Electromagnetism

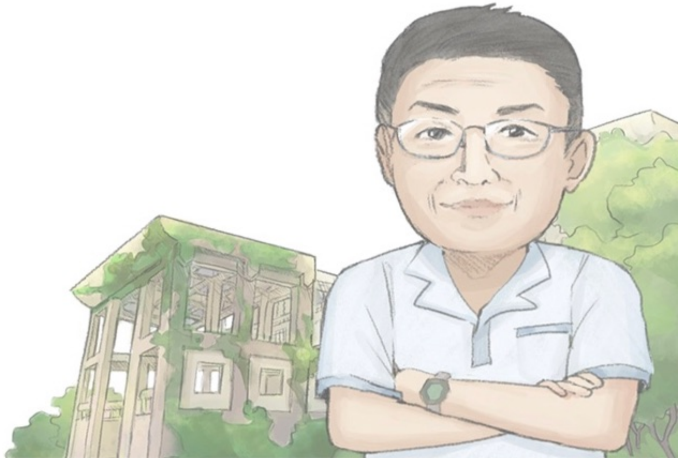
Introduction to Electrodynamics 4th David J. Griffiths

Chap.5

Prof. Tsun Hsu Chang

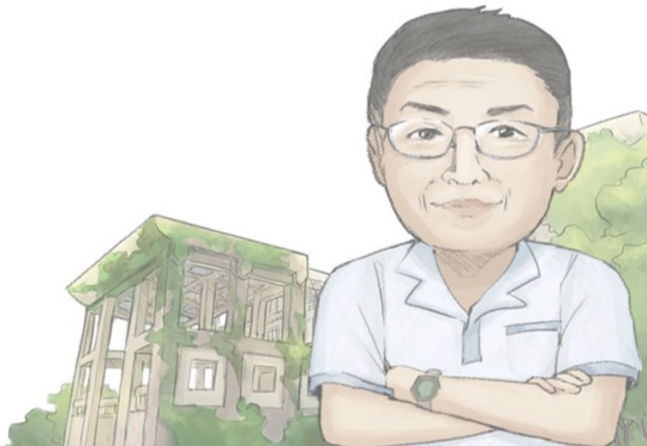
TA: Hung Chun Hsu, Yi Wen Lin, and Tien Fu Yang

2023 Fall



Exercise List

9, 10, 11, 41, 50, 16, 17, 25, 47, 60//



Problem 5.9 Find the magnetic field at point P for each of the steady current configurations shown in Fig. 5.23.

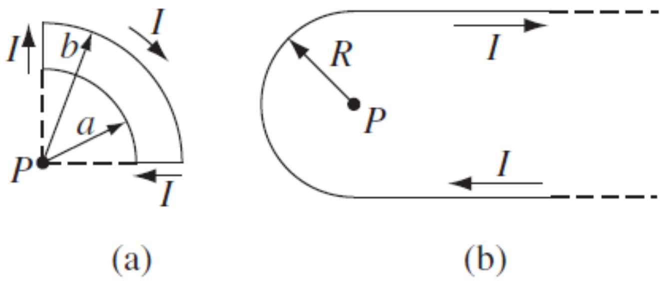


FIGURE 5.23

$$\mathbf{B}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}dl'=\frac{\mu_0 I}{4\pi}\int\frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

(a)

$$\mathbf{B}=\mathbf{B}_{\uparrow}+\mathbf{B}_{\text{outer arc}}+\mathbf{B}_{\leftarrow}+\mathbf{B}_{\text{inner arc}}$$

$$=0+\frac{\mu_0 I}{4\pi}\int_0^{\frac{\pi}{2}}\frac{r\,d\theta}{r^2}\bigg|_{r=b}+0+\frac{\mu_0 I}{4\pi}\int_{\frac{\pi}{2}}^{0}\frac{r\,d\theta}{r^2}\bigg|_{r=a}$$

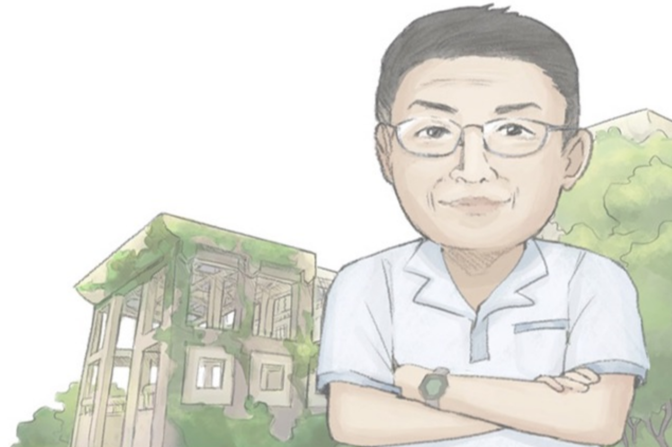
$$=\frac{\mu_0 I}{8}\left(\frac{1}{a}-\frac{1}{b}\right)\text{(pointing outward)}$$

