \infty$?

$$(\mathrm{a})\, V_{in}(\mathbf{r}) = \frac{q}{8\pi\varepsilon_0 R}\left(3-\frac{r^2}{R^2}\right),\quad W=\frac{1}{2}\int \rho V d\tau = \frac{1}{2}\frac{q\rho}{8\pi\varepsilon_0 R}\int_0^R\left(3-\frac{r'^2}{R^2}\right)r'^2 dr'(4\pi)=\frac{q}{4\varepsilon_0 R}\left(\frac{q}{\frac{4}{3}\pi R^3}\right)\left(r'^3-\frac{r'^5}{5R^2}\right)\Bigg|_0^R=\frac{3q}{20\varepsilon_0 R}$$

$$(\mathrm{b})\, \mathbf{E}_{out}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r^2},\ \mathbf{E}_{in}(\mathbf{r})=\frac{qr}{4\pi\varepsilon_0 R^3},\quad W=\frac{\varepsilon_0}{2}\int E^2 d\tau=\frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2\left(\int_R^\infty \frac{1}{r'^4}r'^2 dr'(4\pi)+\int_0^R\frac{r'^2}{R^6}r'^2 dr'(4\pi)\right)$$

$$=\frac{q^2}{8\pi\varepsilon_0}\left(-\frac{1}{r'}\Bigg|_R^\infty+\frac{r'^5}{5R^6}\Bigg|_0^R\right)=\frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{R}+\frac{R^5}{5R^6}\right)=\frac{3q}{20\varepsilon_0 R}$$

$$(\mathrm{c})\, W=\frac{\varepsilon_0}{2}\left(\int_V E^2 d\tau+\oint_S V\mathbf{E}\cdot d\mathbf{a}\right)=\frac{q^2}{8\pi\varepsilon_0}\left(-\frac{1}{r'}\Bigg|_R^a+\frac{r'^5}{5R^6}\Bigg|_0^R\right)+\frac{\varepsilon_0}{2}\frac{q}{4\pi\varepsilon_0 a}\frac{q}{4\pi\varepsilon_0 a^2}a^2(4\pi)=\frac{q^2}{8\pi\varepsilon_0}\left(-\frac{1}{a}+\frac{1}{R}+\frac{1}{5R}+\frac{1}{a}\right)=\frac{3q}{20\varepsilon_0 R}$$

***Problem 2.36** Consider two concentric spherical shells, of radii a and b. Suppose the inner one carries a charge q, and the outer one a charge $-q$ (both of them uniformly distributed over the surface). Calculate the energy of this configuration, (a) using Eq. 2.45, and (b) using Eq. 2.47 and the results of Ex. 2.9.

Take $a > b$

$$(a)W = \frac{\varepsilon_0}{2}\int E^2 d\tau = \frac{\varepsilon_0}{2}\left(\int_a^\infty E_{\text{outside}}^2 r'^2 dr'(4\pi) + \int_b^a E_{\text{gap}}^2 r'^2 dr'(4\pi) + \int_0^b E_{\text{inside}}^2 r'^2 dr'(4\pi)\right)$$

$$= \frac{\varepsilon_0}{2}(4\pi)\left(0 + \int_b^a \left(\frac{q}{4\pi\varepsilon_0 r'^2}\right)^2 r'^2 dr'(4\pi) + 0\right) = \frac{q^2}{8\pi\varepsilon_0}\left(-\frac{1}{a} + \frac{1}{b}\right)$$

$$(b)W_{\text{total}} = W_1 + W_2 + \varepsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 d\tau$$

$$= \frac{\varepsilon_0}{2}\int_a^\infty \left(\frac{-q}{4\pi\varepsilon_0 r'^2}\right)^2 r'^2 dr'(4\pi) + \frac{\varepsilon_0}{2}\int_b^\infty \left(\frac{q}{4\pi\varepsilon_0 r'^2}\right)^2 r'^2 dr'(4\pi) + \varepsilon_0 \int_a^\infty \frac{-q}{4\pi\varepsilon_0 r'^2}\frac{q}{4\pi\varepsilon_0 r'^2} r'^2 dr'(4\pi)$$

$$= \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{a}\right) = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right), \text{ same.}$$

Problem 2.38 A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b, as in Fig. 2.48). The shell carries no net charge.

