國立清華大學

Electromagnetism

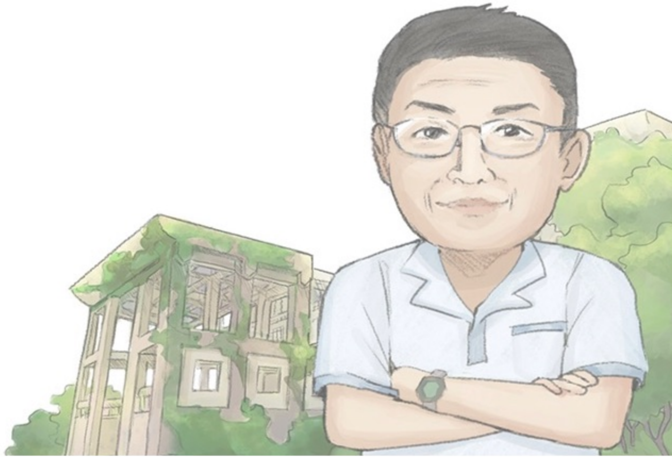
Introduction to Electrodynamics 4th David J. Griffiths

Chap.3

Prof. Tsun Hsu Chang

TA: Hung Chun Hsu, Yi Wen Lin, and Tien Fu Yang

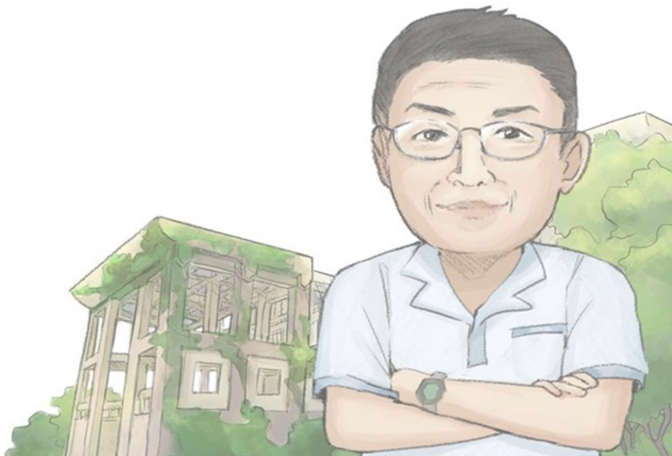
2023 Fall



Exercise List

3.11, 3.13, 3.16, 3.20, 3.27, 3.43, 3.54, 3.56, //

3.7, 3.8, 3.12, 3.19, 3.23, 3.28, 3.29, 3.32, 3.44, 3.49



Problem 3.11 Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q, situated as shown in Fig. 3.15. Set up the image configuration, and calculate the potential in this region. What charges do you need, and where should they be located? What is the force on q? How much work did it take to bring q in from infinity? Suppose the planes met at some angle other than 90°; would you still be able to solve the problem by the method of images? If not, for what particular angles *does* the method work?

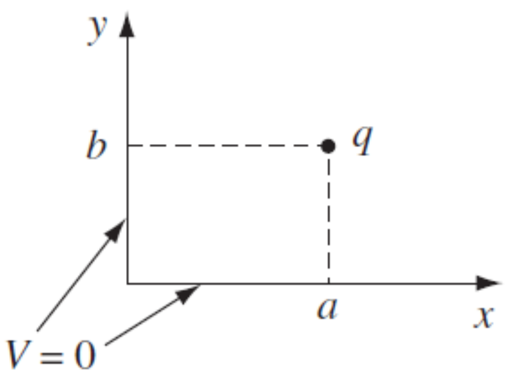


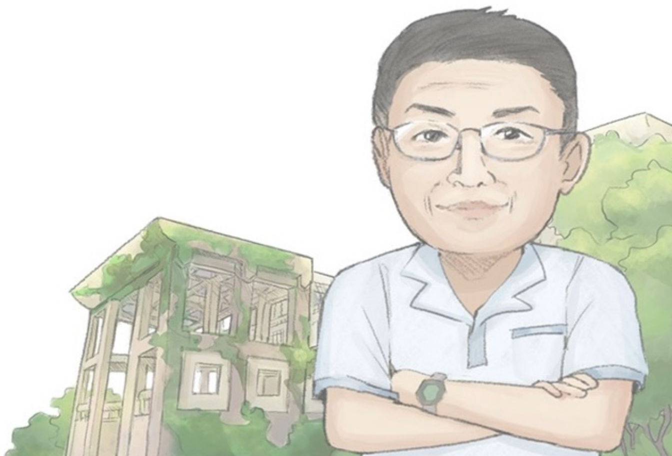
FIGURE 3.15

Image charge: $q' = -q\delta\left[\mathbf{r}-(-a,b)\right] - q\delta\left[\mathbf{r}-(a,-b)\right] + q\delta\left[\mathbf{r}-(-a,-b)\right]$

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q^2}{\left|(a,b)-(-a,b)\right|^2} - \frac{q^2}{\left|(a,b)-(a,-b)\right|^2} + \frac{q^2}{\left|(a,b)-(-a,-b)\right|^2}\right) = \frac{q^2}{4\pi\varepsilon_0}\left(-\frac{1}{4a^2} - \frac{1}{4b^2} + \frac{1}{4a^2+4b^2}\right)$$

$$W = \frac{1}{4}\times\frac{q^2}{4\pi\varepsilon_0}\left(-\frac{1}{2a} - \frac{1}{2b} + \frac{1}{\sqrt{4a^2+4b^2}}\right)$$ The field is only created in first quadrant

More angles: $120°, 180°$



Problem 3.13 Find the potential in the infinite slot of Ex. 3.3 if the boundary at $x = 0$ consists of two metal strips: one, from $y = 0$ to $y = a/2$, is held at a constant potential V_0, and the other, from $y = a/2$ to $y = a$, is at potential $-V_0$.

$$\begin{cases} \text{(i)} & V = 0 \text{ when } y = 0 \\ \text{(ii)} & V = 0 \text{ when } y = a \end{cases} \qquad \begin{matrix} \text{(iii)} & V = V_0(y) \text{ when } x = 0 \\ \text{(iv)} & V \to 0 \text{ as } x \to \infty \end{matrix}$$

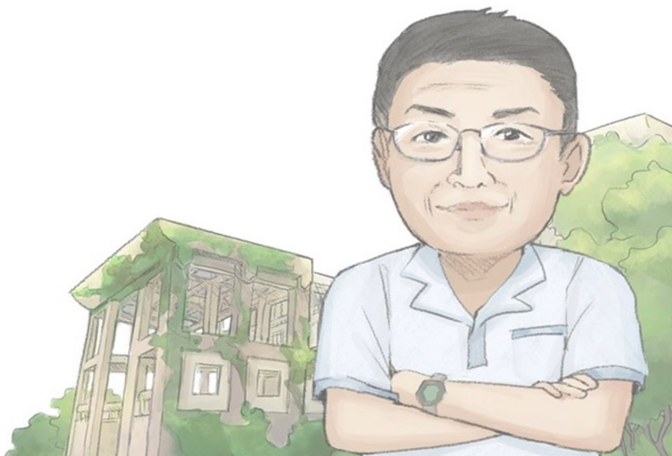
$$\nabla^2 V = 0 = \frac{1}{X}\frac{d^2X}{dX^2} + \frac{1}{Y}\frac{d^2Y}{dY^2} = 0 \Rightarrow V(x,y) = X(x)Y(y) = \left(Ae^{kx} + Be^{-kx}\right)\left(Ce^{iky} + De^{-iky}\right)$$

$$\text{(iv)}: \qquad A = 0 \Rightarrow V(x,y) = Be^{-kx}\left(Ce^{iky} + De^{-iky}\right)$$

$$\text{(i)}: \qquad C + D = 0 \Rightarrow V(x,y) = Be^{-kx}C\left(e^{iky} - e^{-iky}\right) = Be^{-kx}C' \sin ky$$

$$\text{(ii)}: \qquad \sin ka - \sin(-ka) = 2\sin ka = 0 \Rightarrow k = \frac{n\pi}{a} \quad n = 1, 2, 3 \cdots \text{not } 0$$

$$\Rightarrow V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\frac{n\pi y}{a}$$



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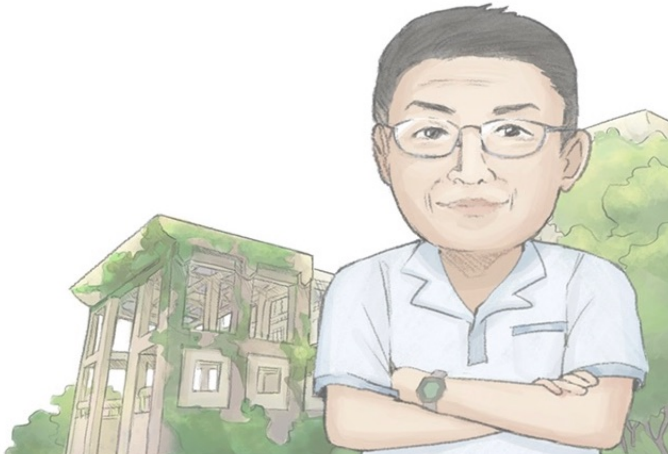
$$\text{(iii)}: \quad \sum_{n=1}^{\infty} C_n \sin\frac{n\pi y}{a} = \begin{cases} V_0 & ,0 < y < \dfrac{a}{2} \\ -V_0 & ,\dfrac{a}{2} < y < a \end{cases}$$

$$\sum_{n=1}^{\infty} C_n \int_0^a \sin\frac{n\pi y}{a}\sin\frac{n'\pi y}{a}\,dy = \frac{a}{2}C_n\delta_{nn'} = \int_0^{\frac{a}{2}} V_0 \sin\frac{n'\pi y}{a}\,dy - \int_{\frac{a}{2}}^{a} V_0 \sin\frac{n'\pi y}{a}\,dy = \frac{aV_0}{n'\pi}\left(-\cos\frac{n'\pi y}{a}\Big|_0^{\frac{a}{2}} + \cos\frac{n'\pi y}{a}\Big|_{\frac{a}{2}}^{a}\right)$$

$$= \frac{aV_0}{n'\pi}\left(-\cos\frac{n'\pi}{2} + 1 + \cos n'\pi - \cos\frac{n'\pi}{2}\right) = \frac{aV_0}{n'\pi}\left[1 + (-1)^{n'} - 2\cos\frac{n'\pi}{2}\right]$$

$$\Rightarrow C_n = \frac{2V_0}{n\pi}\left[1 + (-1)^n - 2\cos\frac{n\pi}{2}\right] = \begin{cases} \dfrac{8V_0}{n\pi}, & \text{if } n = 2, 6, 10, \cdots, 4m-2, \cdots \\ 0, & \text{if } n \text{ else} \end{cases}$$

$$\Rightarrow V(x,y) = \sum_{m=1}^{\infty} \frac{8V_0}{(4m-2)\pi} e^{-\frac{(4m-2)\pi x}{a}} \sin\frac{(4m-2)\pi y}{a}$$



Problem 3.16 A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded (Fig. 3.23). The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box. [What should the potential at the center $(a/2, a/2, a/2)$ be? Check numerically that your formula is consistent with this value.][11]

Focus in the region inside the cube, we have B.C:

