# ***Electromagnetism***

Introduction to Electrodynamics 4th David J. Griffiths

Chap.4

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2022 Fall

# Exercise List

2, 9, 10, 16, 21, 28, 33, 36, 39

6, 13, 14, 15, 18, 19, 22, 24, 26, 31, 32

## Problem 4.2

$$\mathbf{p} = \alpha \mathbf{E}_e$$

$$\oint \mathbf{E}_e \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

$$\Rightarrow E_e(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int_0^r \rho d\tau = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{q}{\pi a^3}\int_0^r e^{-\frac{2r'}{a}} 4\pi r'^2 dr' = \frac{1}{\pi\varepsilon_0}\frac{q}{r^2}\frac{1}{a^3}\int_0^r e^{-\frac{2r'}{a}} r'^2 dr'$$

$$= \frac{1}{\pi\varepsilon_0}\frac{q}{r^2}\frac{1}{a^3}\left\{\left[\left(-\frac{a}{2}\right)e^{-\frac{2r'}{a}} r'^2\right]_0^r - \int_0^r \left(-\frac{a}{2}\right)e^{-\frac{2r'}{a}} 2r'dr'\right\}$$

$$= \frac{1}{\pi\varepsilon_0}\frac{q}{r^2}\frac{1}{a^3}\left\{\left(-\frac{a}{2}\right)e^{-\frac{2r}{a}} r^2 - \left[\left(-\frac{a}{2}\right)^2 e^{-\frac{2r'}{a}}(2r')\right]_0^r + \int_0^r \left(-\frac{a}{2}\right)^2 e^{-\frac{2r'}{a}}(2)dr'\right\}$$

$$= \frac{1}{\pi\varepsilon_0}\frac{q}{r^2}\frac{1}{a^3}\left[\left(-\frac{a}{2}\right)e^{-\frac{2r}{a}} r^2 - \left(-\frac{a}{2}\right)^2 e^{-\frac{2r}{a}}(2r) + \left(-\frac{a}{2}\right)^3 e^{-\frac{2r}{a}}(2) - \left(-\frac{a}{2}\right)^3 (2)\right]$$

$$= \frac{1}{\pi\varepsilon_0}\frac{q}{r^2}\frac{1}{a^3}\left(-\frac{a}{2}\right)^3\left[e^{-\frac{2r}{a}}\left(\frac{4r^2}{a^2} + \frac{4r}{a} + 2\right) - 2\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\left[1 - e^{-\frac{2r}{a}}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right]$$

**Problem 4.2**

$$\mathbf{p} = \alpha \mathbf{E}_e$$

$$E_e(\mathbf{r}) == \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\left[1 - e^{-\frac{2r}{a}}\left(1 + 2\frac{r}{a} + 2\frac{r^2}{a^2}\right)\right]$$

$$\Rightarrow E_e(\mathbf{d}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}\left[1 - e^{-\frac{2d}{a}}\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}\left\{1 - \left[1 + \left(-\frac{2d}{a}\right) + \frac{1}{2}\left(-\frac{2d}{a}\right)^2 + \frac{1}{3!}\left(-\frac{2d}{a}\right)^3 + \cdots\right]\left(1 + 2\frac{d}{a} + 2\frac{d^2}{a^2}\right)\right\}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{d^2}\frac{4}{3}\left(\frac{d}{a}\right)^3 + \text{higher order terms}$$

$$\approx \frac{1}{3\pi\varepsilon_0 a^3} qd = \frac{1}{3\pi\varepsilon_0 a^3} p \Rightarrow \alpha = 3\pi\varepsilon_0 a^3$$

## Problem 4.9

$$(\mathrm{a})\,\mathbf{F}=(\mathbf{p}\cdot\nabla)\mathbf{E}=(\mathbf{p}\cdot\nabla)\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

$$=p_r\partial_r\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)\hat{\mathbf{r}}+p_\theta\frac{1}{r}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)(\partial_\theta\hat{\mathbf{r}})+p_\varphi\frac{1}{r\sin\theta}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)(\partial_\varphi\hat{\mathbf{r}})\ \text{,gradient of a vector field}$$

$$=p_r\partial_r\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)\hat{\mathbf{r}}+p_\theta\frac{1}{r}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)\hat{\boldsymbol{\theta}}+p_\varphi\frac{1}{r\sin\theta}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)(\sin\theta\hat{\boldsymbol{\varphi}})$$

$$=\frac{q}{4\pi\varepsilon_0}\left(-2\frac{p_r}{r^3}\hat{\mathbf{r}}+\frac{p_\theta}{r^3}\hat{\boldsymbol{\theta}}+\frac{p_\varphi}{r^3}\hat{\boldsymbol{\varphi}}\right)$$

$$=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}(\mathbf{p}-3p_r\hat{\mathbf{r}})=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\left[\mathbf{p}-3(\mathbf{p}\cdot\mathbf{r})\hat{\mathbf{r}}\right]$$

$$(\mathrm{b})\,\mathbf{F}=q\mathbf{E}_{\mathrm{dip}}(\mathbf{r})=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}\right]$$

## Problem 4.10

(a) $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = kR,\ \rho_b = -\nabla\cdot\mathbf{P} = -\frac{1}{r^2}\partial_r\left(r^2 P_r\right) = -\frac{1}{r^2}\partial_r\left(r^2 kr\right) = -3k$

(b)

$$r > R: \oint \mathbf{E}\cdot d\mathbf{a} = \frac{1}{\varepsilon_0}Q_{enc} \Rightarrow \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\left(-3k\times\frac{4\pi r^3}{3}\right)\hat{\mathbf{r}} = -\frac{kr}{\varepsilon_0}\hat{\mathbf{r}}$$

$$r < R: \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q_{tot}}{r^2}\hat{\mathbf{r}} = \frac{1}{4\pi\varepsilon_0}\frac{\left(kR\times 4\pi R^2\right)+\left(-3k\times\frac{4\pi R^3}{3}\right)}{r^2}\hat{\mathbf{r}} = 0$$

**Problem 4.16**

$\rho_b = -\nabla \cdot \mathbf{P} = 0 \Rightarrow \mathbf{E}_{\rho_b} = 0$

(a)

the electric field inside a uniformly polarized sphere of radius R:

eq.(4.14): $\mathbf{E}_{\sigma_b}(\mathbf{P}) = -\frac{1}{3\varepsilon_0}\mathbf{P}$

$$\Rightarrow \mathbf{E} = \mathbf{E}_0 + \mathbf{E}_{\sigma_b}(-\mathbf{P}) = \mathbf{E}_0 + \left[-\frac{1}{3\varepsilon_0}(-\mathbf{P})\right] = \mathbf{E}_0 + \frac{1}{3\varepsilon_0}\mathbf{P}$$

$$\Rightarrow \mathbf{D} = \varepsilon_0\mathbf{E} = \varepsilon_0\mathbf{E}_0 + \frac{1}{3}\mathbf{P} = (\mathbf{D}_0 - \mathbf{P}) + \frac{1}{3}\mathbf{P} = \mathbf{D}_0 - \frac{2}{3}\mathbf{P}$$

(b)

$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = P$ but $da \to 0 \Rightarrow \mathbf{E}_{\sigma_b} = 0$

$$\Rightarrow \mathbf{E} = \mathbf{E}_0 \Rightarrow \mathbf{D} = \varepsilon_0\mathbf{E} = \varepsilon_0\mathbf{E}_0 = \mathbf{D}_0 - \mathbf{P}$$

(b)

Electric field of a sheet of surface charge $\sigma : \mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{\mathbf{n}}$

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = P \Rightarrow \mathbf{E}_{\sigma_b} = \frac{\sigma_b}{2\varepsilon_0}\hat{\mathbf{n}} \Rightarrow \mathbf{E}_{\sigma_b}^{\text{wafer}}(\mathbf{P}) = \mathbf{E}_{\sigma_b}^{\text{upper layer}} + \mathbf{E}_{\sigma_b}^{\text{lower layer}} = -\frac{\sigma_b}{2\varepsilon_0}\hat{\mathbf{n}} - \frac{\sigma_b}{2\varepsilon_0}\hat{\mathbf{n}} = -\frac{\sigma_b}{\varepsilon_0}\hat{\mathbf{n}} = -\frac{\mathbf{P}}{\varepsilon_0}$$

$$\Rightarrow \mathbf{E} = \mathbf{E}_0 + \mathbf{E}_{\sigma_b}^{\text{wafer}}(-\mathbf{P}) = \mathbf{E}_0 + \left(-\frac{(-\mathbf{P})}{\varepsilon_0}\right) = \mathbf{E}_0 + \frac{\mathbf{P}}{\varepsilon_0} \Rightarrow \mathbf{D} = \varepsilon_0\mathbf{E} = \varepsilon_0\mathbf{E}_0 + \mathbf{P} = (\mathbf{D}_0 - \mathbf{P}) + \mathbf{P} = \mathbf{D}_0$$

## Problem 4.21

Let $Q$ be the charge of a length $L$ of inner conductor.

$$\oint \mathbf{D}\cdot d\mathbf{a} = D2\pi sL = Q \Rightarrow \mathbf{D} = \frac{Q}{2\pi sL} = \varepsilon\mathbf{E} \Rightarrow \mathbf{E}(\varepsilon) = \frac{Q}{2\pi s\varepsilon L}$$

$$\Rightarrow V = -\int_c^a \mathbf{E}(\varepsilon)\cdot dl = -\int_b^a \frac{Q}{2\pi s\varepsilon_0 L}dl - \int_c^b \frac{Q}{2\pi s\varepsilon_0\varepsilon_r L}\cdot dl = \frac{Q}{2\pi\varepsilon_0 L}\left[\ln\left(\frac{b}{a}\right) + \frac{1}{\varepsilon_r}\ln\left(\frac{c}{b}\right)\right]$$

$$\Rightarrow \frac{C}{L} = \frac{Q}{VL} = 2\pi\varepsilon_0\left[\ln\left(\frac{b}{a}\right) + \frac{1}{\varepsilon_r}\ln\left(\frac{c}{b}\right)\right]^{-1}$$

## Problem 4.28

$$F_{\text{capacitance}} = F_{\text{gravity}} = mg = \rho\pi\left(b^2 - a^2\right)gh$$

$$F_{\text{capacitance}} = \frac{1}{2}\left(\frac{Q}{C}\right)^2 \frac{dC}{dx} = \frac{1}{2}V^2\frac{dC}{dh}$$