(a) Find the surface charge density σ at R, at a, and at b.

(b) Find the potential at the center, using infinity as the reference point.

(c) Now the outer surface is touched to a grounding wire, which drains off charge and lowers its potential to zero (same as at infinity). How do your answers to (a) and (b) change?

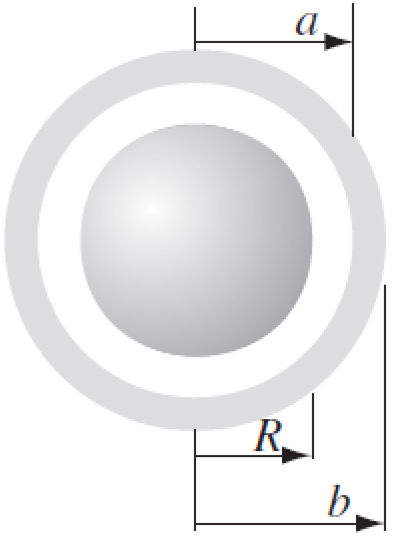


FIGURE 2.48

$$\text{(a)}\ \sigma(R)=\frac{q}{4\pi R^2},\ \sigma(a)=\frac{-q}{4\pi a^2},\ \sigma(b)=\frac{q}{4\pi b^2}$$

$$\text{(b)}\ V=-\int_\infty^0 \mathbf{E}\cdot d\mathbf{l}=-\int_\infty^b \frac{q}{4\pi\varepsilon_0 r'^2}dr'-\int_b^a (0)\,dr-\int_a^R \frac{q}{4\pi\varepsilon_0 r'^2}dr'-\int_R^0 (0)\,dr=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R}-\frac{1}{a}+\frac{1}{b}\right)$$

$$\text{(c)}\ \sigma(R)=\frac{q}{4\pi R^2},\ \sigma(a)=\frac{-q}{4\pi a^2},\ \sigma(b)=0\,;\ \ V=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{R}-\frac{1}{a}\right)$$

*

Problem 2.39 Two spherical cavities, of radii a and b, are hollowed out from the interior of a (neutral) conducting sphere of radius R (Fig. 2.49). At the center of each cavity a point charge is placed—call these charges q_a and q_b.

(a) Find the surface charge densities σ_a, σ_b, and σ_R.

(b) What is the field outside the conductor?

(c) What is the field within each cavity?

(d) What is the force on q_a and q_b?

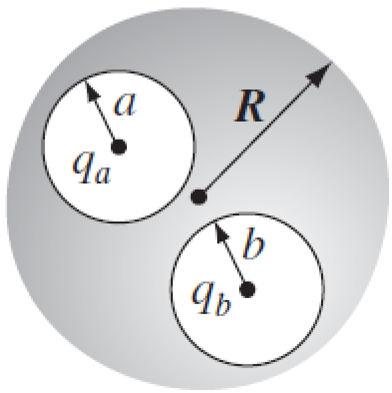


FIGURE 2.49

(a) $\sigma_a = -\dfrac{q_a}{4\pi a^2}, \sigma_b = -\dfrac{q_b}{4\pi b^2}, \sigma_R = \dfrac{q_a + q_b}{4\pi R^2}$

(b) $\mathbf{E}_{\text{outside}} = \dfrac{q_a + q_b}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$

(c) $\mathbf{E}_a = \dfrac{q_a}{4\pi\varepsilon_0 r_a^2}\hat{\mathbf{r}}_a, \mathbf{E}_b = \dfrac{q_b}{4\pi\varepsilon_0 r_b^2}\hat{\mathbf{r}}_b$

(d) 0

(e) Distribution of σ_R and $\mathbf{E}_{\text{outside}}$ will change.