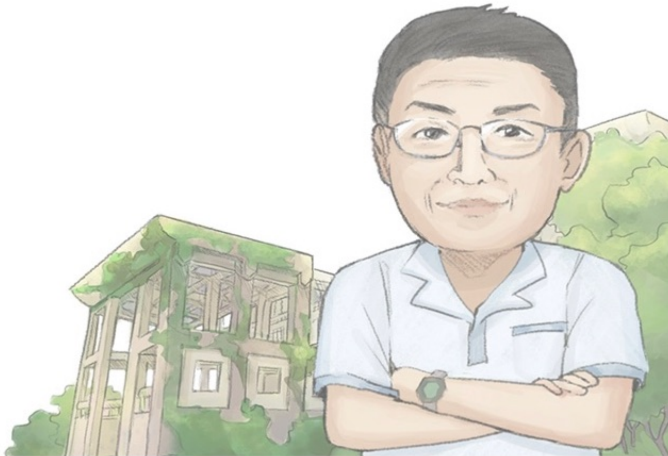
$$\begin{cases} (\text{i}) & V=0 \text{ when } x=0 \quad (\text{iv}) & V=0 \text{ when } y=a \\ (\text{ii}) & V=0 \text{ when } x=a \quad (\text{v}) & V=0 \text{ when } z=0 \\ (\text{iii}) & V=0 \text{ when } y=0 \quad (\text{vi}) & V=V_0 \text{ when } z=a \end{cases}$$

$$\nabla^2 V = 0 = \frac{1}{X}\frac{d^2X}{dX^2} + \frac{1}{Y}\frac{d^2Y}{dY^2} + \frac{1}{Z}\frac{d^2Z}{dZ^2} = 0$$

$$\Rightarrow V(x,y,z) = X(x)Y(y)Z(z) = \left(Ae^{i\alpha x} + Be^{-i\alpha x}\right)\left(Ce^{i\beta y} + De^{-i\beta y}\right)\left(Ee^{\gamma z} + Fe^{-\gamma z}\right)$$

$$(\text{i}): A+B=0 \Rightarrow V(x,y,z) = A\left(e^{i\alpha x} - e^{-i\alpha x}\right)(\cdots)(\cdots) = A(2i\sin\alpha a)(\cdots)(\cdots)$$

$$(\text{ii}): 2i\sin\alpha a = 0 \Rightarrow \alpha = \frac{n\pi}{a},\ \ n = 1,2,3,\ldots \Rightarrow V(x,y,z) = \sum_n A_n \sin\frac{n\pi x}{a}(\cdots)(\cdots)$$



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$$(\text{iii}),(\text{iv}): \beta = \frac{m\pi}{a},\;\; m = 1,2,3,\ldots \Rightarrow V(x,y,z) = \sum_{nm} A_n \sin\frac{n\pi x}{a} B_m \sin\frac{m\pi y}{a}(\cdots)$$

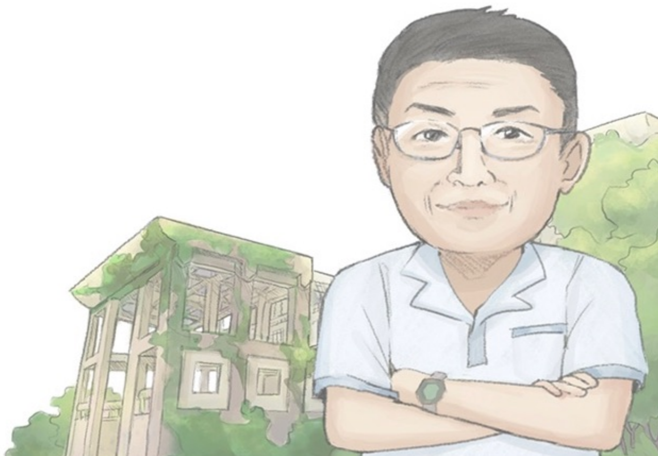
$$(\text{v}): E + F = 0 \Rightarrow V(x,y,z) = \sum_{nm} A_n \sin\frac{n\pi x}{a} B_m \sin\frac{m\pi y}{a} E\left(e^{\gamma z} - e^{-\gamma z}\right) = \sum_{nm} A_n \sin\frac{n\pi x}{a} B_m \sin\frac{m\pi y}{a} E(2\sinh\gamma z)$$

$$= \sum_{nm} C_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \sinh\gamma z$$

$$(\text{vi}): V(x,y,a) = \sum_{nm} C_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \sinh\gamma a = V_0$$

$$\int \sum_{nm} C_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{a} \sinh\gamma a \sin\frac{n'\pi x}{a} \sin\frac{m'\pi y}{a} dxdy = \frac{a^2}{4}\delta_{nn'}\delta_{mm'} C_{nm} \sinh\gamma a$$

$$= \int V_0 \sin\frac{n'\pi x}{a} \sin\frac{m'\pi y}{a} dxdy = V_0 \frac{2a}{n'\pi}\frac{2a}{m'\pi} \cos\frac{n'\pi x}{a}\Big|_0^a \cos\frac{m'\pi y}{a}\Big|_0^a$$



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$$(\text{vi}): V(x,y,a) = V_0 \frac{2a}{n'\pi}\frac{2a}{m'\pi}\cos\frac{n'\pi x}{a}\Big|_0^a \cos\frac{m'\pi y}{a}\Big|_0^a$$

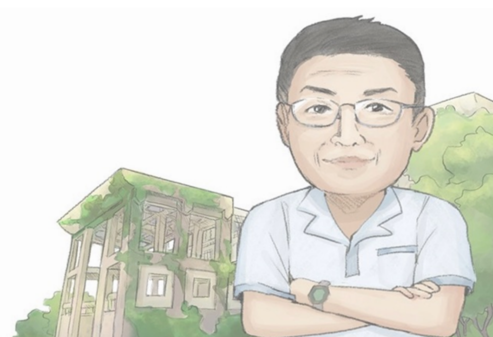
$$= V_0 \frac{a}{n'\pi}\frac{a}{m'\pi}\left[(-1)^{n'}-1\right]\left[(-1)^{m'}-1\right] = \begin{cases} \dfrac{4a^2V_0}{n'm'\pi^2} & ,\text{if } n' \text{ and } m' \text{ both odd} \\ 0 & ,\text{else} \end{cases}$$

$$\Rightarrow C_{nm} = \frac{16V_0}{nm\pi^2 \sinh\gamma a} \quad ,\text{if } n' \text{ and } m' \text{ both odd}$$

$$\Rightarrow V(x,y,z) = \sum_{nm}^{\text{both odd}} \frac{16V_0}{nm\pi^2}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}\frac{\sinh\gamma z}{\sinh\gamma a} \quad ,\gamma = \frac{\pi}{a}\sqrt{n^2+m^2}$$

$$V\left(\frac{a}{2},\frac{a}{2},\frac{a}{2}\right) = \sum_{nm}^{\text{both odd}} \frac{16V_0}{nm\pi^2}\sin\frac{n\pi}{2}\sin\frac{m\pi}{2}\frac{\sinh\frac{\gamma a}{2}}{\sinh\gamma a} = \sum_{nm}^{\text{both odd}} \frac{16V_0}{nm\pi^2}(-1)^{\frac{n+m}{2}+1}\frac{\sinh\frac{\gamma a}{2}}{\sinh\gamma a} \approx \frac{1}{6}V_0$$

n	m	Results
1	1	0.173773
1	3	-0.00376
3	1	-0.00376
3	3	0.00023
1	5	0.000108
5	1	0.000108
3	5	-1.1E-05
5	3	-1.1E-05
5	5	9.73E-07
1	7	-3.5E-06
7	1	-3.5E-06
3	7	4.92E-07
7	3	4.92E-07
5	7	-6.3E-08
7	5	-6.3E-08
7	7	5.84E-09
	SUM	**0.16667**



Problem 3.20 Suppose the potential $V_0(\theta)$ at the surface of a sphere is specified, and there is no charge inside or outside the sphere. Show that the charge density on the sphere is given by

$$\sigma(\theta) = \frac{\epsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta), \tag{3.88}$$

where

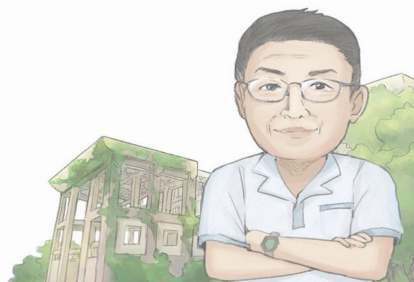
$$C_l = \int_0^{\pi} V_0(\theta) P_l(\cos\theta)\sin\theta\, d\theta. \tag{3.89}$$

$$\nabla^2 V = 0 = \frac{1}{r^2}\partial_r\left(r^2\partial_r V\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta V\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 V$$

$$V(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi) \Rightarrow \frac{1}{R}\partial_r\left(r^2\partial_r R\right) + \frac{1}{\Theta\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Theta\right) + \frac{1}{\Phi\sin^2\theta}\partial_\phi^2\Phi = 0$$

azimuthal symmetry: $\partial_\phi\Phi = 0 \Rightarrow \frac{1}{R}\partial_r\left(r^2\partial_r R\right) + \frac{1}{\Theta\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Theta\right) = 0 \Rightarrow \begin{cases} \frac{1}{R}\partial_r\left(r^2\partial_r R\right) = l(l+1) \\ \frac{1}{\Theta\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Theta\right) = -l(l+1) \end{cases}$

$$\Rightarrow V(r,\theta) = R(r)\Theta(\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)\left[P_l(\cos\theta)\right]$$



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where

$$C_l = \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta \, d\theta. \tag{3.89}$$

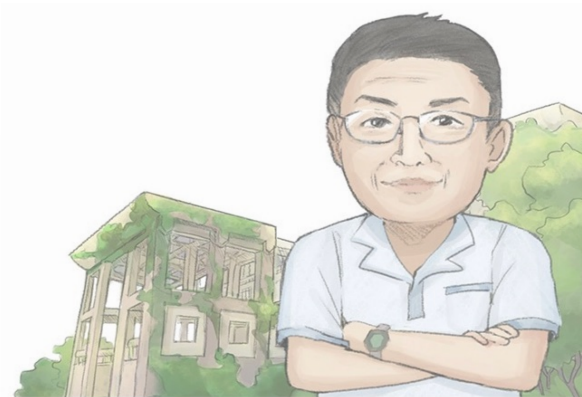
$$V(r,\theta) = R(r)\Theta(\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)\left[P_l(\cos\theta)\right]$$

$r > R$:

$$\begin{cases} r \to \infty : V = 0 \Rightarrow A_l = 0 \Rightarrow V = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta) \\ r = R : V = V_\theta(\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos\theta) \end{cases}$$

$r < R$:

$$\begin{cases} r \to 0 : V \text{ is finite} \Rightarrow B_l = 0 \Rightarrow V = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) \\ r = R : V = V_\theta(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta) \end{cases}$$



Problem 3.20

$r > R:$

$$\begin{cases} \int V_\theta(\theta) P_{l'}(\cos\theta)\sin\theta d\theta = \int \frac{B_l}{R^{l+1}} P_l(\cos\theta) P_{l'}(\cos\theta)\sin\theta d\theta = \frac{B_l}{R^{l+1}}\frac{2}{2l+1}\delta_{ll'} \\ \Rightarrow B_l = \frac{2l+1}{2}R^{l+1}\underbrace{\int V_\theta(\theta) P_l(\cos\theta)\sin\theta d\theta}_{C_l} \Rightarrow V = \sum_{l=0}^{\infty}\frac{2l+1}{2}\frac{R^{l+1}}{r^{l+1}}C_l P_l(\cos\theta) \end{cases}$$

$r < R:$

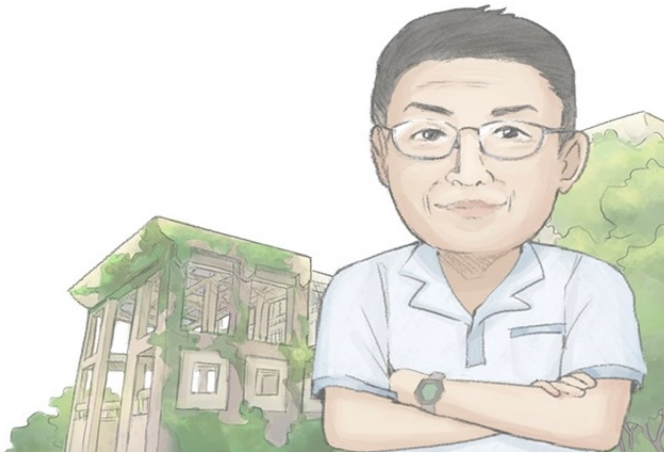
$$\left\{ A_l = \frac{2l+1}{2R^l}C_l \Rightarrow V = \sum_{l=0}^{\infty}\frac{2l+1}{2}\frac{r^l}{R^l}C_l P_l(\cos\theta) \right.$$

Discontinuity:

$$\partial_r V_{\text{outside}}(R)\Big|_{r=R} - \partial_r V_{\text{inside}}(R)\Big|_{r=R} = -\frac{\sigma(\theta)}{\varepsilon_0} = \sum_{l=0}^{\infty}\frac{2l+1}{2}\left[-(l+1)\frac{R^{l+1}}{R^{l+2}} - l\frac{R^{l-1}}{R^l}\right]C_l P_l(\cos\theta)$$

$$= \sum_{l=1}^{\infty} -\frac{(2l+1)^2}{2R}C_l P_l(\cos\theta)$$

$$\Rightarrow \sigma(\theta) = \frac{\varepsilon_0}{2R}\sum_{l=0}^{\infty}(2l+1)^2 C_l P_l(\cos\theta)$$