Gauss Law:

$$\begin{cases} \text{Air part: } E = \dfrac{2\lambda}{4\pi\varepsilon_0 s} \Rightarrow V = \dfrac{2\lambda}{4\pi\varepsilon_0}\ln\dfrac{b}{a} \\ \text{Oil part: } E = \dfrac{2\lambda'}{4\pi\varepsilon s} \Rightarrow V = \dfrac{2\lambda'}{4\pi\varepsilon}\ln\dfrac{b}{a} \end{cases} \Rightarrow \frac{\lambda}{\varepsilon_0} = \frac{\lambda'}{\varepsilon} \Rightarrow \lambda' = \frac{\varepsilon}{\varepsilon_0}\lambda = \varepsilon_r\lambda$$

$$Q = \lambda' h + \lambda(l - h) = \lambda\left[l + \left(\varepsilon_r - 1\right)h\right] = \lambda\left(l + \chi_e h\right) \Rightarrow C = \frac{Q}{V} = \frac{\lambda\left(l + \chi_e h\right)}{\dfrac{2\lambda}{4\pi\varepsilon_0}\ln\dfrac{b}{a}} = \frac{2\pi\varepsilon_0\left(l + \chi_e h\right)}{\ln\dfrac{b}{a}}$$

$$F_{\text{capacitance}} = \frac{1}{2}V^2\frac{dC}{dh} = \frac{1}{2}V^2\frac{2\pi\varepsilon_0\chi_e}{\ln\dfrac{b}{a}} = \rho\pi\left(b^2 - a^2\right)gh \Rightarrow h = \frac{\varepsilon_0\chi_e V^2}{\rho\left(b^2 - a^2\right)g\ln\dfrac{b}{a}}$$

## Problem 4.33

$\mathbf{P} = k\mathbf{r}$,

$$\begin{cases} \rho_b = -\nabla \cdot \mathbf{P} = -3k \Rightarrow Q_{vol} = -3k \times a^3 = -3ka^3 \\ \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = k\left(\frac{a}{2}\right) \Rightarrow Q_{surface} = \frac{ka}{2} \times 6a^2 = 3ka^3 \end{cases} \Rightarrow Q_{total} = 0$$

## Problem 4.36

Continuity $\Rightarrow E_{\|}$ is continuos

No free charge at the boundary $\Rightarrow \sigma_f = 0 \Rightarrow D_{\perp}$ is continuos

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{E_{\|}^2 / E_{\perp}^2}{E_{\|}^1 / E_{\perp}^1} = \frac{E_{\perp}^1}{E_{\perp}^2} = \frac{D_{\perp}^1 / \varepsilon_1}{D_{\perp}^2 / \varepsilon_2} = \frac{\varepsilon_2}{\varepsilon_1}$$

$[$*Comment* : defocus$]$

**Problem 4.39**

(a)$V_0=\dfrac{1}{4\pi\varepsilon_0}\dfrac{Q_{total}}{R}$, $[claim]\rightarrow V(r)=V_0\dfrac{R}{r}=\dfrac{Q_{total}}{4\pi\varepsilon_0 r}$

$$\Rightarrow\begin{cases}\mathbf{E}=-\nabla V=V_0\dfrac{R}{r^2}\hat{\mathbf{r}}\\ \mathbf{P}=\varepsilon_0\chi_e\mathbf{E}=\varepsilon_0\chi_e V_0\dfrac{R}{r^2}\hat{\mathbf{r}}\\ \sigma_b=\mathbf{P}\cdot\hat{\mathbf{n}}=\begin{cases}0, & \text{on northern hemisphere}\\ \varepsilon_0\chi_e V_0\dfrac{R}{r^2}\hat{\mathbf{r}}\cdot(-\hat{\mathbf{r}})\Big|_{r=R}=-\dfrac{\varepsilon_0\chi_e V_0}{R}, & \text{on southern hemisphere}\\ 0, & \text{on the } z=0 \text{ plane}\end{cases}\\ \sigma_f=\dfrac{Q_{total}}{4\pi R^2}-\sigma_b=\begin{cases}\dfrac{\varepsilon_0 V_0}{R}, & \text{on northern hemisphere}\\ \dfrac{\varepsilon_0 V_0}{R}-\left(-\dfrac{\varepsilon_0\chi_e V_0}{R}\right)=\dfrac{\varepsilon_0(1+\chi_e)V_0}{R}, & \text{on southern hemisphere}\end{cases}\end{cases}$$

## Problem 4.39

(b) $V(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q_{total}}{r} = \frac{1}{4\pi\varepsilon_0}\frac{4\pi R\varepsilon_0 V_0}{r} = \frac{V_0 R}{r}$, *Q.E.D.*

(c)Given two solutions:

$$\begin{cases} V_1 \\ \mathbf{E}_1 = -\nabla V_1 \\ \mathbf{D}_1 = \varepsilon \mathbf{E}_1 \end{cases} \quad \begin{cases} V_2 \\ \mathbf{E}_2 = -\nabla V_2 \\ \mathbf{D}_2 = \varepsilon \mathbf{E}_2 \end{cases} \quad \begin{cases} V_3 = V_2 - V_1 \\ \mathbf{E}_3 = -\nabla V_3 = \mathbf{E}_2 - \mathbf{E}_1 \\ \mathbf{D}_3 = \mathbf{D}_2 - \mathbf{D}_1 \end{cases}$$

$$\int_{Vol} \nabla\cdot\left(V_3\mathbf{D}_3\right)d\tau = \oint_{Surf} V_3\mathbf{D}_3\cdot d\mathbf{a} = 0 \left(V_3 = V_1 - V_2 = V_0 - V_0 = 0 \text{ on the surface } r = R\right)$$

$$\Rightarrow \int_{Vol} \nabla\cdot\left(V_3\mathbf{D}_3\right)d\tau = 0 = \int_{Vol}\left(\nabla V_3\right)\cdot\mathbf{D}_3 d\tau + \int_{Vol} V_3\nabla\cdot\mathbf{D}_3 d\tau$$

But $\nabla\cdot\mathbf{D}_3 = \nabla\cdot\left(\mathbf{D}_2 - \mathbf{D}_1\right) = \nabla\cdot\mathbf{D}_2 - \nabla\cdot\mathbf{D}_1 = \rho_f - \rho_f = 0$

$$\Rightarrow \int_{Vol}\left(\nabla V_3\right)\cdot\mathbf{D}_3 d\tau + \int_{Vol} V_3\nabla\cdot\mathbf{D}_3 d\tau = \int_{Vol}\mathbf{E}_3\cdot\mathbf{D}_3 d\tau + 0 = \int_{Vol}\varepsilon\left|\mathbf{E}_3\right|^2 d\tau = 0$$

$\Rightarrow \mathbf{E}_3 = 0 \Rightarrow V_2 - V_1 = const.$ everywhere, and $V_1 = V_2 = V_0$ on the surface $r = R$

$\Rightarrow V_1 = V_2$ everywhere $\Rightarrow$ it's uniquely determined.

(d) Fig.(a): $\sigma_b$ on the dielectric flate surface will not be 0, because of $\mathbf{P}$ is not perpendiculat to $\hat{\mathbf{n}}$;

Fig.(b) works the same way.

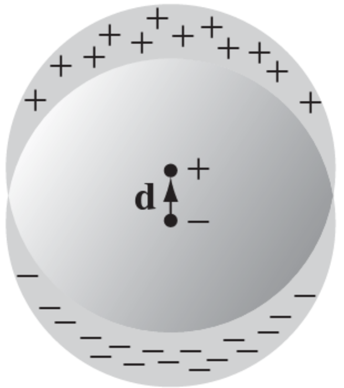
# Exercise List

6, 11, 13, 14, 15, 18, 19, 22, 24, 26, 31, 32

**Problem 4.2** According to quantum mechanics, the electron cloud for a hydrogen atom in the ground state has a charge density

$$\rho(r) = \frac{q}{\pi a^3} e^{-2r/a},$$

where $q$ is the charge of the electron and $a$ is the Bohr radius. Find the atomic polarizability of such an atom. [*Hint*: First calculate the electric field of the electron cloud, $E_e(r)$; then expand the exponential, assuming $r \ll a$.[1]




$$\boxed{\mathbf{p} = \alpha \mathbf{E}_e}$$

Rule of Thumb! Not a fundamental Law!

$$\oint \mathbf{E}_e \cdot d\mathbf{a} = \frac{1}{\varepsilon_0} Q_{enc} \Rightarrow E_e(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int_0^r \rho\, d\tau = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\frac{q}{\pi a^3}\int_0^r e^{-\frac{2r'}{a}} 4\pi r'^2 dr' = \frac{1}{\pi\varepsilon_0}\frac{q}{r^2}\frac{1}{a^3}\int_0^r e^{-\frac{2}{a}r'} r'^2 dr'$$

Feynman's Trick of Integration: Introduct another parameter $\lambda$ to simply the calculation

Consider: $I = \int_0^r e^{-\lambda y} dy = \frac{1}{\lambda}\left(1 - e^{-\lambda r}\right)$

1. $\partial_\lambda I = \partial_\lambda \int_0^r e^{-\lambda y} dy = \int_0^r \partial_\lambda \left(e^{-\lambda y}\right) dy = \int_0^r (-y) e^{-\lambda y} dy = \frac{-1}{\lambda^2}\left(1 - e^{-\lambda r}\right) + \frac{1}{\lambda}\left(1 + \lambda e^{-\lambda r}\right)$

2. $\partial_\lambda^2 I = \partial_\lambda^2 \int_0^r e^{-\lambda y} dy = \int_0^r \partial_\lambda^2 \left(e^{-\lambda y}\right) dy = \int_0^r (-y)^2 e^{-\lambda y} dy = \frac{2}{\lambda^3}\left[1 - e^{-\lambda r}\left(1 + \lambda r + \frac{(\lambda r)^2}{2}\right)\right]$ $\Leftarrow$ Same form as required integration!

$$\therefore E_e(\mathbf{r}) = \frac{1}{\pi\varepsilon_0}\frac{q}{r^2}\frac{1}{a^3}\frac{2}{(2/a)^3}\left[1 - e^{-2r/a}\left(1 + (2/a)r + \frac{(2r/a)^2}{2}\right)\right] \sim \frac{qd}{3\pi\varepsilon_0 a^3} \sim \frac{p}{\alpha}$$