Problem 2.42 A metal sphere of radius R carries a total charge Q. What is the force of repulsion between the "northern" hemisphere and the "southern" hemisphere?

$$\mathbf{F}=\int\frac{1}{2\varepsilon_0}\left(\frac{Q}{4\pi R^2}\right)^2\cos\theta' R^2\sin\theta' d\theta' d\phi'\hat{\mathbf{z}}=\frac{Q^2}{16\pi\varepsilon_0 R^2}\int_0^1\sin\theta' d\sin\theta'=\frac{Q^2}{32\pi\varepsilon_0 R^2}$$

FIGURE 2.53

*__Problem 2.43__ Find the capacitance per unit length of two coaxial metal cylindrical tubes, of radii a and b (Fig. 2.53).

Supposing the charge density on the surface of inner metal cylindrical tube is σ, and the tubes are infinite long.

$\sigma_b=-\frac{a}{b}\sigma, \sigma_a=\sigma$ insuring the quatity of charge on both surfaces are equal.

By Gauss' law, the field $\mathbf{E}(s)$ in the region $b>s>a$ is

$$\mathbf{E}(s)=\frac{(2\pi aL)\sigma}{\varepsilon_0}\frac{1}{2\pi sL}\hat{\mathbf{s}}=\frac{a\sigma}{s\varepsilon_0}\hat{\mathbf{s}}(\text{pointing outward})$$

$$\Rightarrow V_{ab}=\int_b^a-\mathbf{E}\cdot d\mathbf{l}=\lim_{L\to\infty}\int_b^a-\frac{a\sigma}{s\varepsilon_0}ds=\frac{a\sigma}{\varepsilon_0}\ln\frac{b}{a}\Rightarrow C=\frac{Q}{V}=\frac{(2\pi aL)\sigma}{\frac{a\sigma}{\varepsilon_0}\ln\frac{b}{a}}=\frac{2\pi\varepsilon_0 L}{\ln b/a}\Rightarrow C \text{ per unit length is } \frac{2\pi\varepsilon_0}{\ln b/a}$$

* **Problem 2.50** The electric potential of some configuration is given by the expression

$$V(\mathbf{r}) = A\frac{e^{-\lambda r}}{r},$$

where A and λ are constants. Find the electric field $\mathbf{E}(\mathbf{r})$, the charge density $\rho(r)$, and the total charge Q. [*Answer:* $\rho = \epsilon_0 A(4\pi\delta^3(\mathbf{r}) - \lambda^2 e^{-\lambda r}/r)$]

$$\mathbf{E}(\mathbf{r}) = -\nabla V = -\partial_r\left(A\frac{e^{-\lambda r}}{r}\right)\hat{\mathbf{r}} = -A\frac{r(-\lambda)e^{-\lambda r} - e^{-\lambda r}}{r^2}\hat{\mathbf{r}} = \frac{Ae^{-\lambda r}}{r^2}(1+\lambda r)\hat{\mathbf{r}}$$

$$\rho(\mathbf{r}) = -\varepsilon_0\nabla^2 V = -\varepsilon_0\left[\frac{A}{r}\nabla^2\left(e^{-\lambda r}\right) + Ae^{-\lambda r}\nabla^2\left(\frac{1}{r}\right)\right] = -\varepsilon_0\left\{\frac{A}{r}\lambda^2 e^{-\lambda r} + Ae^{-\lambda r}\left[-4\pi\delta^3(\mathbf{r})\right]\right\}$$

$$= \varepsilon_0 A\left[4\pi\delta^3(\mathbf{r}) - \frac{\lambda^2}{r}e^{-\lambda r}\right]$$

$$Q = \int \varepsilon_0 A\left[4\pi\delta^3(\mathbf{r}') - \frac{\lambda^2}{r'}e^{-\lambda r'}\right]r'^2\sin\theta' dr' d\theta' d\phi' = \varepsilon_0 A(4\pi)\left(1 - \lambda^2\int e^{-\lambda r'}r' dr'\right)$$

$$= \varepsilon_0 A(4\pi)\left[1 - \lambda^2\partial_\lambda\left(\int -e^{-\lambda r'}dr'\right)\right] = \varepsilon_0 A(4\pi)\left[1 - \lambda^2\partial_\lambda\left(\frac{1}{\lambda}e^{-\lambda r'}\right)_0^\infty\right]$$

$$= \varepsilon_0 A(4\pi)\left[1 - \lambda^2\partial_\lambda\left(-\frac{1}{\lambda}\right)\right] = \varepsilon_0 A(4\pi)\left[1 - \lambda^2\left(\frac{1}{\lambda^2}\right)\right] = 0$$