Problem 3.27 A sphere of radius R, centered at the origin, carries charge density

$$\rho(r,\theta) = k\frac{R}{r^2}(R-2r)\sin\theta,$$

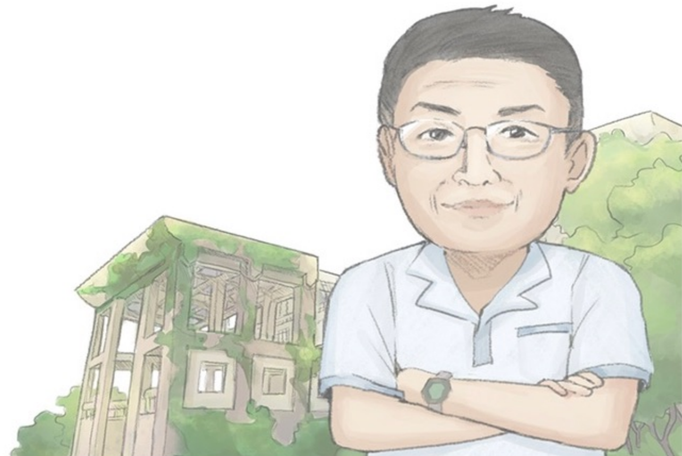
where k is a constant, and r, θ are the usual spherical coordinates. Find the approximate potential for points on the z axis, far from the sphere.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau', \qquad (3.95)$$

$$\lim_{r\to\infty} V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\theta')k\frac{R}{r'^2}(R-2r')\sin\theta' d\tau'$$

$$= \lim_{r\to\infty}\frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{kR}{r^{(n+1)}}\int_0^R (r')^n\frac{(R-2r')}{r'^2}r'^2 dr' \underbrace{\int_0^{\pi} P_n(\cos\theta')\sin\theta'\sin\theta' d\theta'}_{\text{□□□□□□□□□□}}\int_0^{2\pi} d\phi',$$

$$= \lim_{r\to\infty}\frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{kR}{r^{(n+1)}}\left(\frac{R}{n+1}r'^{n+1} - \frac{2}{n+2}r'^{n+2}\right)\Bigg|_0^R \int_0^{\pi}\left(\delta_{n,0} + \cos\theta'\delta_{n,1} + \frac{3\cos^2\theta'-1}{2}\delta_{n,2} + \dots\right)\sin^2\theta' d\theta'(2\pi)$$



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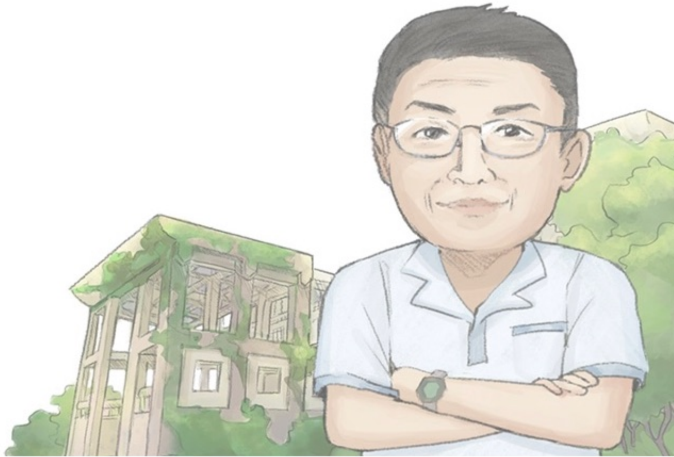
$$= \lim_{r\to\infty}\frac{1}{2\varepsilon_0}\left[\overbrace{\frac{kR}{r}\underbrace{\left(Rr'-r'^2\right)\Big|_0^R}_{=0}\int_0^\pi \sin\theta'\sin\theta' d\theta'}^{n=0} + \overbrace{\frac{kR}{r^2}\left(\frac{R}{2}r'^2-\frac{2}{3}r'^3\right)\Big|_0^R\underbrace{\int_0^\pi \cos\theta'\sin^2\theta' d\theta'}_{=0}}^{n=1} \right.$$
$$\left. + \overbrace{\frac{kR}{r^3}\left(\frac{R}{2}r'^3-\frac{2}{3}r'^4\right)\Big|_0^R\int_0^\pi \frac{3\cos^2\theta'-1}{2}\sin^2\theta' d\theta'}^{n=2} + \sum_{n=3}^{\infty}(\ldots)\right]$$

$$= \lim_{r\to\infty}\frac{1}{2\varepsilon_0}\left[\frac{-1}{12}\frac{kR^5}{r^3}\underbrace{\int_0^\pi\left(2\sin^2\theta'-3\sin^4\theta'\right)d\theta'}_{=2\left(\frac{\pi}{2}\right)-3\left(\frac{3\pi}{8}\right)=-\frac{\pi}{8}} + \sum_{n=3}^{\infty}(\ldots)\right] \approx \frac{1}{4\pi\varepsilon_0}\frac{1}{48}\frac{k\pi^2R^5}{r^3}$$



Problem 3.43 A conducting sphere of radius a, at potential V_0, is surrounded by a thin concentric spherical shell of radius b, over which someone has glued a surface charge

$$\sigma(\theta) = k\cos\theta,$$

where k is a constant and θ is the usual spherical coordinate.

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$$V_{\text{outside}} = \sum_{l=0}^{\infty}\frac{B_l^{r>b}}{r^{l+1}}P_l(\cos\theta)$$

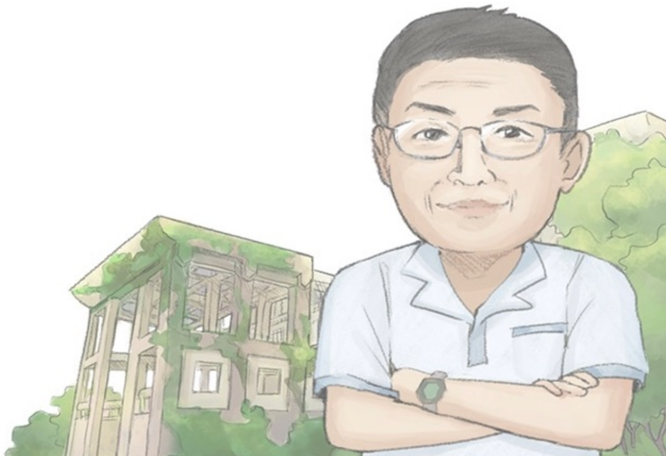
(a) Find the potential in each region: (i) $r > b$, and (ii) $a < r < b$.

$$V_{r>b}(r=b) = V_{b>r>a}(r=b) \Rightarrow \sum_{l=0}^{\infty}\frac{B_l^{r>b}}{b^{l+1}}P_l(\cos\theta) = \sum_{l=0}^{\infty}\left(A_l^{b>r>a}b^l + \frac{B_l^{b>r>a}}{b^{l+1}}\right)P_l(\cos\theta)$$

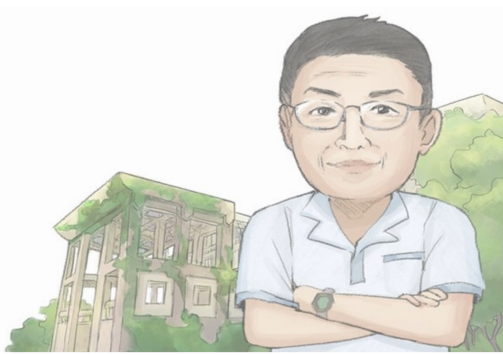
$$\Rightarrow \frac{B_l^{r>b}}{b^{l+1}} = A_l^{b>r>a}b^l + \frac{B_l^{b>r>a}}{b^{l+1}} \Rightarrow B_l^{r>b} = A_l^{b>r>a}b^{2l+1} + B_l^{b>r>a}$$

$$V_{b>r>a}(r=a) = V_0 = \sum_{l=0}^{\infty}\left(A_l^{b>r>a}a^l + \frac{B_l^{b>r>a}}{a^{l+1}}\right)P_l(\cos\theta)$$

$$\Rightarrow \begin{cases} A_0^{b>r>a} + \dfrac{B_0^{b>r>a}}{a} = V_0 \\ A_{l\neq 0}^{b>r>a}a^l + \dfrac{B_{l\neq 0}^{b>r>a}}{a^{l+1}} = 0 \end{cases} \Rightarrow \begin{cases} A_0^{b>r>a} + \dfrac{B_0^{b>r>a}}{a} = V_0 \\ A_{l\neq 0}^{b>r>a}a^{2l+1} + B_{l\neq 0}^{b>r>a} = 0 \end{cases}$$



Problem 3.43



$$\left.\frac{\partial V_{r>b}}{\partial r}\right|_{r=b}-\left.\frac{\partial V_{b>r>a}}{\partial r}\right|_{r=b}=-\frac{\sigma(\theta)}{\varepsilon_0}$$

$$=\left.\sum_{l=0}^{\infty}-(l+1)\frac{B_l^{r>b}}{r^{l+2}}P_l(\cos\theta)\right|_{r=b}-\left.\sum_{l=0}^{\infty}\left(lA_l^{b>r>a}r^{l-1}-(l+1)\frac{B_l^{b>r>a}}{r^{l+2}}\right)P_l(\cos\theta)\right|_{r=b}=-\frac{k\cos\theta}{\varepsilon_0}$$

$$\Rightarrow\begin{cases}-\dfrac{2}{b^3}B_1^{r>b}-A_1^{b>r>a}+\dfrac{2}{b^3}B_1^{b>r>a}=-\dfrac{k}{\varepsilon_0}\\ -\dfrac{l+1}{b^{l+2}}B_{l\neq1}^{r>b}-lb^{l-1}A_{l\neq1}^{b>r>a}+\dfrac{l+1}{b^{l+2}}B_{l\neq1}^{b>r>a}=0\end{cases}$$

$$B_l^{r>b}=A_l^{b>r>a}b^{2l+1}+B_l^{b>r>a}$$

$$\begin{cases}A_0^{b>r>a}+\dfrac{B_0^{b>r>a}}{a}=V_0\\ A_{l\neq0}^{b>r>a}a^{2l+1}+B_{l\neq0}^{b>r>a}=0\end{cases}$$

$l=0$

$$\begin{cases}B_0^{r>b}=A_0^{b>r>a}b+B_0^{b>r>a}\\ -\dfrac{1}{b^2}B_0^{r>b}+\dfrac{1}{b^2}B_0^{b>r>a}=0\\ A_0^{b>r>a}+\dfrac{B_0^{b>r>a}}{a}=V_0\end{cases}\Rightarrow\begin{cases}B_0^{r>b}=aV_0\\ A_0^{b>r>a}=0\\ B_0^{b>r>a}=aV_0\end{cases}$$

$l=1$

$$\begin{cases}B_1^{r>b}=A_1^{b>r>a}b^3+B_1^{b>r>a}\\ -\dfrac{2}{b^3}B_1^{r>b}-A_1^{b>r>a}+\dfrac{2}{b^3}B_1^{b>r>a}=-\dfrac{k}{\varepsilon_0}\\ A_1^{b>r>a}a^3+B_1^{b>r>a}=0\end{cases}\Rightarrow\begin{cases}B_1^{r>b}=\dfrac{k}{3\varepsilon_0}\left(b^3-a^3\right)\\ A_1^{b>r>a}=\dfrac{k}{3\varepsilon_0}\\ B_1^{b>r>a}=-\dfrac{a^3k}{3\varepsilon_0}\end{cases}$$

Problem 3.43

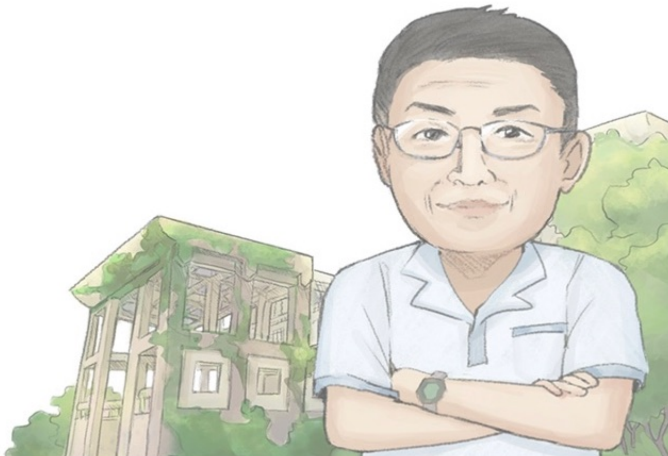
$$B_l^{r>b} = A_l^{b>r>a} b^{2l+1} + B_l^{b>r>a}$$

$$\begin{cases} A_0^{b>r>a} + \dfrac{B_0^{b>r>a}}{a} = V_0 \\ A_{l\neq 0}^{b>r>a} a^{2l+1} + B_{l\neq 0}^{b>r>a} = 0 \end{cases} \qquad \begin{cases} -\dfrac{2}{b^3} B_1^{r>b} - A_1^{b>r>a} + \dfrac{2}{b^3} B_1^{b>r>a} = -\dfrac{k}{\varepsilon_0} \\ -\dfrac{l+1}{b^{l+2}} B_{l\neq 1}^{r>b} - l b^{l-1} A_{l\neq 1}^{b>r>a} + \dfrac{l+1}{b^{l+2}} B_{l\neq 1}^{b>r>a} = 0 \end{cases}$$