Experiment

Hydrogen atom ~ 0.667e-30

$$\begin{cases} \dfrac{\alpha_{qm}}{4\pi\varepsilon_0} \sim \dfrac{3}{4} a^3 \sim 0.09 \times 10^{-30}\, m^3 \\ \dfrac{\alpha_{cl}}{4\pi\varepsilon_0} \sim a^3 \sim 0.12 \times 10^{-30}\, m^3 \end{cases}$$

# Polarizability and susceptibility

$$\mathbf{E}_{macro} = \mathbf{E}_{self} + \mathbf{E}_{ext}$$

In a linear dielectric, the polarization is said to be proportional to the filed $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$ ← Macroscopic

If the material consists of atoms (or nonpolar molecules), the induced dipole moment $\mathbf{p} = \alpha \mathbf{E}$ ← Microscopic

$$\mathbf{P} = N\mathbf{p} = N\alpha\mathbf{E} \overset{?}{\Rightarrow} \chi_e = \frac{N\alpha}{\varepsilon_0}$$

If the density of atoms is low, it's not far off. However, the fields used are from different viewpoints!

$$\mathbf{E}_{self} = \frac{-\mathbf{p}}{4\pi\varepsilon_0 R^3} \Rightarrow \mathbf{E}_{macro} = \frac{-\alpha}{4\pi\varepsilon_0 R^3}\mathbf{E}_{ext} + \mathbf{E}_{ext} = \left(1 - \frac{N\alpha}{3\varepsilon_0}\right)\mathbf{E}_{ext} = \frac{\mathbf{P}}{\varepsilon_0 \chi_e} = \frac{N\alpha}{\varepsilon_0 \chi_e}\mathbf{E}_{ext}$$

$$\therefore \chi_e = \frac{N\alpha/\varepsilon_0}{1 - N\alpha/3\varepsilon_0} \Rightarrow \alpha = \frac{3\varepsilon_0}{N}\frac{\chi_e}{3+\chi_e} = \frac{3\varepsilon_0}{N}\frac{\varepsilon_r - 1}{\varepsilon_r + 2} \approx \frac{3\varepsilon_0}{N}\frac{n^2 - 1}{n^2 + 2}$$

Clausius-Mossotti formula     Lorentz-Lorenz relation

What about polar substance?

# Polarizability and susceptibility

Energy of a dipole in an external field $u = -\mathbf{p}\cdot\mathbf{E} = -pE\cos\theta$

Statistical mechanics says that for a material in equilibrium at absolute temperature, the probability of a given molecule having energy is proportional to the Boltzmann factor

$$\exp\left(-u/kT\right)$$

The average energy of the dipoles is therefore

$$\langle u\rangle = \frac{\int ue^{-u/kT}\,d\Omega}{\int e^{-u/kT}\,d\Omega} = \frac{\int_{-pE}^{pE} ue^{-u/kT}\,du}{\int_{-pE}^{pE} e^{-u/kT}\,du} = kT - pE\left[\frac{e^{pE/kT}+e^{-pE/kT}}{e^{pE/kT}-e^{-pE/kT}}\right] = kT - pE\coth\left(pE/kT\right)$$

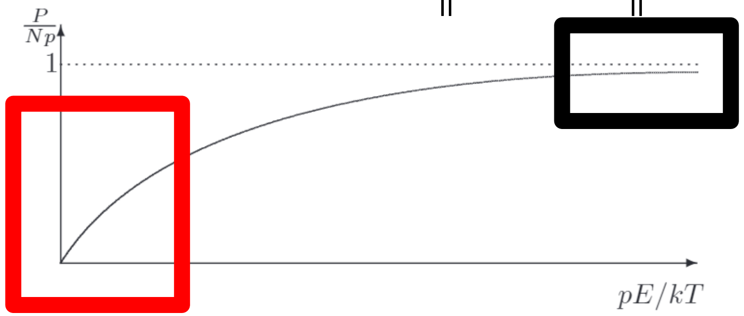
$$\therefore \|\mathbf{P}\| = N\|\langle\mathbf{p}\rangle\| = N\left\|\langle\mathbf{p}\cdot\mathbf{E}\rangle\frac{\hat{\mathbf{E}}}{E}\right\| = -Np\frac{\langle u\rangle}{pE} = Np\left\{\coth\left(\frac{pE}{kT}\right) - \frac{kT}{pE}\right\}$$

Langevin equation

Linear region

$$P \approx \frac{Np^2}{3kT}E = \varepsilon_0\chi_e E$$

$$\Rightarrow \chi_e = \frac{Np^2}{3\varepsilon_0 kT}$$




Comment: For large fields/low temperatures, all the molecules are lined up, and the material is nonlinear.

**Problem 4.6** A (perfect) dipole **p** is situated a distance $z$ above an infinite grounded conducting plane (Fig. 4.7). The dipole makes an angle $\theta$ with the perpendicular to the plane. Find the torque on **p**. If the dipole is free to rotate, in what orientation will it come to rest?

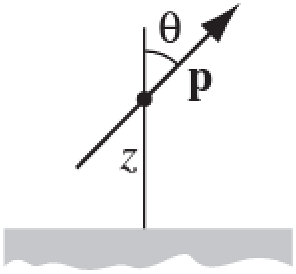
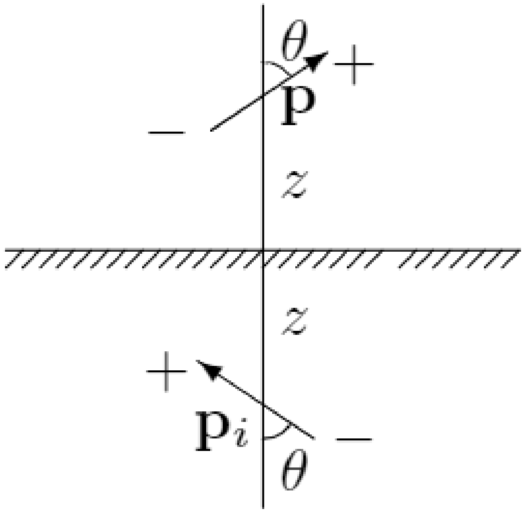



**FIGURE 4.7**

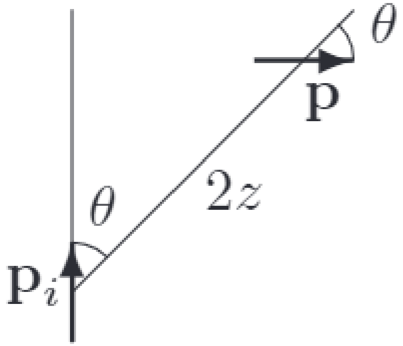



1. Use Image Dipole to free the boundary as shown in the figure.
2. Try to describe the system with respect to the origin of the image dipole

$$V_{dip}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}\cdot\int \mathbf{r}'\rho(\mathbf{r}')\,d\tau' = \frac{1}{4\pi\varepsilon_0}\frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}$$

$$\Rightarrow \begin{cases} E_r = -\partial_r V = \dfrac{2p\cos\theta}{4\pi\varepsilon_0 r^3} \\ E_\theta = -\dfrac{1}{r}\partial_\theta V = \dfrac{p\sin\theta}{4\pi\varepsilon_0 r^3} \end{cases} \Rightarrow \mathbf{E}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right)$$

Come to rest
when angle = 0 or 90 deg



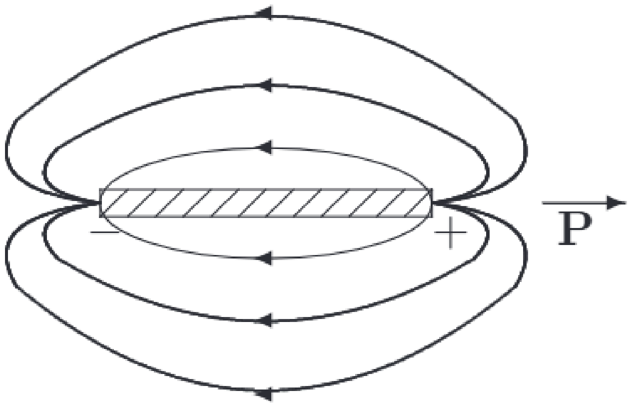

$$\begin{cases} \mathbf{E}_i\big|_{@p} = \dfrac{p}{4\pi\varepsilon_0(2z)^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) \\ \mathbf{p} = p\left(\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right) \end{cases} \Rightarrow \mathbf{N} = \mathbf{p}\times\mathbf{E}_i = \frac{p^2}{8\pi\varepsilon_0 r^3}\sin(2\theta)\left(-\hat{\boldsymbol{\phi}}\right)$$

$$\hat{\boldsymbol{\theta}}\times\hat{\mathbf{r}} = -\hat{\boldsymbol{\phi}}$$

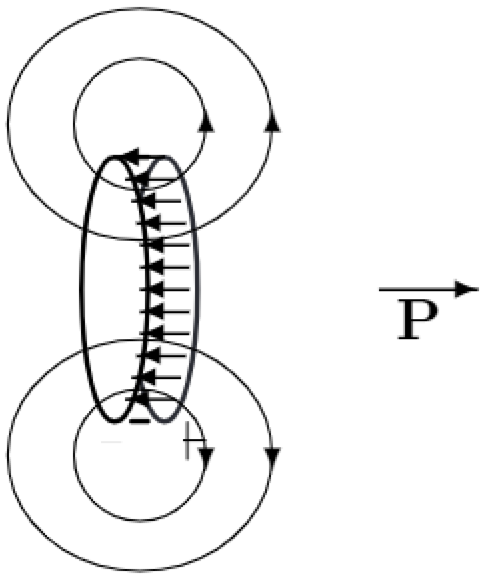
**Problem 4.11** A short cylinder, of radius $a$ and length $L$, carries a "frozen-in" uniform polarization $\mathbf{P}$, parallel to its axis. Find the bound charge, and sketch the electric field (i) for $L \gg a$, (ii) for $L \ll a$, and (iii) for $L \approx a$. [This is known as a **bar electret**; it is the electrical analog to a bar magnet. In practice, only very special materials—barium titanate is the most "familiar" example—will hold a permanent electric polarization. That's why you can't buy electrets at the toy store.]