*

Problem 2.53 In a vacuum diode, electrons are "boiled" off a hot **cathode**, at potential zero, and accelerated across a gap to the **anode**, which is held at positive potential V_0. The cloud of moving electrons within the gap (called **space charge**) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then on, a steady current I flows between the plates.

Suppose the plates are large relative to the separation ($A \gg d^2$ in Fig. 2.55), so that edge effects can be neglected. Then V, ρ, and v (the speed of the electrons) are all functions of x alone.

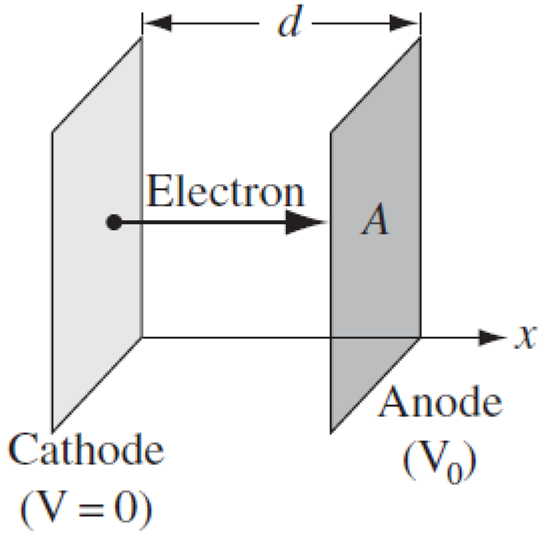


FIGURE 2.55

(a) Write Poisson's equation for the region between the plates.

$$\nabla^2 V = \frac{d^2V}{dx^2} = -\frac{\rho}{\varepsilon_0}$$

(b) Assuming the electrons start from rest at the cathode, what is their speed at point x, where the potential is $V(x)$?

$$qV = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2qV}{m}}$$

(c) In the steady state, I is independent of x. What, then, is the relation between ρ and v?

$$dq = A\rho dx, \qquad I = \frac{dq}{dt} = A\rho\frac{dx}{dt} = A\rho v$$

* **Problem 2.53** In a vacuum diode, electrons are "boiled" off a hot **cathode**, at potential zero, and accelerated across a gap to the **anode**, which is held at positive potential V_0. The cloud of moving electrons within the gap (called **space charge**) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then on, a steady current I flows between the plates.

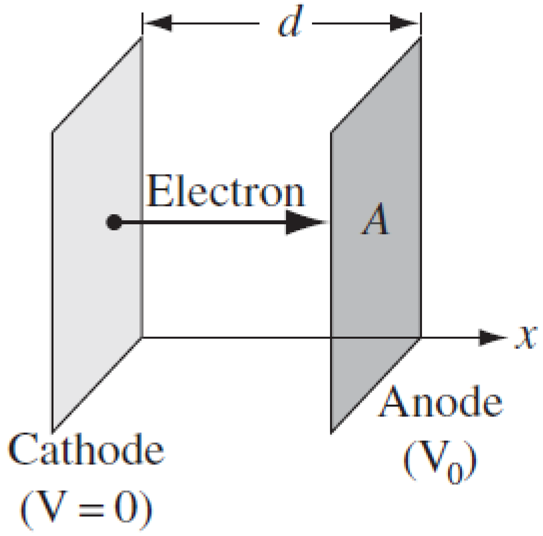


FIGURE 2.55

Suppose the plates are large relative to the separation ($A \gg d^2$ in Fig. 2.55), so that edge effects can be neglected. Then V, ρ, and v (the speed of the electrons) are all functions of x alone.