$l \neq 0,1$

$$\begin{cases} B_{l\neq 0,1}^{r>b} = A_{l\neq 0,1}^{b>r>a} b^{2l+1} + B_{l\neq 0,1}^{b>r>a} \\ A_{l\neq 0,1}^{b>r>a} a^{2l+1} + B_{l\neq 0,1}^{b>r>a} = 0 \\ -\dfrac{l+1}{b^{l+2}} B_{l\neq 0,1}^{r>b} - l b^{l-1} A_{l\neq 0,1}^{b>r>a} + \dfrac{l+1}{b^{l+2}} B_{l\neq 0,1}^{b>r>a} = 0 \end{cases} \Rightarrow \begin{cases} B_{l\neq 0,1}^{r>b} = A_{l\neq 0,1}^{b>r>a} \left(b^{2l+1} - a^{2l+1} \right) \\ B_{l\neq 0,1}^{b>r>a} = -A_{l\neq 0,1}^{b>r>a} a^{2l+1} \\ A_{l\neq 0,1}^{b>r>a} \left[-\dfrac{l+1}{b^{l+2}} \left(b^{2l+1} - a^{2l+1} \right) - l b^{l-1} - \dfrac{l+1}{b^{l+2}} a^{2l+1} \right] = 0 \end{cases}$$

$$\Rightarrow B_{l\neq 0,1}^{r>b} = B_{l\neq 0,1}^{b>r>a} = A_{l\neq 0,1}^{b>r>a} = 0$$

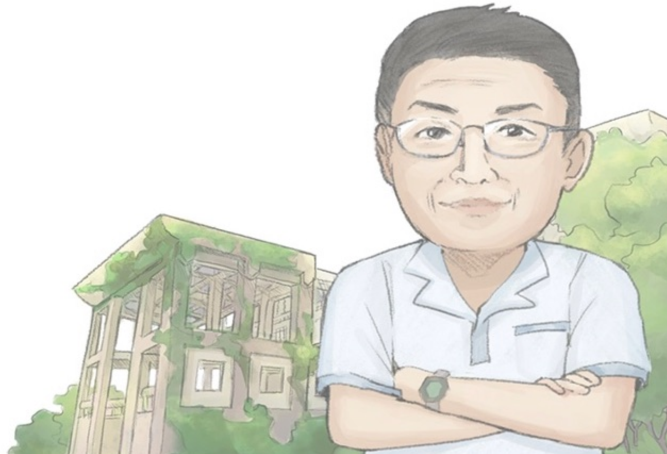


Problem 3.43

$$A_l^{r>b} = 0$$

$$\begin{cases} B_0^{r>b} = aV_0 \\ A_0^{b>r>a} = 0 \\ B_0^{b>r>a} = aV_0 \end{cases} \quad \begin{cases} B_1^{r>b} = \dfrac{k}{3\varepsilon_0}\left(b^3 - a^3\right) \\ A_1^{b>r>a} = \dfrac{k}{3\varepsilon_0} \\ B_1^{b>r>a} = -\dfrac{a^3 k}{3\varepsilon_0} \end{cases} \quad \begin{cases} B_{l\neq 0,1}^{r>b} = 0 \\ A_{l\neq 0,1}^{b>r>a} = 0 \\ B_{l\neq 0,1}^{b>r>a} = 0 \end{cases}$$

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta) = \begin{cases} \dfrac{aV_0}{r} + \dfrac{\left(b^3 - a^3\right)k\cos\theta}{3\varepsilon_0 r^2}, & r > b \\ \dfrac{aV_0}{r} + \dfrac{\left(r^3 - a^3\right)k\cos\theta}{3\varepsilon_0 r^2}, & b > r > a \end{cases}$$



Problem 3.43 A conducting sphere of radius a, at potential V_0, is surrounded by a thin concentric spherical shell of radius b, over which someone has glued a surface charge

$$\sigma(\theta) = k\cos\theta,$$

where k is a constant and θ is the usual spherical coordinate.

$$V(r,\theta) = \begin{cases} \dfrac{aV_0}{r} + \dfrac{\left(b^3 - a^3\right)k\cos\theta}{3\varepsilon_0 r^2}, & r > b \\ \dfrac{aV_0}{r} + \dfrac{\left(r^3 - a^3\right)k\cos\theta}{3\varepsilon_0 r^2}, & b > r > a \end{cases}$$

(b) Find the induced surface charge $\sigma_i(\theta)$ on the conductor.

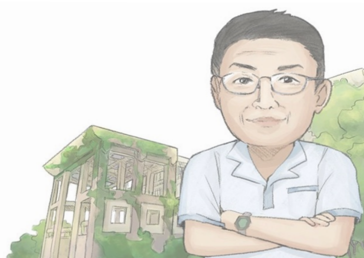
$$\left.\frac{\partial V_{b>r>a}}{\partial r}\right|_{r=a} = -\frac{\sigma_i(\theta)}{\varepsilon_0} = -\frac{aV_0}{a^2} - \frac{2\left(a^3 - a^3\right)k\cos\theta}{3\varepsilon_0 a^3} + \frac{3a^2 k\cos\theta}{3\varepsilon_0 a^2} \Rightarrow \sigma_i(\theta) = \frac{\varepsilon_0 V_0}{a} - k\cos\theta$$

(c) What is the total charge of this system? Check that your answer is consistent with the behavior of V at large r.

$$Q = \int \sigma(\theta) b^2 \sin\theta d\theta d\phi + \int \sigma_i(\theta) a^2 \sin\theta d\theta d\phi = \int kb^2 \cos\theta \sin\theta d\theta d\phi + \int \left(\frac{\varepsilon_0 V_0}{a} - k\cos\theta\right) a^2 \sin\theta d\theta d\phi = 4\pi a\varepsilon_0 V_0$$

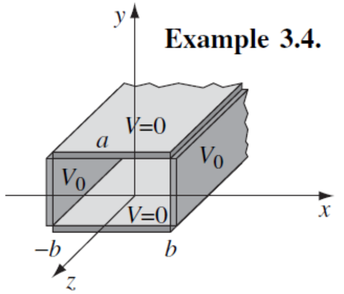
$$\lim_{r\to\infty} V_{r>b} = \lim_{r\to\infty}\left(\frac{aV_0}{r} + \frac{\left(b^3 - a^3\right)k\cos\theta}{3\varepsilon_0 r^2}\right) \approx \frac{aV_0}{r}$$

Regarding this system as a point charge with $Q = 4\pi a\varepsilon_0 V_0$: $V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{4\pi a\varepsilon_0 V_0}{r} = \dfrac{aV_0}{r}$, same.



Problem 3.54 For the infinite rectangular pipe in Ex. 3.4, suppose the potential on the bottom ($y = 0$) and the two sides ($x = \pm b$) is zero, but the potential on the top ($y = a$) is a nonzero constant V_0. Find the potential inside the pipe. [*Note:* This is a rotated version of Prob. 3.15(b), but set it up as in Ex. 3.4, using sinusoidal functions in y and hyperbolics in x. It is an unusual case in which $k = 0$ must be included. Begin by finding the general solution to Eq. 3.26 when $k = 0$.][26]

[*Answer:* $V_0\left(\frac{y}{a} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)}\sin(n\pi y/a)\right)$. Alternatively, using sinusoidal functions of x and hyperbolics in y, $-\frac{2V_0}{b}\sum_{n=1}^{\infty}\frac{(-1)^n\sinh(\alpha_n y)}{\alpha_n\sinh(\alpha_n a)}\cos(\alpha_n x)$, where $\alpha_n \equiv (2n-1)\pi/2b$]



Example 3.4.

(i) $V = 0$ when $x = -b$

(ii) $V = 0$ when $x = b$

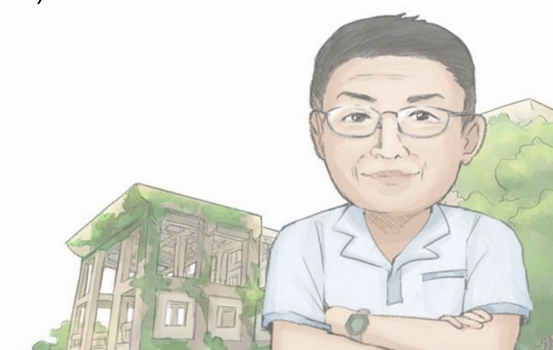
(iii) $V = 0$ when $y = 0$

(iv) $V = V_0$ when $y = a$

$$\nabla^2 V = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \Rightarrow \partial_x^2 X = k^2 X \quad , \partial_y^2 Y = -k^2 Y$$

$$\Rightarrow V = A\cosh kx \sin ky \Rightarrow \begin{cases} A\cosh kb \sin ky = 0 \\ A\cosh kx \sin ka = V_0 \end{cases} \text{No Fourier's trick}$$

$$\Rightarrow V = \sum_k X_k Y_k = \underbrace{const. + \left(\bar{a}x + \bar{b}\right)\left(\bar{c}y + \bar{d}\right) + \bar{e}xy}_{k=0} + \sum_{k\neq 0}\left(\bar{A}\sinh kx + \bar{B}\cosh kx\right)\left(\bar{C}\sin ky + \bar{D}\cos ky\right)$$



Problem 3.54

$$V=\sum_k X_k Y_k=\underbrace{const.+\left(\bar{a}x+\bar{b}\right)\left(\bar{c}y+\bar{d}\right)+\bar{e}xy}_{k=0}+\sum_{k\neq 0}\left(\bar{A}\sinh kx+\bar{B}\cosh kx\right)\left(\bar{C}\sin ky+\bar{D}\cos ky\right)$$

$$(\text{iii}):const.+\left(\bar{a}x+\bar{b}\right)\left(\bar{d}\right)+\sum_{k\neq 0}\left(\bar{A}\sinh kx+\bar{B}\cosh kx\right)\left(\bar{D}\right)=0\Rightarrow \bar{a}=\bar{D}=0 \qquad ,const.+\overline{bd}=0$$

$$\Rightarrow V=\bar{b}\bar{c}y+\bar{e}xy+\sum_{k\neq 0}\left(\bar{A}\sinh kx+\bar{B}\cosh kx\right)\left(\bar{C}\sin ky\right)$$

$$(\text{iv}):\bar{b}\bar{c}a+\bar{e}ax+\sum_{k\neq 0}\left(\bar{A}\sinh kx+\bar{B}\cosh kx\right)\left(\bar{C}\sin ka\right)=V_0\Rightarrow \bar{e}=\bar{C}\sin ka=0 \qquad ,k=\frac{n\pi}{a}\ ,\bar{b}\bar{c}a=V_0$$

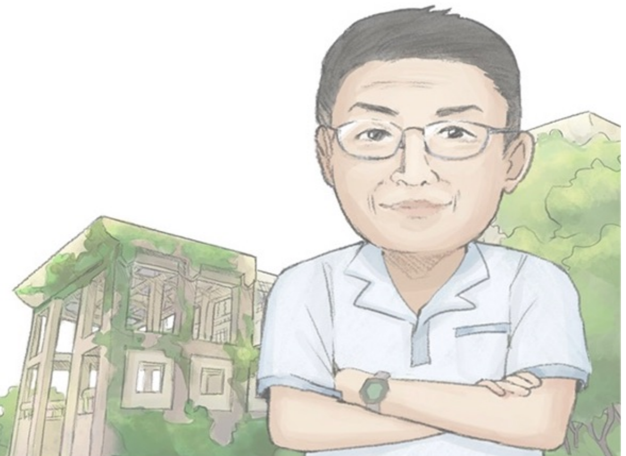
$$\Rightarrow V=\frac{V_0}{a}y+\sum_{n=1}^{\infty}\left(A_n\sinh\frac{n\pi x}{a}+B_n\cosh\frac{n\pi x}{a}\right)\sin\frac{n\pi y}{a}$$