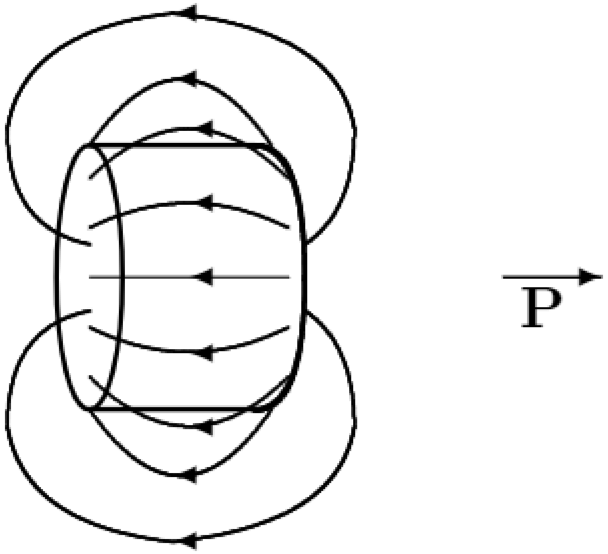
Electron + magnet → Electret

$$\begin{cases} \rho_b = -\nabla \cdot \mathbf{P} = 0 \\ \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \pm P \end{cases}$$




$L \gg a \Rightarrow$ Like Dipole




$L \ll a \Rightarrow$ Like Parallel Plate Capacitor

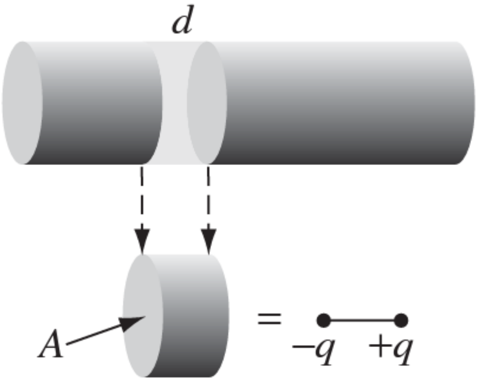
Field is nearly uniform inside with fringing field at edges.

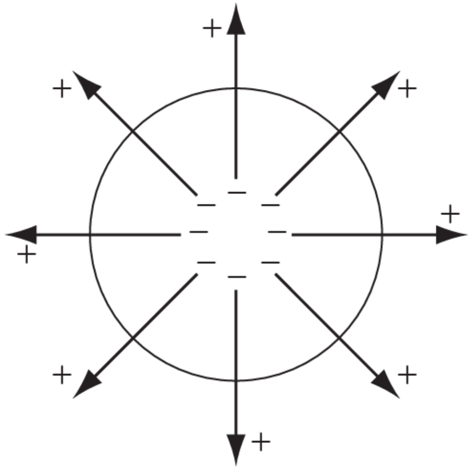
**Problem 4.13** A very long cylinder, of radius $a$, carries a uniform polarization $\mathbf{P}$ perpendicular to its axis. Find the electric field inside the cylinder. Show that the field *outside* the cylinder can be expressed in the form

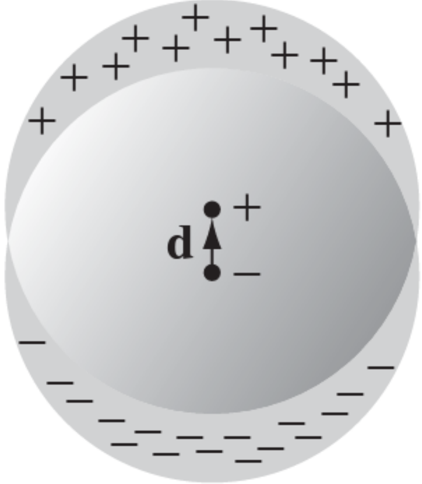
$$\begin{cases} \rho_b = -\nabla \cdot \mathbf{P} \\ \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} \end{cases}$$

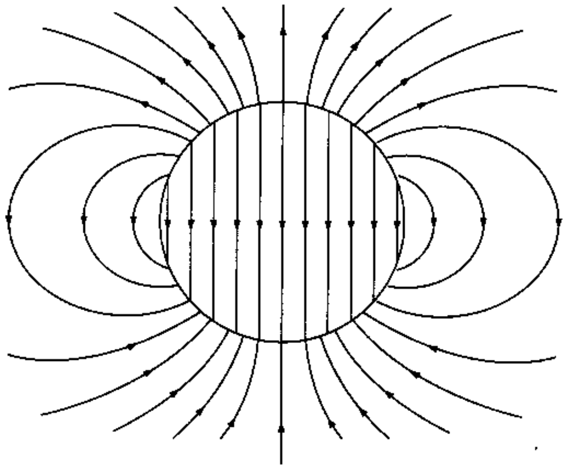
$$\mathbf{E}(\mathbf{r}) = \frac{a^2}{2\epsilon_0 s^2}[2(\mathbf{P} \cdot \hat{\mathbf{s}})\hat{\mathbf{s}} - \mathbf{P}].$$

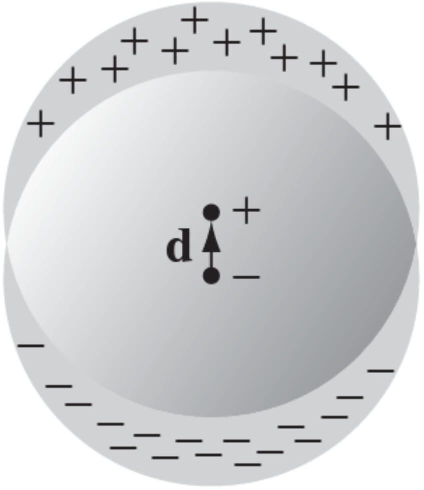
[*Careful:* I said "uniform," not "radial"!]

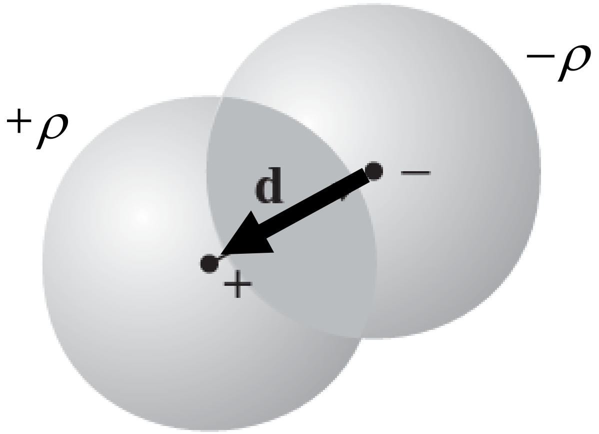



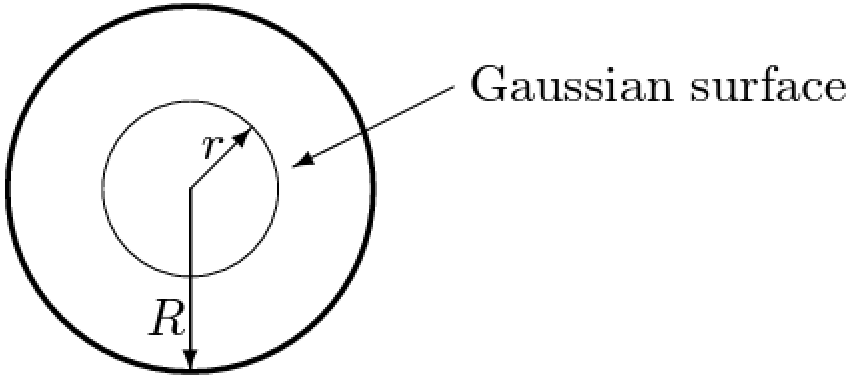




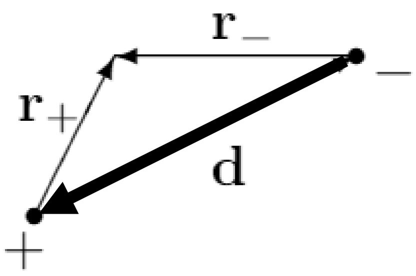




Prob. 2.18




$$\oint \mathbf{E}\cdot d\mathbf{a} = E \cdot 4\pi r^2 = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3 \rho. \quad \text{So}$$

$$\boxed{\mathbf{E} = \frac{1}{3\epsilon_0}\rho r \hat{\mathbf{r}}.}$$




$$\mathbf{E}_+ = \frac{\rho}{3\varepsilon_0}\mathbf{r}_+$$

$$\mathbf{E}_- = -\frac{\rho}{3\varepsilon_0}\mathbf{r}_-$$

For field in the region of overlap

$$\mathbf{E}_+ + \mathbf{E}_- = \frac{\rho}{3\varepsilon_0}\left(\mathbf{r}_+ - \mathbf{r}_-\right) = -\frac{\rho}{3\varepsilon_0}\mathbf{d} = -\frac{1}{4\pi\varepsilon_0}\frac{q\mathbf{d}}{R^3} = \frac{-1}{3\varepsilon_0}\mathbf{P}$$

Points outside →All the charge on each sphere were concentrated at the respective center.

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}$$

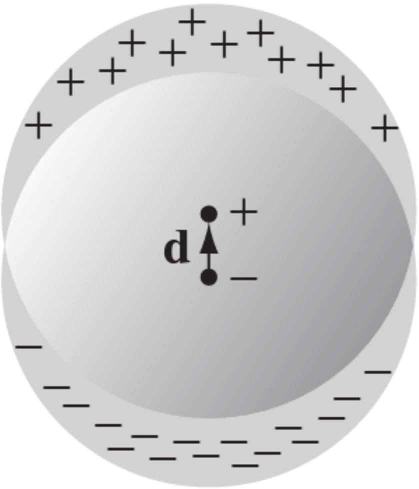
**Problem 4.13** A very long cylinder, of radius $a$, carries a uniform polarization $\mathbf{P}$ perpendicular to its axis. Find the electric field inside the cylinder. Show that the field *outside* the cylinder can be expressed in the form

$$\mathbf{s}_{\pm} = \mathbf{s} \mp \frac{\mathbf{d}}{2} \Rightarrow \frac{\mathbf{s}_{\pm}}{\mathbf{s}_{\pm}^2} \cong \frac{1}{s^2}\left(s \pm s\frac{\mathbf{s}\cdot\mathbf{d}}{s^2} \mp \frac{\mathbf{d}}{2}\right)$$

$$\begin{cases} \rho_b = -\nabla\cdot\mathbf{P} \\ \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} \end{cases}$$

$$\mathbf{E}(\mathbf{r}) = \frac{a^2}{2\epsilon_0 s^2}[2(\mathbf{P}\cdot\hat{\mathbf{s}})\hat{\mathbf{s}} - \mathbf{P}].$$

[*Careful:* I said "uniform," not "radial"!]