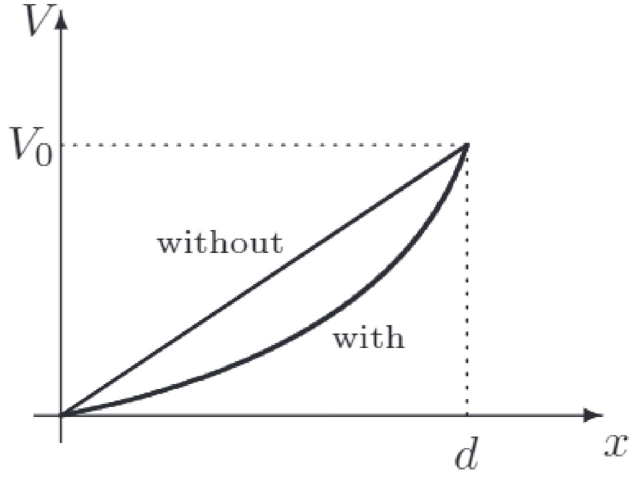
(d) Use these three results to obtain a differential equation for V, by eliminating ρ and v.

$$\frac{d^2V}{dx^2} = \beta V^{-1/2}, \quad \beta = -\frac{I}{\varepsilon_0 A}\sqrt{\frac{m}{2q}} \Rightarrow V \sim x^{4/3} \qquad O-2 = -\frac{O}{2} \Rightarrow O = \frac{4}{3}$$

Matching $V(0)=0,\ V(d)=V_0 \Rightarrow V = V_0\left(\frac{x}{d}\right)^{4/3}$ No space-charge: $V = V_0\left(\frac{x}{d}\right)$

$$\rho = -\varepsilon_0 \frac{d^2V}{dx^2} = -\varepsilon_0 V_0 \frac{1}{d^{4/3}}\frac{4}{3}\frac{1}{3}x^{-2/3} = -\frac{4\varepsilon_0 V}{9\left(d^2x\right)^{2/3}}$$

$$v = \sqrt{\frac{2qV}{m}} = \sqrt{\frac{2qV_0}{m}}\left(\frac{x}{d}\right)^{2/3}$$



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Problem 2.53 In a vacuum diode, electrons are "boiled" off a hot **cathode**, at potential zero, and accelerated across a gap to the **anode**, which is held at positive potential V_0. The cloud of moving electrons within the gap (called **space charge**) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then on, a steady current I flows between the plates.

Suppose the plates are large relative to the separation ($A \gg d^2$ in Fig. 2.55), so that edge effects can be neglected. Then V, ρ, and v (the speed of the electrons) are all functions of x alone.

(e) Solve this equation for V as a function of x, V_0, and d. Plot $V(x)$, and compare it to the potential *without* space-charge. Also, find ρ and v as functions of x.

$$dq = A\rho dx, \qquad I = \frac{dq}{dt} = A\rho\frac{dx}{dt} = A\rho v$$

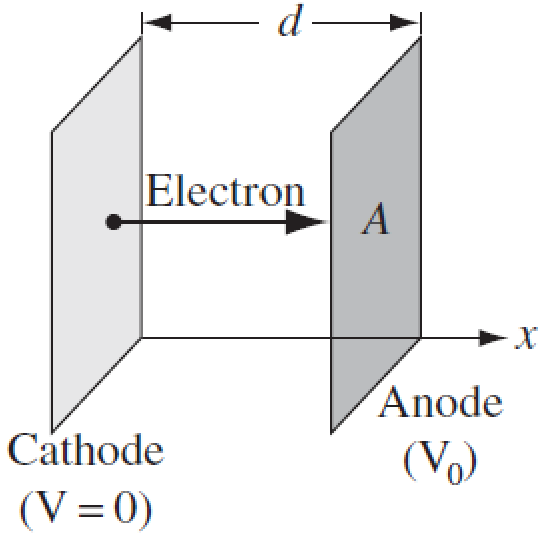

FIGURE 2.55

***Problem 2.53** In a vacuum diode, electrons are "boiled" off a hot **cathode**, at potential zero, and accelerated across a gap to the **anode**, which is held at positive potential V_0. The cloud of moving electrons within the gap (called **space charge**) quickly builds up to the point where it reduces the field at the surface of the cathode to zero. From then on, a steady current I flows between the plates.