(i) $V=0$ when $x=-b$

(ii) $V=0$ when $x=b$

(iii) $V=0$ when $y=0$

(iv) $V=V_0$ when $y=a$



Problem 3.54

$$V=\frac{V_0}{a}y+\sum_{n=1}^{\infty}\left(A_n\sinh\frac{n\pi x}{a}+B_n\cosh\frac{n\pi x}{a}\right)\sin\frac{n\pi y}{a}$$

(i) $V=0$ when $x=-b$

(ii) $V=0$ when $x=b$

(iii) $V=0$ when $y=0$

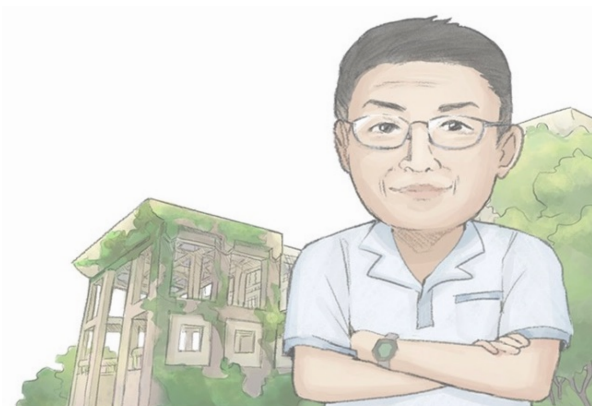
(iv) $V=V_0$ when $y=a$

$$(\mathrm{i}),(\mathrm{ii}):\begin{cases}\frac{V_0}{a}y+\sum_{n=1}^{\infty}\left(A_n\sinh\frac{n\pi b}{a}+B_n\cosh\frac{n\pi}{a}\right)\sin\frac{n\pi y}{a}=0\\ \frac{V_0}{a}y+\sum_{n=1}^{\infty}\left(-A_n\sinh\frac{n\pi b}{a}+B_n\cosh\frac{n\pi b}{a}\right)\sin\frac{n\pi y}{a}=0\end{cases}\Rightarrow\begin{cases}A_n=0\\ \sum_{n=1}^{\infty}B_n\left(\cosh\frac{n\pi b}{a}\right)\sin\frac{n\pi y}{a}=-\frac{V_0}{a}y\end{cases}$$

$$\Rightarrow\int\sin\frac{n'\pi y}{a}\sum_{n=1}^{\infty}B_n\left(\cosh\frac{n\pi b}{a}\right)\sin\frac{n\pi}{a}y\,dy=-\frac{V_0}{a}\int y\sin\frac{n\pi y}{a}dy$$

$$\Rightarrow B_n\cosh\frac{n\pi b}{a}=-\frac{V_0}{a}\frac{2}{a}\left(-\frac{a}{n\pi}y\cos\frac{n\pi y}{a}\bigg|_0^a+\int_0^a\frac{a}{n\pi}\cos\frac{n\pi y}{a}\right)=-\frac{2V_0}{a^2}\left(-\frac{a^2}{n\pi}(-1)^n\right)=\frac{2V_0}{n\pi}(-1)^n$$

$$\Rightarrow V=\frac{V_0}{a}y+\sum_{n=1}^{\infty}\frac{2V_0}{n\pi}(-1)^n\frac{\cosh\frac{n\pi x}{a}}{\cosh\frac{n\pi b}{a}}\sin\frac{n\pi}{a}y=V_0\left[\frac{y}{a}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\cosh\frac{n\pi x}{a}}{\cosh\frac{n\pi b}{a}}\sin\frac{n\pi}{a}y\right]$$



Problem 3.54

(i) $V = 0$ when $x = -b$

(ii) $V = 0$ when $x = b$

(iii) $V = 0$ when $y = 0$

(iv) $V = V_0$ when $y = a$

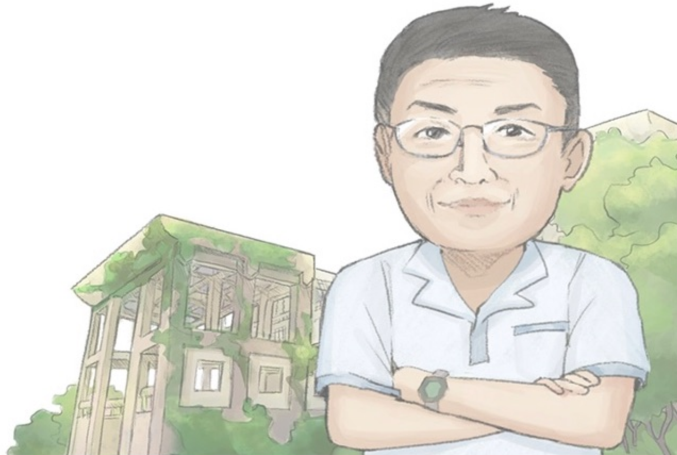
$$\nabla^2 V = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} = 0 \Rightarrow V = \sum_n \sin\frac{n\pi(x+b)}{2b} C_n \sinh\frac{n\pi y}{2b}$$

$$(\text{iv}) \rightarrow \sum_n C_n \sin\frac{n\pi(x+b)}{2b}\sinh\frac{n\pi a}{2b} = V_0$$

$$\Rightarrow \int_{-b}^{b} \sin\frac{n'\pi(x+b)}{2b} \sum_n C_n \sin\frac{n\pi(x+b)}{2b}\sinh\frac{n\pi a}{2b} dx = V_0 \int_{-b}^{b} \sin\frac{n'\pi(x+b)}{2b} dx$$

$$\Rightarrow C_n \sinh\frac{n\pi a}{2b} = \frac{2V_0}{2b}\int_0^{2b} \sin\frac{n\pi x'}{2b} dx' = \frac{2V_0}{2b}\frac{2b}{n\pi}(1-\cos n\pi) = \begin{cases} 0 & \text{, if } n \text{ even} \\ \dfrac{4V_0}{n\pi} & \text{, if } n \text{ odd} \end{cases}$$

$$\Rightarrow V = \sum_{n \text{ odd}} \frac{4V_0}{n\pi} \frac{\sinh\frac{n\pi y}{2b}}{\sinh\frac{n\pi a}{2b}} \sin\frac{n\pi(x+b)}{2b}$$



Problem 3.54

(i) $V=0$ when $x=-b$

(ii) $V=0$ when $x=b$

(iii) $V=0$ when $y=0$

(iv) $V=V_0$ when $y=a$

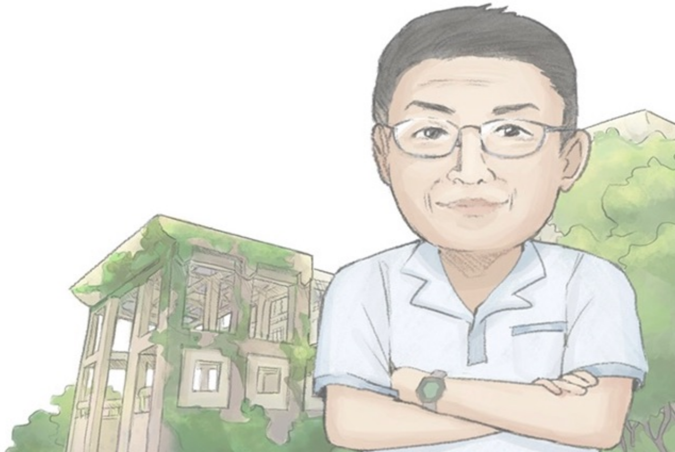
$$\sin(A+B)=\sin A\cos B+\sin B\cos A$$

$$V=\sum_{n\ \mathrm{odd}}\frac{4V_0}{n\pi}\frac{\sinh\frac{n\pi y}{2b}}{\sinh\frac{n\pi a}{2b}}\sin\frac{n\pi(x+b)}{2b}=\sum_{m=1}^{\infty}\frac{4V_0}{(2m-1)\pi}\frac{\sinh\frac{(2m-1)\pi y}{2b}}{\sinh\frac{(2m-1)\pi a}{2b}}\sin\frac{(2m-1)\pi(x+b)}{2b}$$

$$=\sum_{m=1}^{\infty}\frac{2V_0}{b\alpha_m}\frac{\sinh\alpha_m y}{\sinh\alpha_m a}\sin\alpha_m(x+b),\ \ \alpha_m=\frac{(2m-1)\pi}{2b}$$

$$=\sum_{m=1}^{\infty}\frac{2V_0}{b\alpha_m}\frac{\sinh\alpha_m y}{\sinh\alpha_m a}\left[\sin\alpha_m x\underbrace{\cos\frac{(2m-1)\pi}{2}}_{=0}+\cos\alpha_m x\underbrace{\sin\frac{(2m-1)\pi}{2}}_{=-(-1)^m}\right]$$

$$=-\frac{2V_0}{b}\sum_{m=1}^{\infty}\frac{(-1)^m\sinh\alpha_m y}{\alpha_m\sinh\alpha_m a}\cos\alpha_m x$$



Problem 3.56 An ideal electric dipole is situated at the origin, and points in the z direction, as in Fig. 3.36. An electric charge is released from rest at a point in the xy plane. Show that it swings back and forth in a semi-circular arc, as though it were a pendulum supported at the origin.[28]

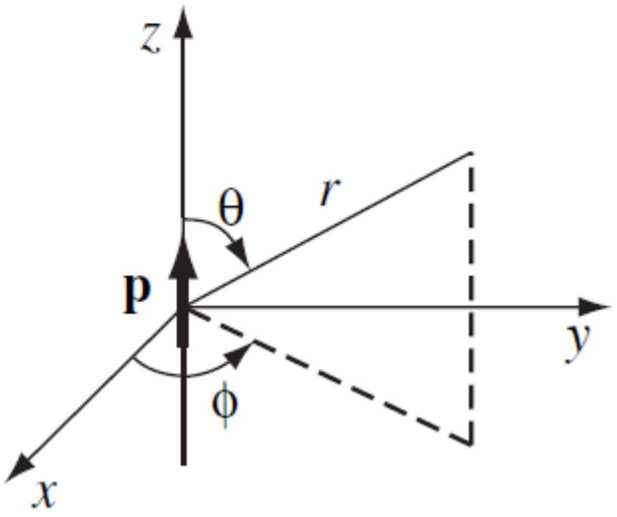


FIGURE 3.36

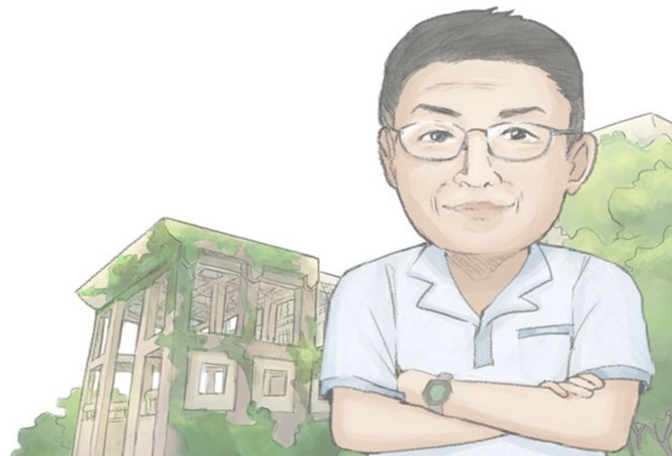
$$\mathbf{E}_{\text{dip}}(r,\theta)=\frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

$$\mathbf{F}=q\mathbf{E}_{\text{dip}}=\frac{qp}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

$$\mathbf{F}=mg\hat{\mathbf{z}}+\mathbf{T}\qquad T+mg\cos\theta=\frac{mv^2}{r},\qquad \frac{1}{2}mv^2+mgr\cos\theta=const.$$

$$\Rightarrow T=\frac{mv^2}{r}-mg\cos\theta=\frac{2mgr\cos\theta}{r}-mg\cos\theta=mg\cos\theta$$

$$\Rightarrow \mathbf{F}=mg\hat{\mathbf{z}}+\mathbf{T}=mg\left(\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}\right)+mg\cos\theta\,\hat{\mathbf{r}}=mg\left(2\cos\theta\,\hat{\mathbf{r}}+\sin\theta\,\hat{\boldsymbol{\theta}}\right)$$