Again by Gauss's Law, $E(2\pi s)\ell = \dfrac{\rho\pi s^2\ell}{\varepsilon_0} \Rightarrow \mathbf{E} = \dfrac{\rho s}{2\varepsilon_0}\hat{\mathbf{s}}$

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho}{2\varepsilon_0}(\mathbf{s}_+ - \mathbf{s}_-) = -\frac{\rho}{2\varepsilon_0}\mathbf{d} = \frac{-1}{2\varepsilon_0}\mathbf{P} \ (\text{for } s < a)$$

$$\mathbf{P}\left(\pi a^2\ell\right) = \left(\rho\pi a^2\ell\right)\mathbf{d}$$

For points outside,

$$E(2\pi s)\ell = \frac{\rho\pi a^2\ell}{\varepsilon_0} \Rightarrow \mathbf{E} = \frac{\rho a^2}{2\varepsilon_0 s}\hat{\mathbf{s}} \qquad \therefore \mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho a^2}{2\varepsilon_0}\left(\frac{\hat{\mathbf{s}}_+}{\mathbf{s}_+} - \frac{\hat{\mathbf{s}}_-}{\mathbf{s}_-}\right)$$

Key: Consider two cylinders of opposite uniform charge density whose cross sections are as shown!

**Problem 4.14** When you polarize a neutral dielectric, the charge moves a bit, but the *total* remains zero. This fact should be reflected in the bound charges $\sigma_b$ and $\rho_b$. Prove from Eqs. 4.11 and 4.12 that the total bound charge vanishes.

$$Q_{total} = \int_v \rho_b d\tau + \oint_{\partial V} \sigma_b da = \int_v (-\nabla \cdot \mathbf{P}) d\tau + \oint_{\partial V} (\mathbf{P} \cdot \hat{\mathbf{n}}) da = 0$$

---

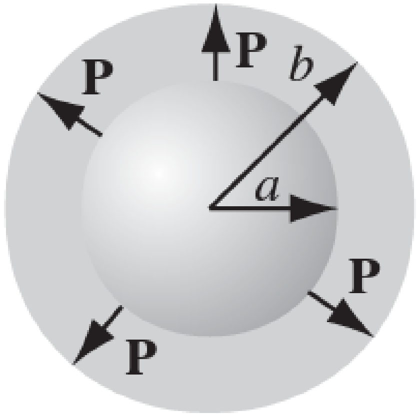
**Problem 4.15** A thick spherical shell (inner radius $a$, outer radius $b$) is made of dielectric material with a "frozen-in" polarization

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\hat{\mathbf{r}},$$

where $k$ is a constant and $r$ is the distance from the center (Fig. 4.18). (There is no *free* charge in the problem.) Find the electric field in all three regions by two different methods:

(a) Locate all the bound charge, and use Gauss's law (Eq. 2.13) to calculate the field it produces.

(b) Use Eq. 4.23 to find $\mathbf{D}$, and then get $\mathbf{E}$ from Eq. 4.21. [Notice that the second method is much faster, and it avoids any explicit reference to the bound charges.]




No free charge!

$$\begin{cases} \rho_b = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2}\partial_r\left(r^2 \frac{k}{r}\right) = -\frac{k}{r^2} \\ \sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} = \begin{cases} +\mathbf{P} \cdot \hat{\mathbf{r}} = k/b & (r = b) \\ -\mathbf{P} \cdot \hat{\mathbf{r}} = -k/a & (r = a) \end{cases} \end{cases}$$

$$\text{Gauss's Law} \Rightarrow \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{r^2}\hat{\mathbf{r}}$$

$$\Rightarrow \mathbf{E} = 0 \text{ for } r < a \text{ and } r > b$$

$$\text{For } a < r < b,\ Q_{enc} = \frac{-k}{a}\left(4\pi a^2\right) + \int_a^r \left(\frac{-k}{r'^2}\right) 4\pi r'^2 dr' = -4\pi k r \Rightarrow \mathbf{E} = -\frac{k}{\varepsilon_0 r}\hat{\mathbf{r}}$$

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{enc}} = 0 \Rightarrow \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P} = 0 \text{ everywhere} \Rightarrow \mathbf{E} = \frac{-\mathbf{P}}{\varepsilon_0}$$

**Problem 4.18** The space between the plates of a parallel-plate capacitor (Fig. 4.24) is filled with two slabs of linear dielectric material. Each slab has thickness $a$, so the total distance between the plates is $2a$. Slab 1 has a dielectric constant of 2, and slab 2 has a dielectric constant of 1.5. The free charge density on the top plate is $\sigma$ and on the bottom plate $-\sigma$.

(a) Find the electric displacement $\mathbf{D}$ in each slab.

(b) Find the electric field $\mathbf{E}$ in each slab.

(c) Find the polarization $\mathbf{P}$ in each slab.

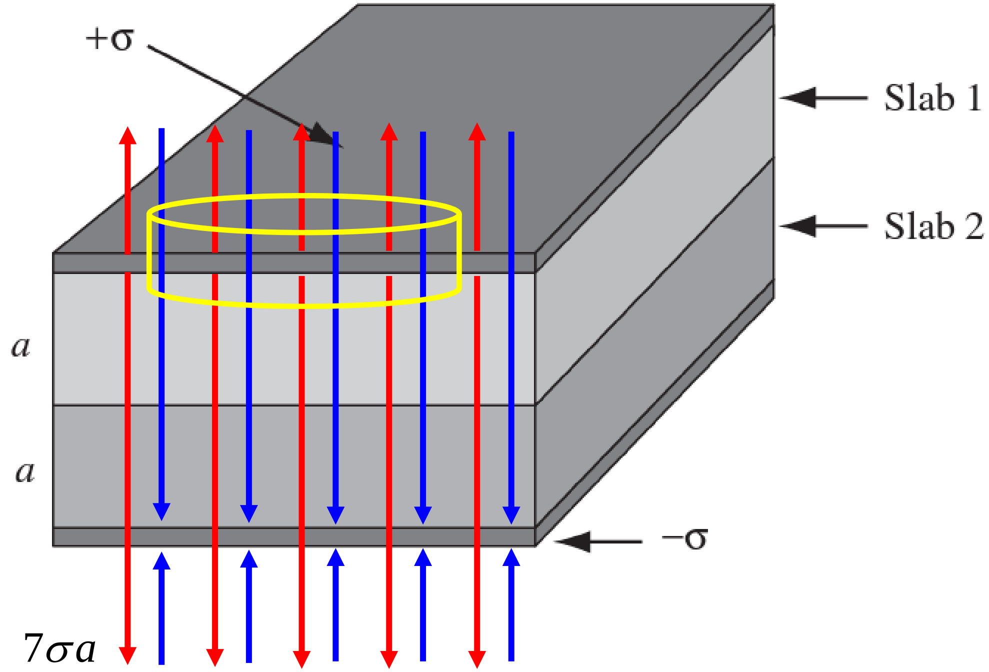
(d) Find the potential difference between the plates.

(e) Find the location and amount of all bound charge.

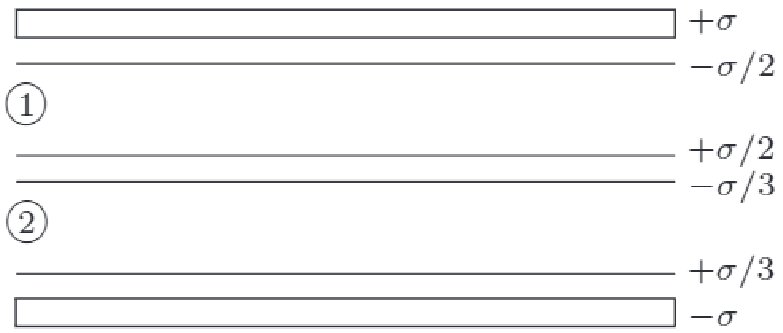
(f) Now that you know all the charge (free and bound), recalculate the field in each slab, and confirm your answer to (b).

$$\oint_S \mathbf{D}\cdot d\mathbf{a} = Q_{f_{enc}} \Rightarrow 2DA = \sigma A \Rightarrow \mathbf{D} = \frac{\sigma}{2}\hat{\mathbf{n}}$$

$$\therefore \mathbf{D} = \begin{cases} \mathbf{0}, \text{ outside the plates} \\ \dfrac{-\sigma}{2}\hat{\mathbf{z}} + \dfrac{-\sigma}{2}\hat{\mathbf{z}} = -\sigma\hat{\mathbf{z}} \end{cases}$$




$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} \Rightarrow \begin{cases} \mathbf{E}_1 = -\dfrac{\sigma}{2\varepsilon_0}\hat{\mathbf{z}}, \text{ for slab 1} \\ \mathbf{E}_2 = -\dfrac{2\sigma}{3\varepsilon_0}\hat{\mathbf{z}}, \text{ for slab 2} \end{cases} \Rightarrow \Delta V = -\int_-^+ \mathbf{E}\cdot d\vec{\ell} = -\left[\left(-\frac{2\sigma}{3\varepsilon_0}\right)a + \left(-\frac{\sigma}{2\varepsilon_0}\right)a\right] = \frac{7\sigma a}{6\varepsilon_0}$$