Suppose the plates are large relative to the separation ($A \gg d^2$ in Fig. 2.55), so that edge effects can be neglected. Then V, ρ, and v (the speed of the electrons) are all functions of x alone.

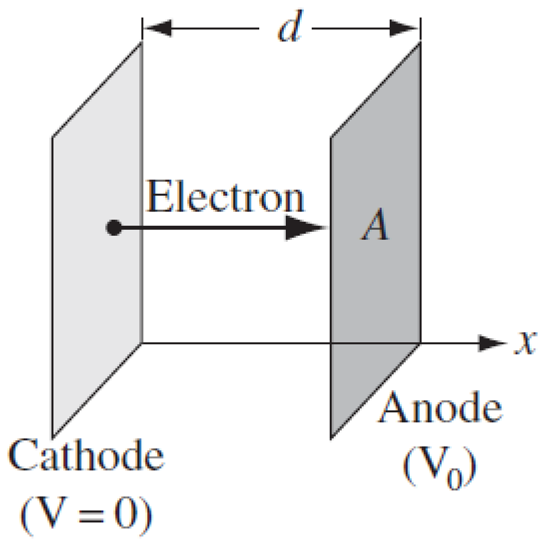

FIGURE 2.55

(f) Show that

$$I = KV_0^{3/2}, \tag{2.56}$$

and find the constant K. (Equation 2.56 is called the **Child-Langmuir law**. It holds for other geometries as well, whenever space-charge limits the current. Notice that the space-charge limited diode is *nonlinear*—it does not obey Ohm's law.)

$$V(d) = V_0 = \left(\frac{81 I^2 m}{32\varepsilon_0^2 A^2 q}\right)^{1/3} d^{4/3} \Rightarrow I = \sqrt{\frac{32\varepsilon_0^2 A^2 q}{81 m d^4}} V_0^{3/2} = KV_0^{3/2}, \qquad K = \frac{4\varepsilon_0 A}{9d^2}\sqrt{\frac{2q}{m}}$$

Problem 2.60 A point charge q is at the center of an uncharged spherical conducting shell, of inner radius a and outer radius b. *Question:* How much work would it take to move the charge out to infinity (through a tiny hole drilled in the shell)? [*Answer:* $(q^2/8\pi\epsilon_0)(1/a - 1/b)$.]

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \dfrac{q}{4\pi\varepsilon_0 r^2} & ,r<a \\ 0 & ,a<r<b \\ \dfrac{q}{4\pi\varepsilon_0 r^2} & ,b<r \end{cases}$$

$$W_{\text{total}} = \frac{\varepsilon_0}{2}\left(\frac{q}{4\pi\varepsilon_0}\right)^2\left[\int_0^a \frac{1}{r'^4} r'^2 dr'(4\pi) + 0 + \int_b^\infty \frac{1}{r'^4} r'^2 dr'(4\pi)\right]$$

$$= \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{b} - \frac{1}{a}\right)\text{(excluding the self-energy of the point charge)}$$

Recall **Prob. 2.36**(a) for $a = b' > b = a'$

$$W = \frac{\varepsilon_0}{2}\int E^2 d\tau = \frac{\varepsilon_0}{2}\left(\int_{b'}^\infty E_{\text{outside}}^2 r'^2 dr'(4\pi) + \int_{a'}^{b'} E_{\text{gap}}^2 r'^2 dr'(4\pi) + \int_0^{a'} E_{\text{inside}}^2 r'^2 dr'(4\pi)\right)$$

$$= \frac{\varepsilon_0}{2}(4\pi)\left(0 + \int_{a'}^{b'}\left(\frac{q}{4\pi\varepsilon_0 r'^2}\right)^2 r'^2 dr'(4\pi) + 0\right) = \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{a'} - \frac{1}{b'}\right)$$