Electromagnetism

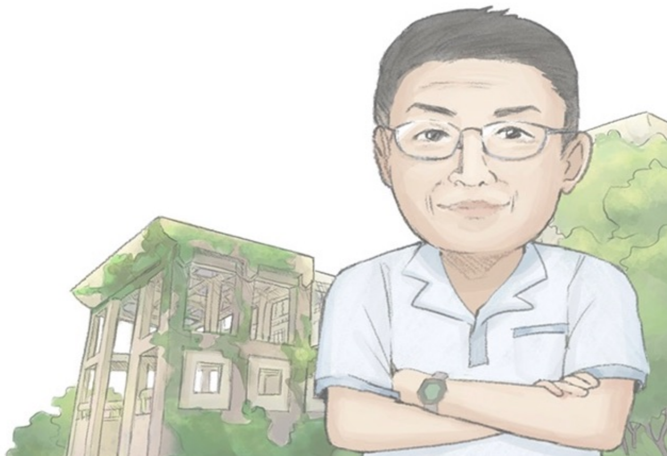
Introduction to Electrodynamics 4th David J. Griffiths

Chap.3

Prof. Tsun Hsu Chang

TA: Hung Chun Hsu, Yi Wen Lin, and Tien Fu Yang

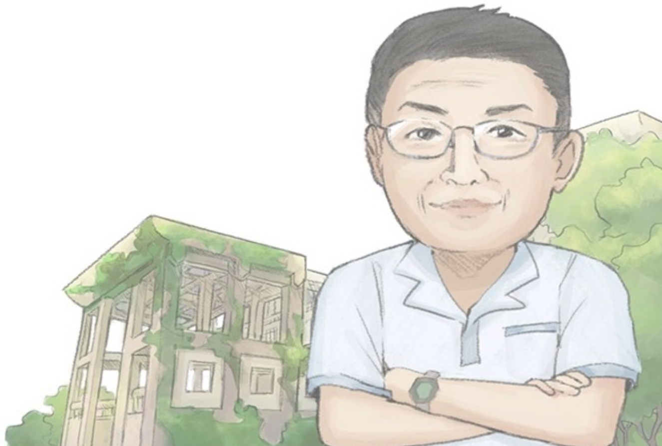
2022 Fall



Exercise List

11, 13, 16, 20, 27, 43, 54, 56,

7, 8, 12, 19, 23, 28, 29, 32, 44, 49



Problem 3.7 Find the force on the charge $+q$ in Fig. 3.14. (The xy plane is a grounded conductor.)

To keep $V=0$ on the xy plane $\Rightarrow$ need two image charges: charge $2q$ at $(0,0,-d)$ and charge $-q$ at $(0,0,-3d)$

$\therefore$ Force on charge q becomes

$$\mathbf{F}=\frac{1}{4\pi\varepsilon_0}\left[\frac{-2q^2}{(2d)^2}+\frac{2q^2}{(4d)^2}+\frac{-q^2}{(6d)^2}\right]\hat{z}=\frac{1}{4\pi\varepsilon_0}\frac{-q^2}{d^2}\frac{29}{72}$$

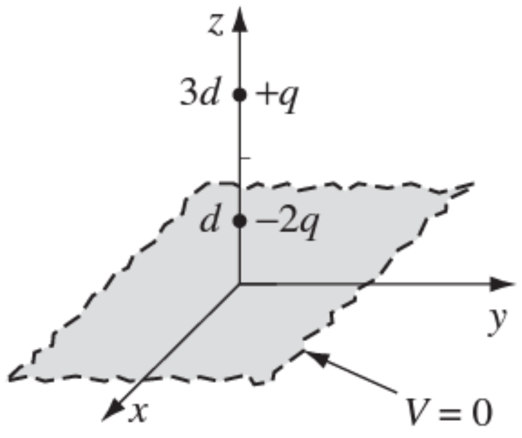
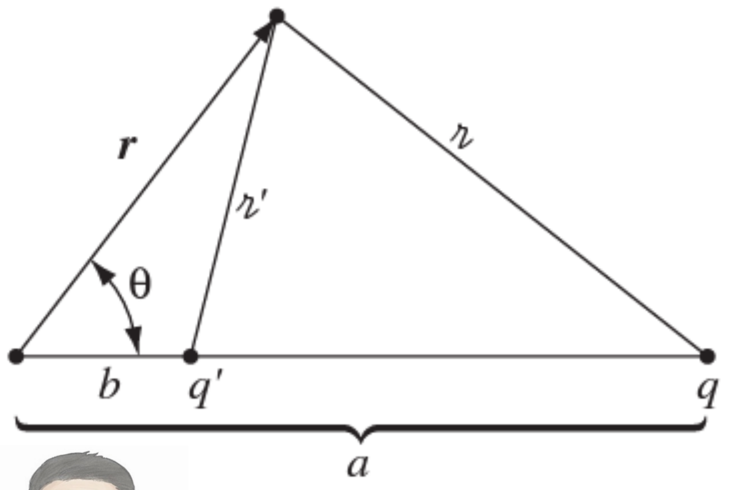


FIGURE 3.14

Problem 3.8

(a) With (3.15)-(3.17) + Law of cosine, it's straightforward.

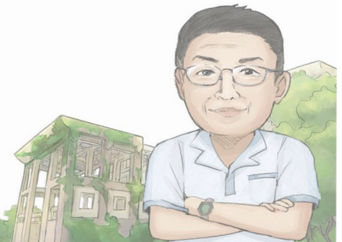


(b) Induced Surface Charge $\sigma=-\varepsilon_0\left.\frac{\partial V}{\partial n}\right|_{r=R}$

$$\left.\frac{\partial V}{\partial n}\right|_{r=R}=\frac{1}{4\pi\varepsilon_0}\frac{1}{2}\frac{q\left(\frac{2a^2}{R}-2R\right)}{\left(R^2+a^2-2aR\cos\theta\right)^{3/2}}\Rightarrow\sigma=\frac{q}{4\pi R}\frac{\left(R^2-a^2\right)}{\left(R^2+a^2-2aR\cos\theta\right)^{3/2}}$$

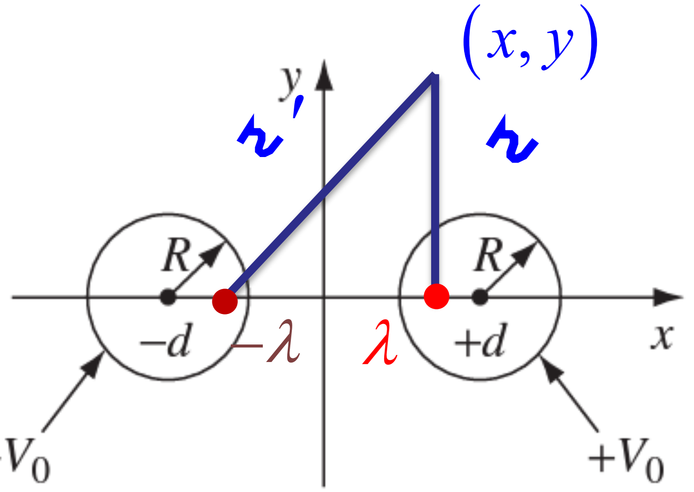
$$q'=\int\sigma R^2\sin\theta d\theta d\phi=\frac{q}{4\pi}\left(R^3-Ra^2\right)\cdot 2\pi\int_{-1}^{1}dx\left(R^2+a^2-2aRx\right)^{-3/2}=-\frac{R}{a}q$$

Consistent with (3.15)!



Problem 3.8

(c) $W=\int \mathbf{F}\cdot d\mathbf{l}=\int_{r=a}^{\infty}\frac{-Rq^2 r}{4\pi\varepsilon_0}\frac{1}{\left(r^2-R^2\right)^2}dr=-\frac{Rq^2}{8\pi\varepsilon_0}\frac{1}{a^2-R^2}$



Problem 3.12 Two straight parallel pipes

(1) Symmetry: $V=0$ at the center

(2) Circular pipes: Consider two thin straight lines $\mathbf{E}=\frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\hat{r}\Rightarrow V=\lim_{s\to\infty}\frac{\lambda}{2\pi\varepsilon_0}\ln\left|\frac{s}{a}\right|$ (Prob 2.22)

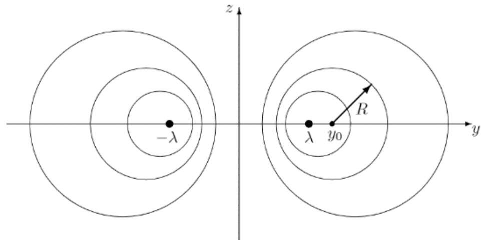
$$V=V_+ + V_- = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{\mathscr{r}'}{\mathscr{r}}=\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right)$$

Equi-potential V_0 constraints: $V_0=\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right)\Rightarrow\frac{(x+a)^2+y^2}{(x-a)^2+y^2}=e^{(4\pi\varepsilon_0 V_0/\lambda)}=k=\text{Const.}$

$$\Rightarrow x^2+2ax+a^2+y^2=k\left(x^2-2ax+a^2+y^2\right)\Rightarrow x^2(k-1)+y^2(k-1)+a^2(k-1)-2ax(k+1)=0$$

$\Rightarrow x^2+y^2+a^2-2ax\left(\frac{k+1}{k-1}\right)=0 \Leftarrow$ Two boundaries of the pipes are circles: $(x-x_0)^2+y^2=R^2$

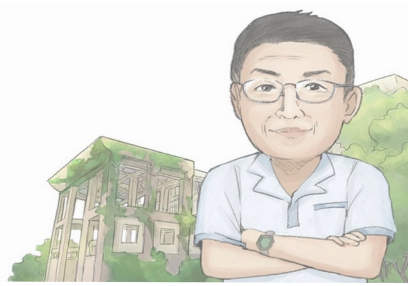
Problem 3.12 Two straight parallel pipes



$$x^2+y^2+a^2-2ax\left(\frac{k+1}{k-1}\right)=0 \Leftarrow \text{Two boundaries of the pipes are circles: } (x-x_0)^2+y^2=R^2$$

$$\left[x-a\left(\frac{k+1}{k-1}\right)\right]^2+y^2=\left(a\frac{k+1}{k-1}\right)^2-a^2 \Leftarrow x_0=a\left(\frac{k+1}{k-1}\right)=d \;\&\; R^2=a^2\left[\left(\frac{k+1}{k-1}\right)^2-1\right]\Rightarrow a=\sqrt{d^2-R^2}$$

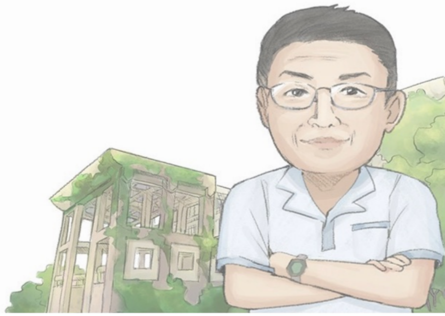
Recall: Equi-potential V_0 constraints: $V_0=\dfrac{\lambda}{4\pi\varepsilon_0}\ln\left(\dfrac{(x+a)^2+y^2}{(x-a)^2+y^2}\right)\Rightarrow\dfrac{(a+x)^2+y^2}{(a-x)^2+y^2}=e^{(4\pi\varepsilon_0V_0/\lambda)}=k=\text{Const.}$



$$\therefore\begin{cases}\dfrac{k+1}{k-1}=\dfrac{e^{(4\pi\varepsilon_0V_0/\lambda)}+1}{e^{(4\pi\varepsilon_0V_0/\lambda)}-1}=\dfrac{\cosh(2\pi\varepsilon_0V_0/\lambda)}{\sinh(2\pi\varepsilon_0V_0/\lambda)}=\coth\left(\dfrac{2\pi\varepsilon_0V_0}{\lambda}\right)=\dfrac{d}{a}\\[2ex]\left(\dfrac{k+1}{k-1}\right)^2-1=\dfrac{4k}{(k-1)^2}=\dfrac{4}{e^{(4\pi\varepsilon_0V_0/\lambda)}-2+e^{-(4\pi\varepsilon_0V_0/\lambda)}}=\dfrac{4}{\left[e^{(2\pi\varepsilon_0V_0/\lambda)}-e^{-(2\pi\varepsilon_0V_0/\lambda)}\right]^2}=\dfrac{1}{\sinh^2\left(\dfrac{2\pi\varepsilon_0V_0}{\lambda}\right)}\end{cases}$$

$$\Rightarrow\begin{cases}d=a\coth\left(\dfrac{2\pi\varepsilon_0V_0}{\lambda}\right)\\[2ex]R=a\dfrac{1}{\sinh(2\pi\varepsilon_0V_0/\lambda)}\end{cases}\Rightarrow\frac{d}{R}=\cosh\left(\frac{2\pi\varepsilon_0V_0}{\lambda}\right)\Rightarrow\lambda=\frac{2\pi\varepsilon_0V_0}{\cosh^{-1}(d/R)}\Rightarrow V=\frac{V_0}{2\cosh^{-1}(d/R)}\ln\left|\frac{\left(\sqrt{d^2-R^2}+x\right)^2+y^2}{\left(\sqrt{d^2-R^2}-x\right)^2+y^2}\right|$$

Problem 3.19



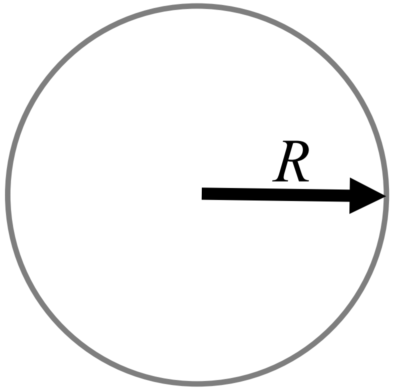
Laplace equation: $\nabla^2 V = 0 \overset{\text{Azimuthal symmetry}}{\Rightarrow} V(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta)$

Given: $V(R,\theta) = k\cos 3\theta = k\left(4\cos^3\theta - 3\cos\theta\right) = k\left[\alpha P_3(\cos\theta) + \beta P_1(\cos\theta)\right] \Leftarrow \because$ Odd function!