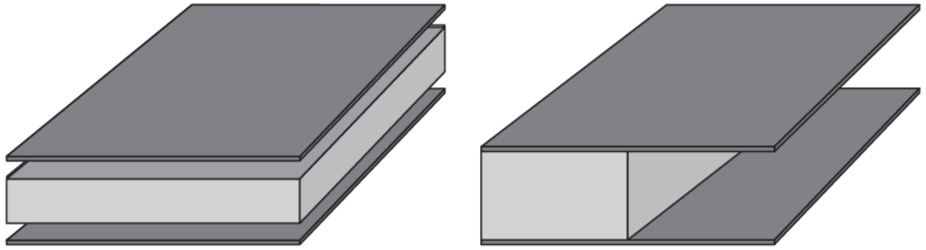



$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} = \varepsilon_0(\varepsilon_r - 1)\mathbf{E} = \varepsilon_0(\varepsilon_r - 1)\frac{-\sigma}{\varepsilon_0\varepsilon_r}\hat{\mathbf{z}} = -\sigma\left(1 - \frac{1}{\varepsilon_r}\right)\hat{\mathbf{z}}$$

$$\therefore \mathbf{P}_1 = -\frac{\sigma}{2}\hat{\mathbf{z}},\ \ \mathbf{P}_2 = -\frac{\sigma}{3}\hat{\mathbf{z}} \Rightarrow \nabla\cdot\mathbf{P} = 0 \Rightarrow \rho_b = 0 \text{ everywhere}$$

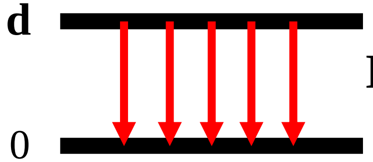
$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \mp\dfrac{\sigma}{2}, \text{ for slab 1} \\ \mp\dfrac{\sigma}{3}, \text{ for slab 2} \end{cases}$$

**Problem 4.19** Suppose you have enough linear dielectric material, of dielectric constant $\epsilon_r$, to *half*-fill a parallel-plate capacitor (Fig. 4.25). By what fraction is the capacitance increased when you distribute the material as in Fig. 4.25(a)? How about Fig. 4.25(b)? For a given potential difference $V$ between the plates, find **E**, **D**, and **P**, in each region, and the free and bound charge on all surfaces, for both cases.



(a) (b)

Without dielectric,




$$\mathbf{E} = -\frac{\sigma}{\varepsilon_0}\hat{\mathbf{z}}$$

$$V = -\int_{(-)}^{(+)} \mathbf{E}\cdot d\vec{\ell} = -\int_0^d \left(-\frac{\sigma}{\varepsilon_0}\right) dz = \frac{\sigma}{\varepsilon_0} d \Rightarrow C = \frac{Q}{V} = \frac{\sigma A}{\sigma d/\varepsilon_0} = \frac{\varepsilon_0 A}{d}$$

In (a),

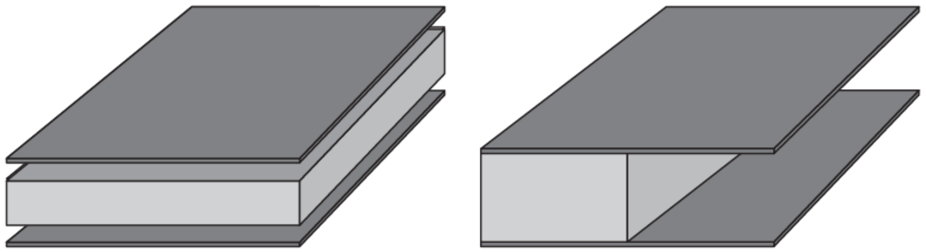
$$\mathbf{D} = -\sigma\hat{\mathbf{z}} \Rightarrow \mathbf{E} = \begin{cases} -\sigma/\varepsilon_0\,\hat{\mathbf{z}} \text{ ,in air (vaccum)} \\ -\sigma/\varepsilon\,\hat{\mathbf{z}} \text{ ,in dielectric} \end{cases}$$

$$V = \frac{\sigma}{\varepsilon_0}\left(\frac{d}{2}\right) + \frac{\sigma}{\varepsilon}\left(\frac{d}{2}\right) = \left(\frac{\sigma d}{2\varepsilon_0}\right)\left(1+\varepsilon_r^{-1}\right) \Rightarrow \sigma = \frac{2\varepsilon_0 V}{d\left(1+\varepsilon_r^{-1}\right)} \Rightarrow C_a = \frac{Q}{V} = \frac{\sigma A}{\left(\frac{\sigma d}{2\varepsilon_0}\right)\left(1+\varepsilon_r^{-1}\right)} = \frac{\varepsilon_0 A}{d}\left(\frac{2}{1+\varepsilon_r^{-1}}\right) \Rightarrow \frac{C_a}{C_0} = \frac{2}{1+\varepsilon_r^{-1}}$$

$$\Rightarrow \mathbf{E} = \begin{cases} -\frac{2V}{d\left(1+\varepsilon_r^{-1}\right)}\hat{\mathbf{z}} \text{ ,in air (vaccum)} \\ -\frac{2V}{d\left(\varepsilon_r+1\right)}\hat{\mathbf{z}} \text{ ,in dielectric} \end{cases} \Rightarrow \begin{cases} \mathbf{P} = \varepsilon_0\left(\varepsilon_r-1\right)\mathbf{E} = \begin{cases} 0, \text{ in air (vaccum)} \\ -\frac{2V\varepsilon_0\left(\varepsilon_r-1\right)}{d\left(\varepsilon_r+1\right)}\hat{\mathbf{z}}, \text{ in dielectric} \end{cases} \\ \mathbf{D} = \varepsilon\mathbf{E} = \begin{cases} -\frac{2V\varepsilon_0\varepsilon_r}{d\left(\varepsilon_r+1\right)}\hat{\mathbf{z}}, \text{ in air (vaccum)} \\ -\frac{2V\varepsilon_0\varepsilon_r}{d\left(\varepsilon_r+1\right)}\hat{\mathbf{z}} \text{ ,in dielectric} \end{cases} \end{cases}$$

$$\begin{cases} \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = -\frac{2V\varepsilon_0\left(\varepsilon_r-1\right)}{d\left(\varepsilon_r+1\right)} \text{ (at top surface of dielectric)} \\ \sigma_f = \varepsilon_0 E_{air} = \frac{2V\varepsilon_0\varepsilon_r}{d\left(\varepsilon_r+1\right)} \text{ (on top plate above dielectric)} \end{cases}$$

**Problem 4.19** Suppose you have enough linear dielectric material, of dielectric constant $\epsilon_r$, to *half*-fill a parallel-plate capacitor (Fig. 4.25). By what fraction is the capacitance increased when you distribute the material as in Fig. 4.25(a)? How about Fig. 4.25(b)? For a given potential difference $V$ between the plates, find **E**, **D**, and **P**, in each region, and the free and bound charge on all surfaces, for both cases.



(a) (b)

In (b),

$$E=\frac{V}{d}\Rightarrow\sigma=\varepsilon_0E=\varepsilon_0\frac{V}{d}\ \text{ in air (vaccum)}$$

$$P=\varepsilon_0\chi_eE=\varepsilon_0\chi_e\frac{V}{d}\ \text{ in dielectric}\Rightarrow\sigma_b=-\varepsilon_0\chi_e\frac{V}{d}\ \ (\text{at top surface of dielectric})$$

$$\therefore\sigma_{tot}=\frac{\varepsilon_0V}{d}=\sigma_b+\sigma_f\Rightarrow\sigma_f=\frac{\varepsilon_0V}{d}(1+\chi_e)=\frac{\varepsilon V}{d}\ \ (\text{on top plate above dielectric})$$

$$\Rightarrow C_b=\frac{Q}{V}=\frac{1}{V}\left(\frac{\varepsilon_0V}{d}\frac{A}{2}+\frac{\varepsilon V}{d}\frac{A}{2}\right)=\frac{\varepsilon_0A}{d}\left(\frac{1+\varepsilon_r}{2}\right)\Rightarrow\frac{C_b}{C_0}=\frac{1+\varepsilon_r}{2}$$

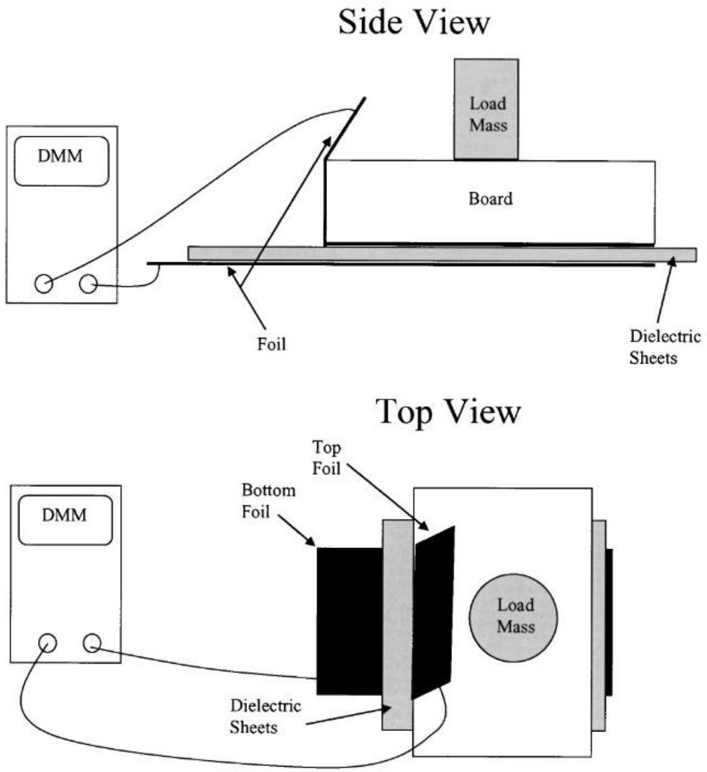
$$\Rightarrow\mathbf{E}=\begin{cases}-\frac{V}{d}\hat{\mathbf{z}}\ \text{,in air (vaccum)}\\-\frac{V}{d}\hat{\mathbf{z}}\ \text{,in dielectric}\end{cases}\Rightarrow\begin{cases}\mathbf{P}=\varepsilon_0(\varepsilon_r-1)\mathbf{E}=\begin{cases}0,\ \text{in air (vaccum)}\\(\varepsilon_r-1)\frac{-\varepsilon_0V}{d}\hat{\mathbf{z}},\ \text{in dielectric}\end{cases}\\\mathbf{D}=\varepsilon\mathbf{E}=\begin{cases}-\frac{V}{d}\varepsilon_0\hat{\mathbf{z}},\ \text{in air (vaccum)}\\-\frac{V}{d}\varepsilon_0\varepsilon_r\hat{\mathbf{z}}\ \text{,in dielectric}\end{cases}\end{cases}$$