$\left[\text{Recall: } P_0(x) = 0,\ P_1(x) = x,\ P_2(x) = \frac{3}{2}x^2 - \frac{1}{2},\ P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x\right]$

$\therefore V(R,\theta) = \frac{k}{5}\left[8P_3(\cos\theta) - 3P_1(\cos\theta)\right]$

$\Rightarrow V(r,\theta) = \left(A_1 r + \frac{B_1}{r^2}\right)P_1(\cos\theta) + \left(A_3 r^3 + \frac{B_3}{r^4}\right)P_3(\cos\theta)$

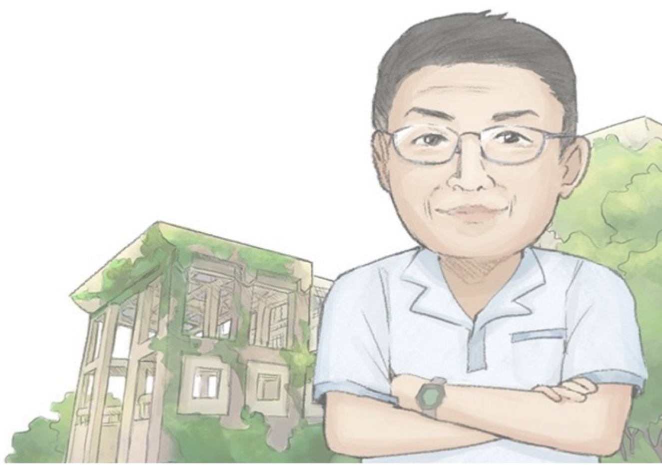


$V(R,\theta) = k\cos 3\theta$

$$\Rightarrow \begin{cases} V_{in} \text{ should be finite when } r \to 0 \Rightarrow V_{in} = A_1 r P_1(\cos\theta) + A_3 r^3 P_3(\cos\theta) \\ V_{out} \text{ should be finite when } r \to \infty \Rightarrow V_{out} = \frac{B_1}{r^2}P_1(\cos\theta) + \frac{B_3}{r^4}P_3(\cos\theta) \end{cases}$$

(Based on physics!)

Continuious constraint on the surface: $V_{in}\big|_{r=R} = V_{out}\big|_{r=R} \Rightarrow \begin{cases} A_1 R = \frac{B_1}{R^2} = \frac{-3}{5}k \\ A_3 R^3 = \frac{B_3}{R^4} = \frac{8}{5}k \end{cases} \Rightarrow \begin{cases} V_{in} = \frac{-3kr}{5R}P_1(\cos\theta) + \frac{8kr^3}{5R^3}P_3(\cos\theta) \\ V_{out} = \frac{-3kR}{5r^2}P_1(\cos\theta) + \frac{8kR^4}{5r^4}P_3(\cos\theta) \end{cases}$



Problem 3.19

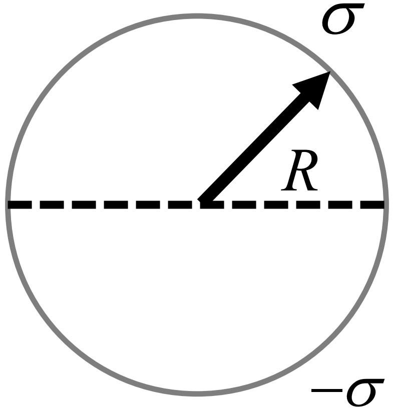
$$\begin{cases} V_{in} = \dfrac{-3kr}{5R} P_1(\cos\theta) + \dfrac{8kr^3}{5R^3} P_3(\cos\theta) \\ V_{out} = \dfrac{-3kR^2}{5r^2} P_1(\cos\theta) + \dfrac{8kR^4}{5r^4} P_3(\cos\theta) \end{cases}$$

To obtain the surface charge density

$\Rightarrow$ recall that radial derivative of V suffers a discontinuity at the surface. $\left.\dfrac{\partial V_{out}}{\partial r}\right|_{r=R} - \left.\dfrac{\partial V_{in}}{\partial r}\right|_{r=R} = -\dfrac{\sigma}{\varepsilon_0}$

$$\therefore \sigma(R,\theta) = \varepsilon_0\left[-\frac{9k}{5R} P_1(\cos\theta) + \frac{56k}{5R} P_3(\cos\theta)\right] = \frac{\varepsilon_0 k}{5R}\cos\theta\left[140\cos^2\theta - 93\right]$$

Problem 3.23



$$\nabla^2 V = 0 \Rightarrow \begin{cases} V_{in} = \sum_\ell \dfrac{B_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \\ V_{out} = \sum_\ell A_\ell r^\ell P_\ell(\cos\theta) \end{cases} \Leftarrow \text{Finite fact of physics}$$

B.C.1: $\left.V_{in}\right|_{r=R} = \left.V_{out}\right|_{r=R} \Rightarrow B_\ell = A_\ell R^{2\ell+1}$

B.C.2: $\left.\dfrac{\partial V_{out}}{\partial r}\right|_{r=R} - \left.\dfrac{\partial V_{in}}{\partial r}\right|_{r=R} = -\dfrac{\sigma}{\varepsilon_0} \Rightarrow \sigma(\theta) = \sum_\ell \varepsilon_0 (2\ell+1) A_\ell R^{\ell-1} P_\ell(\cos\theta)$

Problem 3.23

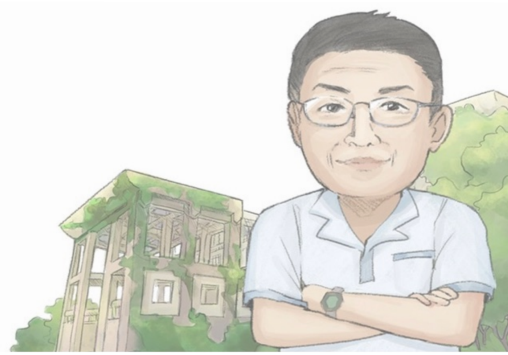
$$\sigma(\theta)=\sum_{\ell}\varepsilon_0(2\ell+1)A_\ell R^{\ell-1}P_\ell(\cos\theta)$$

$$\int_{-1}^{1}P_\ell(\cos\theta)P_{\ell'}(\cos\theta)d(\cos\theta)=\frac{2}{2\ell+1}\delta_{\ell,\ell'}$$

Now, the coefficients can be determined by Fourier's trick:

$$\int_{-1}^{1}\sigma(\theta)P_{\ell'}(\cos\theta)d(\cos\theta)=\int_{-1}^{1}\sum_{\ell}\varepsilon_0(2\ell+1)A_\ell R^{\ell-1}P_\ell(\cos\theta)P_{\ell'}(\cos\theta)d(\cos\theta)$$

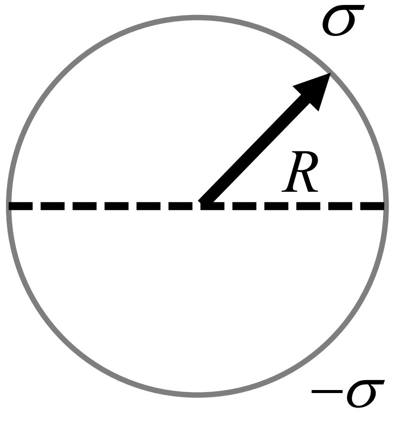
$$\Rightarrow A_\ell=\frac{1}{\varepsilon_0}\frac{1}{2\ell+1}\frac{1}{R^{\ell-1}}\frac{2\ell+1}{2}\int_0^{\pi}\sigma(\theta)P_\ell(\cos\theta)\sin\theta d\theta$$



Recall the given surface condition

$$A_\ell=\frac{\sigma_0}{2\varepsilon_0R^{\ell-1}}\left(\int_0^{\pi/2}P_\ell(\cos\theta)\sin\theta d\theta-\int_{\pi/2}^{\pi}P_\ell(\cos\theta)\sin\theta d\theta\right)=\frac{\sigma_0}{2\varepsilon_0R^{\ell-1}}\left(\int_0^{1}P_\ell(x)dx-\int_{-1}^{0}P_\ell(x)dx\right)$$

$$\Rightarrow A_\ell=\begin{cases}0, & \text{if } \ell \text{ is even}\\ \dfrac{\sigma_0}{\varepsilon_0R^{\ell-1}}\displaystyle\int_0^1P_\ell(x)dx, & \text{if } \ell \text{ is odd}\end{cases}\Rightarrow\begin{cases}A_0=A_2=A_4=\ldots\ldots=0\\ A_1=\dfrac{\sigma_0}{2\varepsilon_0},\ A_3=\dfrac{-\sigma_0}{8\varepsilon_0R^2},\ A_5=\dfrac{\sigma_0}{16\varepsilon_0R^4},\ \ldots\ldots\end{cases}$$

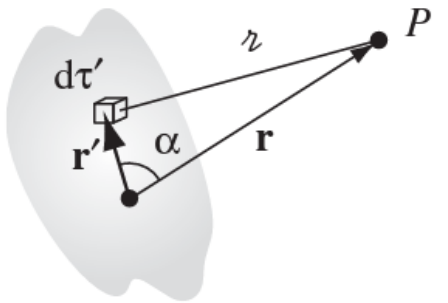


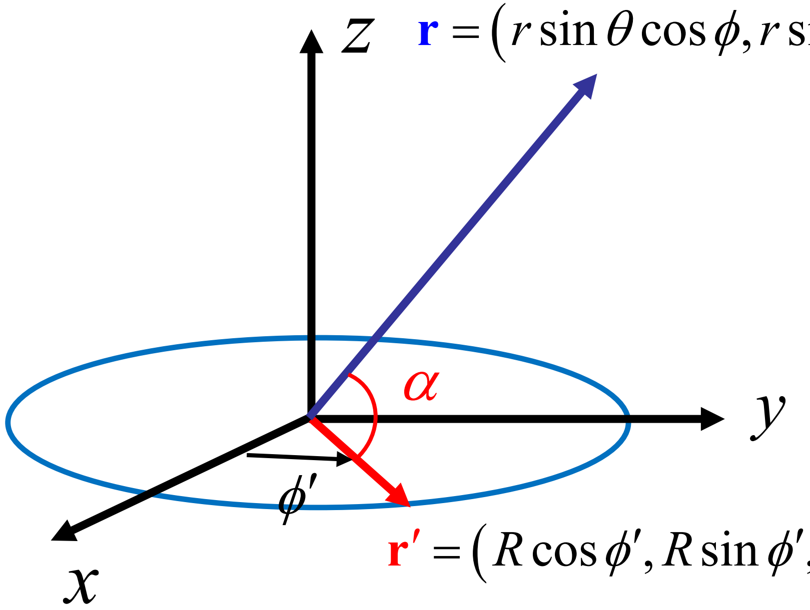
$$\because B_\ell=A_\ell R^{2\ell+1}\Rightarrow B_1=\frac{\sigma_0R^3}{2\varepsilon_0},\ B_3=\frac{-\sigma_0R^5}{8\varepsilon_0},\ B_5=\frac{\sigma_0R^7}{16\varepsilon_0}$$