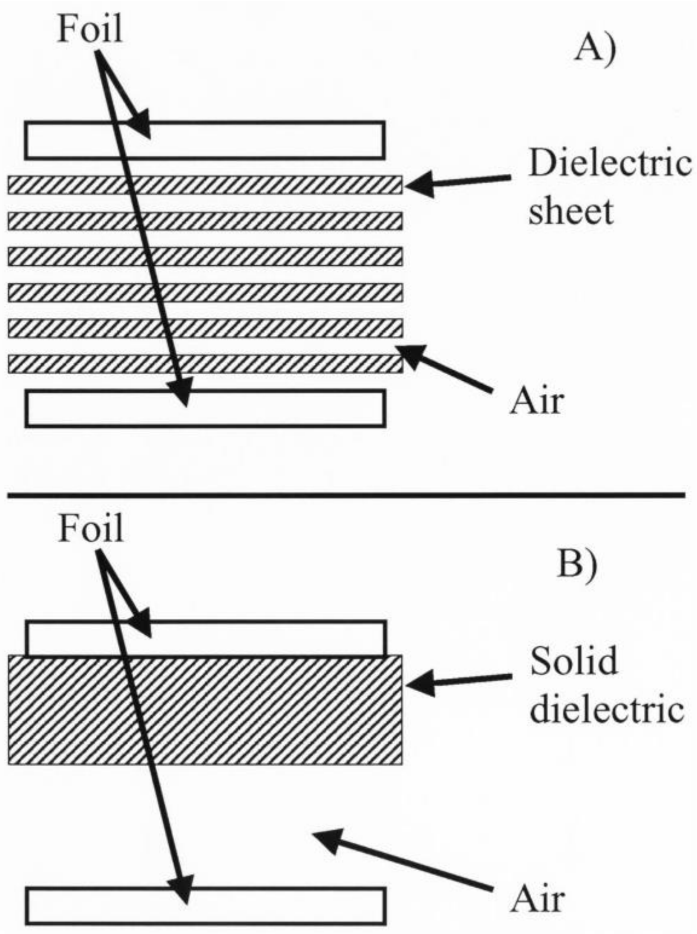
| | **E** | **D** | **P** |
|---|---|---|---|
| (a) air | $\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{V}{d}\hat{\mathbf{x}}$ | $\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0V}{d}\hat{\mathbf{x}}$ | $0$ |
| (a) dielectric | $\frac{2}{(\epsilon_r+1)}\frac{V}{d}\hat{\mathbf{x}}$ | $\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0V}{d}\hat{\mathbf{x}}$ | $\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0V}{d}\hat{\mathbf{x}}$ |
| (b) air | $\frac{V}{d}\hat{\mathbf{x}}$ | $\frac{\epsilon_0V}{d}\hat{\mathbf{x}}$ | $0$ |
| (b) dielectric | $\frac{V}{d}\hat{\mathbf{x}}$ | $\epsilon_r\frac{\epsilon_0V}{d}\hat{\mathbf{x}}$ | $(\epsilon_r-1)\frac{\epsilon_0V}{d}\hat{\mathbf{x}}$ |

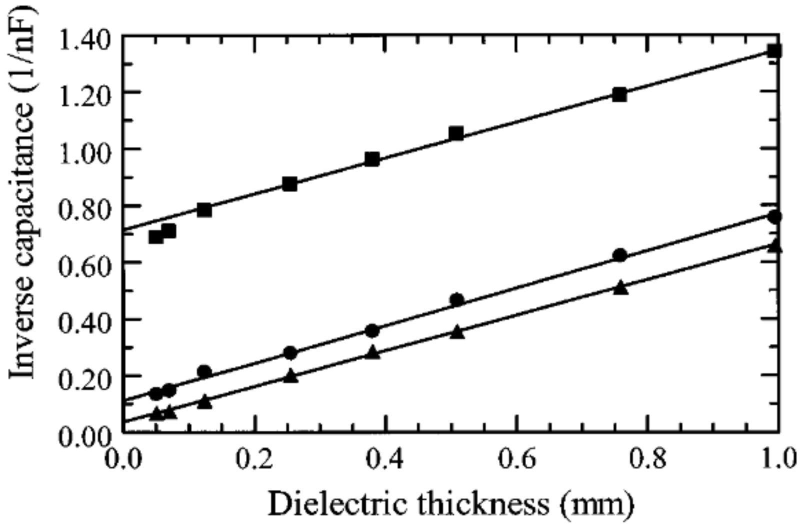
| | $\sigma_b$ (top surface) | $\sigma_f$ (top plate) |
|---|---|---|
| (a) | $-\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0V}{d}$ | $\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0V}{d}$ |
| (b) | $-(\epsilon_r-1)\frac{\epsilon_0V}{d}$ | $\epsilon_r\frac{\epsilon_0V}{d}$ (left); $\frac{\epsilon_0V}{d}$ (right) |

$$\left(\frac{C_b}{C_0}=\frac{1+\varepsilon_r}{2}\right)>\left(\frac{C_a}{C_0}=\frac{2}{1+\varepsilon_r^{-1}}\right)$$

# Parallel Plate Method




$$C=\frac{\varepsilon_r\varepsilon_0 A}{d}$$

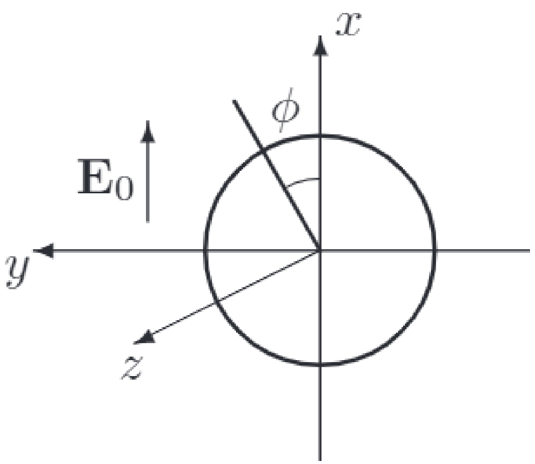
$$\Rightarrow \frac{1}{C_{measured}}=\frac{D_{teflon}}{\varepsilon_r\varepsilon_0 A}+\frac{d_{air}}{\varepsilon_0 A}$$

$$\Rightarrow \frac{1}{C_{measured}}=\frac{ND_{teflon}}{\varepsilon_r\varepsilon_0 A}+\frac{2d_{a,f}+(N-1)d_{d,f}}{\varepsilon_0 A}$$

$$\Rightarrow \frac{1}{C_{measured}}=\frac{ND_{teflon}}{\varepsilon_{r,eff}\,\varepsilon_0 A}+\frac{2d_{a,f}-d_{d,f}}{\varepsilon_0 A}$$

Ref: *AJP* **73**, 52 (2005)

**Problem 4.22** A very long cylinder of linear dielectric material is placed in an otherwise uniform electric field $\mathbf{E}_0$. Find the resulting field within the cylinder. (The radius is $a$, the susceptibility $\chi_e$, and the axis is perpendicular to $\mathbf{E}_0$.)



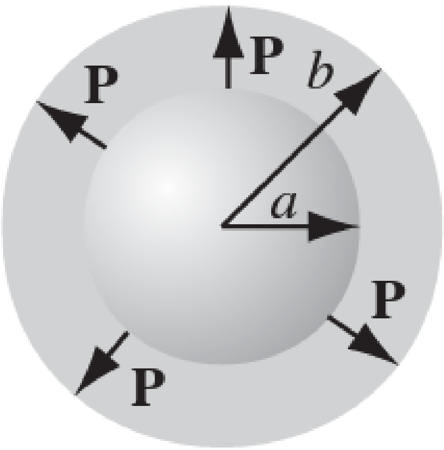

$$\begin{cases} V_{in} = V_{out} & \text{at } s = a \\ \varepsilon \partial_s V_{in} = \varepsilon_0 \partial_s V_{out} & \text{at } s = a \\ V_{out} \to -E_0 s \cos\phi & \text{for } s \gg a \end{cases} \Rightarrow \begin{cases} V_{in}(s,\phi) = \sum_{k=1}^{\infty} s^k \left( a_k \cos k\phi + b_k \sin k\phi \right) \\ V_{out}(s,\phi) = -E_0 s \cos\phi + \sum_{k=1}^{\infty} s^{-k} \left( c_k \cos k\phi + d_k \sin k\phi \right) \end{cases}$$

$$\begin{cases} \sum_{k=1}^{\infty} a^k \left( a_k \cos k\phi + \cancel{b_k \sin k\phi} \right) = -E_0 a \cos\phi + \sum_{k=1}^{\infty} a^{-k} \left( c_k \cos k\phi + \cancel{d_k \sin k\phi} \right) \\ \varepsilon_r \sum_{k=1}^{\infty} k a^{k-1} \left( a_k \cos k\phi + \cancel{b_k \sin k\phi} \right) = -E_0 \cos\phi - \sum_{k=1}^{\infty} k a^{-k-1} \left( c_k \cos k\phi + \cancel{d_k \sin k\phi} \right) \end{cases}$$

$$\Rightarrow \begin{cases} a_1 a = -E_0 a + \frac{c_1}{a}, \text{ for } k = 1 \\ a^k a_k = a^{-k} c_k \Rightarrow c_k = a^{2k} a_k, \text{ for } k \neq 1 \\ \varepsilon_r a_1 = -E_0 - \frac{c_1}{a^2}, \text{ for } k = 1 \\ \varepsilon_r k a^{k-1} a_k = -k a^{-k-1} c_k \Rightarrow \varepsilon_r a_k = -a^{-2k} c_k, \text{ for } k \neq 1 \end{cases} \qquad \Rightarrow a_k = c_k = 0 \text{ unless } k = 1$$

$$\therefore a_1 = -\frac{E_0}{1 + \chi_e/2} \Rightarrow V_{in}(s,\phi) = -\frac{E_0}{1 + \chi_e/2} s \cos\phi \Rightarrow \mathbf{E}(s,\phi) = \frac{\mathbf{E}_0}{1 + \chi_e/2}$$