Problem 3.28

Generating function of Legendre polynomials

Multiple expansion: $V(\mathbf{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau' = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau'$





$\mathbf{r} = (r\sin\theta\cos\phi, r\sin\theta\sin\phi, \cos\theta)$

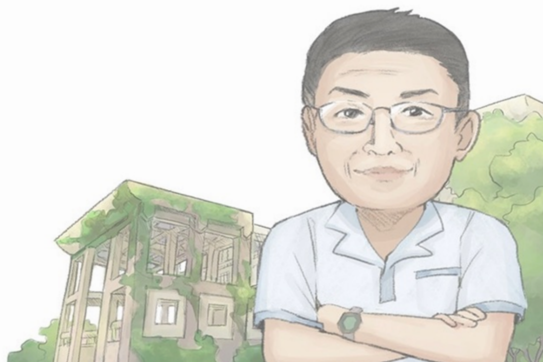
$\mathbf{r}' = (R\cos\phi', R\sin\phi', 0)$

$$\cos\alpha = \frac{\mathbf{r}\cdot\mathbf{r}'}{|\mathbf{r}||\mathbf{r}'|} = \sin\theta(\cos\phi\cos\phi' + \sin\phi\sin\phi')$$

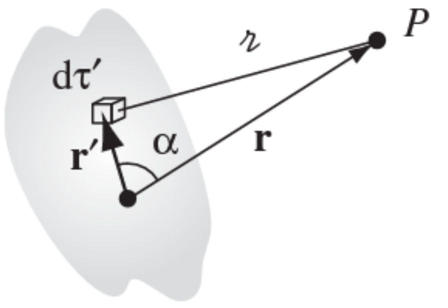
$$(1)\ n=0:\ V_0 = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int \rho(\mathbf{r}')\,d\tau' \rightarrow V_0 = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int_0^{2\pi}\lambda R\,d\phi' = \frac{\lambda}{2\varepsilon_0}\frac{R}{r}P_0(\cos\theta)$$

$$(2)\ n=1:\ V_1 = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int_0^{2\pi} R(\cos\alpha)\lambda R\,d\phi' = 0$$

$$(3)\ n=2:\ V_2 = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\int_0^{2\pi} R^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\lambda R\,d\phi' = -\frac{\lambda}{4\varepsilon_0}\left(\frac{R}{r}\right)^3 P_2(\cos\theta)$$



Problem 3.44

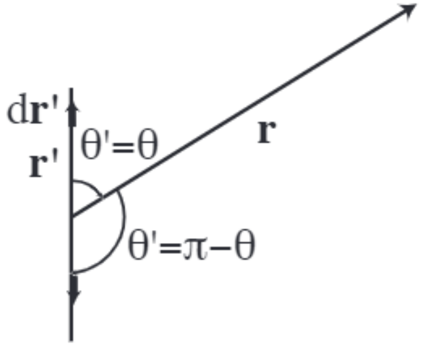


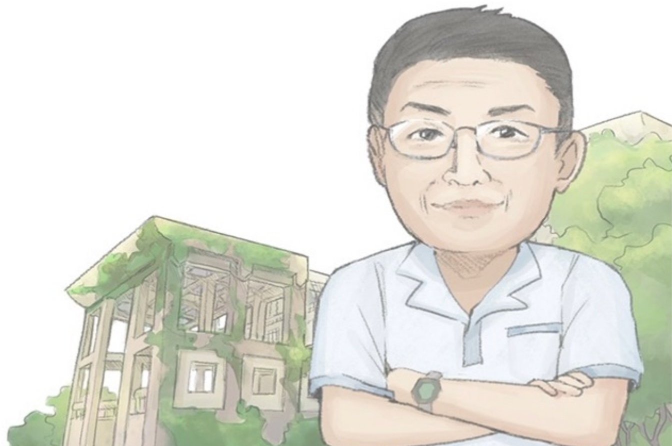
Multiple expansion: $V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\rho(\mathbf{r}')\,d\tau' = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)\rho(\mathbf{r}')\,d\tau'$

$$\therefore V = \frac{1}{4\pi\varepsilon_0}\int_{-a}^{a}\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)\lambda\,dz = \frac{\lambda}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int_{-a}^{a}(r')^n P_n(\cos\alpha)\,dz \equiv \frac{\lambda}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}} I$$

$$I = \int_{-a}^{0}(-z)^n P_n(\cos(\pi-\theta))\,dz + \int_{0}^{a} z^n P_n(\cos\theta)\,dz = \int_{a}^{0} y^n P_n(-\cos\theta)(-dy) + \int_{0}^{a} z^n P_n(\cos\theta)\,dz$$

$$\Rightarrow I = \left[1+(-1)^n\right]P_n(\cos\theta)\frac{a^{n+1}}{n+1} \Rightarrow V = \frac{Q}{8\pi\varepsilon_0 a}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\left[1+(-1)^n\right]P_n(\cos\theta)\frac{a^{n+1}}{n+1}$$



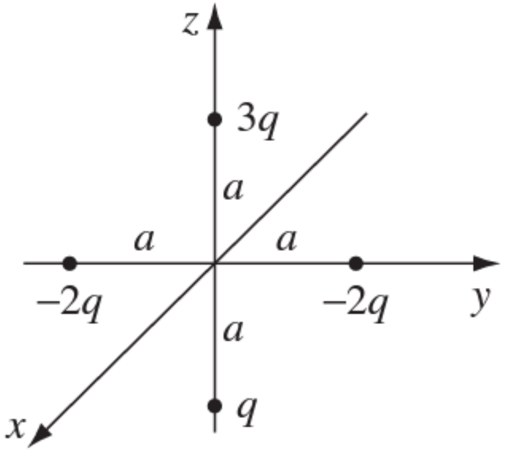


Problem 3.29

Observe this system at points far from the origin $\Rightarrow Q_{tot}=0 \Rightarrow$ No monopole!

$\therefore$ First non-vanishing term is dipole term. $\Rightarrow \mathbf{p}=\left(3qa-qa\right)\hat{z}+\left(-2qa-2q\left(-a\right)\right)\hat{y}=2qa\hat{z}$

$$\Rightarrow\ V \cong \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}=\frac{1}{4\pi\varepsilon_0}\frac{2qa\cos\theta}{r^2}$$



Problem 3.32

(a) monopole: $2q$ dipole: $3qa\hat{\mathbf{z}} \Rightarrow V \cong \dfrac{2q}{4\pi\varepsilon_0 r}+\dfrac{1}{4\pi\varepsilon_0}\dfrac{3qa\cos\theta}{r^2}$

(b) monopole: $2q$ dipole: $-qa\left(-\hat{\mathbf{z}}\right) \Rightarrow V \cong \dfrac{2q}{4\pi\varepsilon_0 r}+\dfrac{1}{4\pi\varepsilon_0}\dfrac{qa\cos\theta}{r^2}$

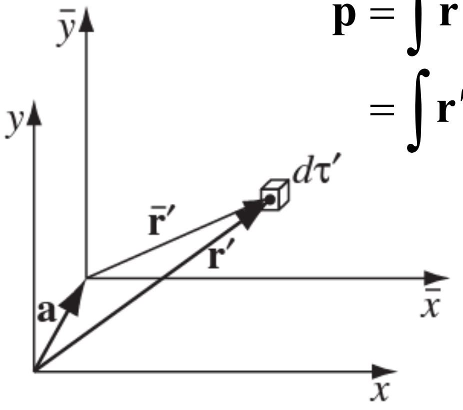
(c) monopole: $2q$ dipole: $3qa\hat{\mathbf{y}} \Rightarrow V \cong \dfrac{2q}{4\pi\varepsilon_0 r}+\dfrac{1}{4\pi\varepsilon_0}\dfrac{3qa\sin\theta\sin\phi}{r^2}$

$$\mathbf{p}\equiv\int \mathbf{r}'\rho\left(\mathbf{r}'\right)d\tau'=\sum_{i=1}^{n}q_i\mathbf{r}_i'$$

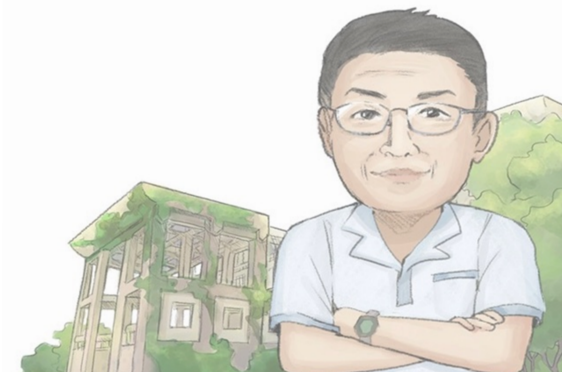
Collection of point charges

#Comments on moving the origin (charges):

a. Monopole moment doesn’t change, since the total charge is independent of the coordinate system.
b. Dipole moment changes when the origin is shifted if and only if the total charge is non-zero.

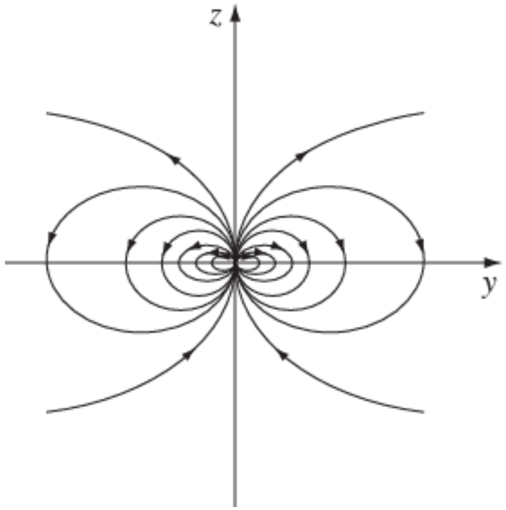


$$\bar{\mathbf{p}}=\int \bar{\mathbf{r}}'\rho\left(\mathbf{r}'\right)d\tau'=\int\left(\mathbf{r}'-\mathbf{a}\right)\rho\left(\mathbf{r}'\right)d\tau'$$
$$=\int \mathbf{r}'\rho\left(\mathbf{r}'\right)d\tau'-\mathbf{a}\int\rho\left(\mathbf{r}'\right)d\tau'=\mathbf{p}-Q\mathbf{a}$$



Problem 3.49 Correct form for dipole field $\mathbf{E}_{dip}$

Before going to the detail of the problem, let's examine the case on the handout.



On page 60, it's given that $\mathbf{E}=\dfrac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\boldsymbol{\theta}}\right)$.

If you consider the case at the origin, the dipole field blows up at $r=0$.

Recall the way we did in Chap.1 $\Rightarrow$ introduce the dirac delta function $\delta(x)$

$$\Rightarrow \mathbf{E}_{dip}=\frac{1}{4\pi\varepsilon_0}\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}}{r^3}-\frac{\mathbf{p}}{3\varepsilon_0}\delta^3(\mathbf{r})$$

Now, go back to the original question, where the surface charge density $\sigma=k\cos\theta$

First tackle the problem by multiple expansion: by symmetry, the dipole moment is clearly along $\hat{\mathbf{z}}-$axis.

$$\Rightarrow \mathbf{p}=\hat{\mathbf{z}}\int z\sigma da=\hat{\mathbf{z}}\int(R\cos\theta)(k\cos\theta)R^2\sin\theta d\theta d\phi=\frac{4\pi R^3k}{3}\hat{\mathbf{z}}$$

$$\Rightarrow V_{dip,out}=\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}=\frac{4\pi R^3k}{3}\frac{\cos\theta}{r^2}=V_{exact}$$ $\Leftarrow$ All multiple moments, except dipole, of this distribution are ZERO!

Recall eq. (3.86), the potential inside is $V=\dfrac{kr}{3\varepsilon_0}\cos\theta=\dfrac{kz}{3\varepsilon_0}\Rightarrow \mathbf{E}=-\dfrac{k}{3\varepsilon_0}\hat{\mathbf{z}}=-\dfrac{\mathbf{p}}{4\pi\varepsilon_0R^3}$ $\Leftarrow$ Blow up as $R\to 0$

Consider its volume integral: $\displaystyle\int\mathbf{E}d\tau=-\frac{\mathbf{p}}{3\varepsilon_0}$ $\Leftarrow$ Independent of R! $\Rightarrow$ Match the delta-function term!