**Problem 4.24** An uncharged conducting sphere of radius $a$ is coated with a thick insulating shell (dielectric constant $\epsilon_r$) out to radius $b$. This object is now placed in an otherwise uniform electric field $\mathbf{E}_0$. Find the electric field in the insulator.




$$\begin{cases} V_{in}(r,\theta)=0 & \text{for } r\le a \\ V_{med}=0 & \text{at } r=a \\ V_{med}=V_{out} & \text{at } r=b \\ \varepsilon\partial_r V_{med}=\varepsilon_0\partial_r V_{out} & \text{at } r=b \\ V_{out}\to -E_0 r\cos\theta & \text{for } r\gg b \end{cases} \Rightarrow \begin{cases} V_{in}(r,\theta)=0 \\ V_{med}(r,\theta)=\sum\left(A_\ell r^\ell+\dfrac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \\ V_{out}(r,\theta)=-E_0 r\cos\theta+\sum\dfrac{C_\ell}{r^{\ell+1}}P_\ell(\cos\theta) \end{cases}$$

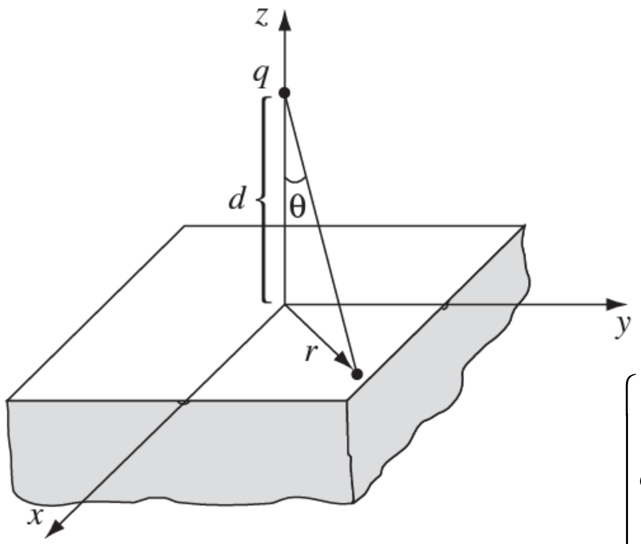
$$\Rightarrow \begin{cases} -E_0 b\cos\theta+\sum\dfrac{C_\ell}{b^{\ell+1}}P_\ell(\cos\theta)=\sum\left(A_\ell b^\ell+\dfrac{B_\ell}{b^{\ell+1}}\right)P_\ell(\cos\theta) \\ \varepsilon_r\sum\left(\ell A_\ell b^{\ell-1}-(\ell+1)\dfrac{B_\ell}{b^{\ell+2}}\right)P_\ell(\cos\theta)=-E_0\cos\theta-\sum(\ell+1)\dfrac{C_\ell}{b^{\ell+2}}P_\ell(\cos\theta) \\ A_\ell a^\ell+\dfrac{B_\ell}{a^{\ell+1}}=0\Rightarrow B_\ell=-a^{2\ell+1}A_\ell \end{cases}$$

$$\Rightarrow \begin{cases} A_\ell=B_\ell=0 \quad \text{for } \ell\neq 1 \\ B_\ell=A_\ell\left(b^{2\ell+1}-a^{2\ell+1}\right) \\ A_1=\dfrac{-3E_0}{2\left[1-(a/b)^3\right]+\varepsilon_r\left[1+2(a/b)^3\right]} \Rightarrow V_{med}=\ldots\ldots\Rightarrow \mathbf{E}=-\nabla V=\ldots\ldots \\ B_1=A_1\left(b^3-a^3\right)+E_0 b^3 \end{cases}$$

**Problem 4.25** Suppose the region *above* the $xy$ plane in Ex. 4.8 is *also* filled with linear dielectric but of a different susceptibility $\chi_e'$. Find the potential everywhere.

$$\because \rho_b = -\nabla\cdot\mathbf{P} = -\nabla\cdot\left(\varepsilon_0\chi_e\frac{\mathbf{D}}{\varepsilon}\right) = -\left(\frac{\chi_e}{1+\chi_e}\right)\rho_f$$

$$\therefore q_b = -\left(\frac{\chi_e}{1+\chi_e}\right)q \Rightarrow q_t = q_b + q = \frac{q}{\varepsilon_r'}$$




Four charges involved:

(i) $q$

(ii) polarization charge surrounding $q_p$

(iii) surface charge $\sigma_b$ on top surface of lower dielectric

(iv) surface charge $\sigma_b'$ on lower surface of upper dielectric

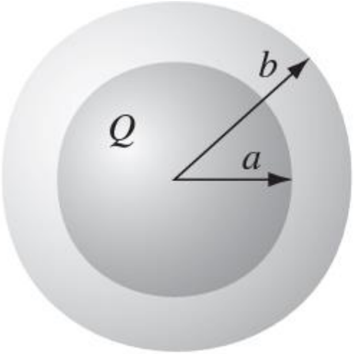
$$\begin{cases} \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \varepsilon_0\chi_e E_z = \varepsilon_0\chi_e\left[\dfrac{-1}{4\pi\varepsilon_0}\dfrac{qd/\varepsilon_r'}{\left(r^2+d^2\right)^{3/2}} - \dfrac{\sigma_b}{2\varepsilon_0} - \dfrac{\sigma_b'}{2\varepsilon_0}\right] \\ \sigma_b' = \mathbf{P}\cdot\hat{\mathbf{n}} = -\varepsilon_0\chi_e' E_z = \varepsilon_0\chi_e'\left[\dfrac{1}{4\pi\varepsilon_0}\dfrac{qd/\varepsilon_r'}{\left(r^2+d^2\right)^{3/2}} - \dfrac{\sigma_b}{2\varepsilon_0} - \dfrac{\sigma_b'}{2\varepsilon_0}\right] \end{cases} \Rightarrow \begin{cases} \sigma_b = \dfrac{-1}{4\pi}\dfrac{qd}{\left(r^2+d^2\right)^{3/2}}\dfrac{\chi_e}{\left[1+\left(\chi_e+\chi_e'\right)/2\right]} \\ \sigma_b' = \dfrac{1}{4\pi}\dfrac{qd}{\left(r^2+d^2\right)^{3/2}}\dfrac{\varepsilon_r\chi_e'/\varepsilon_r'}{\left[1+\left(\chi_e+\chi_e'\right)/2\right]} \end{cases}$$

$$\Rightarrow \sigma_t = \sigma_b + \sigma_b' = \frac{1}{4\pi}\frac{qd}{\left(r^2+d^2\right)^{3/2}}\frac{\left(\chi_e'-\chi_e\right)}{\varepsilon_r'\left[1+\left(\chi_e+\chi_e'\right)/2\right]}\left(=0 \text{ when } \chi_e' = \chi_e\,!\right)$$

$$V_{z<0} = \frac{1}{4\pi\varepsilon_0}\frac{q/\varepsilon_r' + q_t}{\sqrt{x^2+y^2+\left(z-d\right)^2}}$$

$$\Rightarrow q_t = \frac{\left(\chi_e'-\chi_e\right)q}{2\varepsilon_r'\left[1+\left(\chi_e+\chi_e'\right)/2\right]} = \left(\frac{\varepsilon_r'-\varepsilon_r}{\varepsilon_r'+\varepsilon_r}\right)\frac{q}{\varepsilon_r'} \Rightarrow V_{z>0} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q/\varepsilon_r'}{\sqrt{x^2+y^2+\left(z-d\right)^2}} + \frac{q_t}{\sqrt{x^2+y^2+\left(z+d\right)^2}}\right\}$$

**Problem 4.26** A spherical conductor, of radius $a$, carries a charge $Q$ (Fig. 4.29). It is surrounded by linear dielectric material of susceptibility $\chi_e$, out to radius $b$. Find the energy of this configuration (Eq. 4.58).



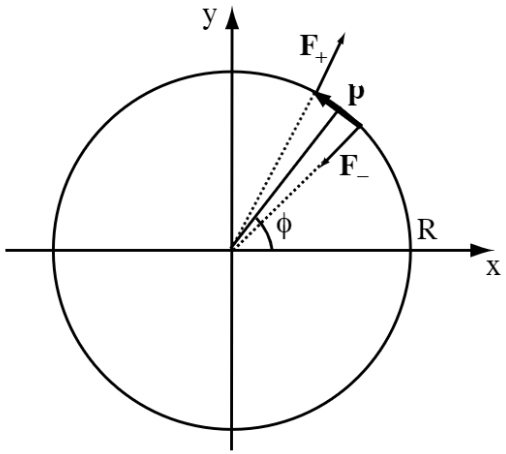

$$\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{enc}} \Rightarrow \mathbf{D} = \begin{cases} 0, & r<a \\ \dfrac{Q}{4\pi r^2}, & r>a \end{cases} \Rightarrow \mathbf{E} = \begin{cases} 0, & r<a \\ \dfrac{Q}{4\pi\varepsilon r^2}, & b>r>a \\ \dfrac{Q}{4\pi\varepsilon_0 r^2}, & r>a \end{cases}$$

$$\therefore W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}\,d\tau = \frac{1}{2}\frac{Q^2}{(4\pi)^2}4\pi\left(\frac{1}{\varepsilon}\int_a^b r^{-4}r^2dr + \frac{1}{\varepsilon_0}\int_b^\infty r^{-4}r^2dr\right) = \frac{Q^2}{8\pi\varepsilon_0}\left(\frac{1}{a}+\frac{\chi_e}{b}\right)$$

**Problem 4.31** A point charge $Q$ is "nailed down" on a table. Around it, at radius $R$, is a frictionless circular track on which a dipole $\mathbf{p}$ rides, constrained always to point tangent to the circle. Use Eq. 4.5 to show that the electric force on the dipole is

$$\mathbf{F} = \frac{Q}{4\pi\epsilon_0}\frac{\mathbf{p}}{R^3}.$$

Notice that this force is always in the "forward" direction (you can easily confirm this by drawing a diagram showing the forces on the two ends of the dipole). Why isn't this a perpetual motion machine?[21]




$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} = \left(\frac{p}{s}\partial_\phi\right)\frac{1}{4\pi\varepsilon_0}\frac{Q}{s^2}\hat{\mathbf{s}} = \frac{pQ}{4\pi\varepsilon_0 s^3}\hat{\phi} = \frac{Q}{4\pi\varepsilon_0 R^3}\mathbf{p}$$