(b)

$$\mathbf{B}=\mathbf{B}_{\leftarrow}+\mathbf{B}_{\text{arc}}+\mathbf{B}_{\rightarrow}$$

$$=\frac{\mu_0 I}{4\pi}\int_0^{\frac{\pi}{2}}\frac{\cos\theta\,d\theta}{s}\bigg|_{s=R}+\frac{\mu_0 I}{4\pi}\int_0^{\pi}\frac{r\,d\theta}{r^2}\bigg|_{r=R}+\frac{\mu_0 I}{4\pi}\int_0^{\frac{\pi}{2}}\frac{\cos\theta\,d\theta}{s}\bigg|_{s=R}$$

$$=\frac{\mu_0 I}{4\pi R}+\frac{\mu_0 I}{4R}+\frac{\mu_0 I}{4\pi R}=\frac{\mu_0 I}{4R}\left(1+\frac{2}{\pi}\right)\text{(pointing inward)}$$



Problem 5.10

(a) Find the force on a square loop placed as shown in Fig. 5.24(a), near an infinite straight wire. Both the loop and the wire carry a steady current I.

(b) Find the force on the triangular loop in Fig. 5.24(b).

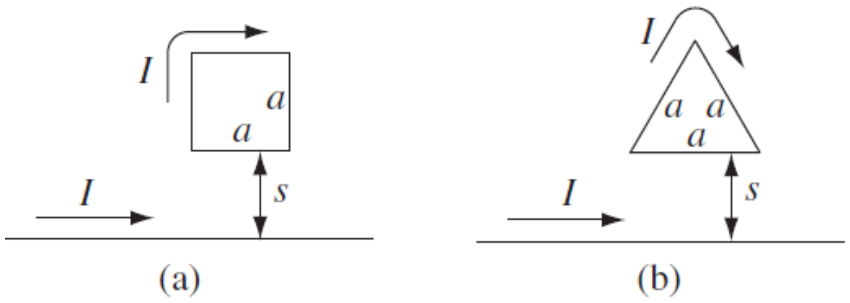


FIGURE 5.24

Infinite long straight wire: $\mathbf{B} = \hat{\mathbf{z}}\dfrac{\mu_0 I}{4\pi}\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \dfrac{\cos\theta d\theta}{y} = \dfrac{\mu_0 I}{2\pi y}\hat{\mathbf{z}}$

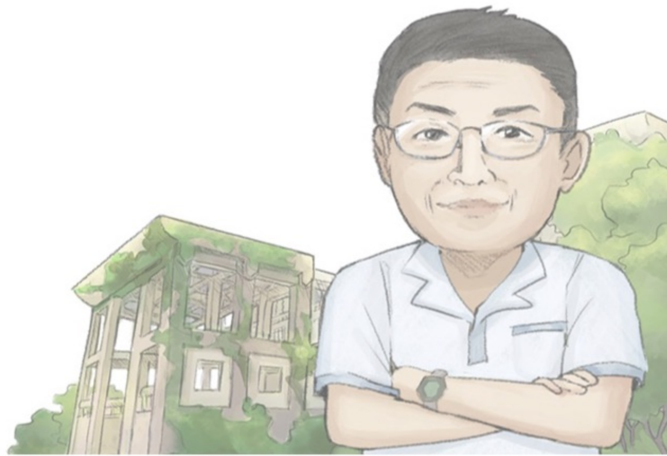
$$d\mathbf{F} = I\left(d\mathbf{l}\times\mathbf{B}\right) = I\left(dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}\right)\times\left(\frac{\mu_0 I}{2\pi y}\hat{\mathbf{z}}\right) = \frac{\mu_0 I^2}{2\pi y}\left(-dx\hat{\mathbf{y}} + dy\hat{\mathbf{x}}\right)$$

(a)

$$\mathbf{F} = \mathbf{F}_{\uparrow} + \mathbf{F}_{\rightarrow} + \mathbf{F}_{\downarrow} + \mathbf{F}_{\leftarrow}$$

$$= \int_s^{s+a}\frac{\mu_0 I^2}{2\pi y}\left(dy\hat{\mathbf{x}}\right) + \int_0^a \frac{\mu_0 I^2}{2\pi y}\left(-dx\hat{\mathbf{y}}\right)\Bigg|_{y=s+a} + \int_{s+a}^{s}\frac{\mu_0 I^2}{2\pi y}\left(dy\hat{\mathbf{x}}\right) + \int_a^0 \frac{\mu_0 I^2}{2\pi y}\left(-dx\hat{\mathbf{y}}\right)\Bigg|_{y=s}$$

$$= \frac{\mu_0 I^2}{2\pi}\left[\hat{\mathbf{x}}\ln\frac{s+a}{s} + \left(-\hat{\mathbf{y}}\right)\frac{a}{s+a} + \hat{\mathbf{x}}\ln\frac{s}{s+a} + \left(-\hat{\mathbf{y}}\right)\frac{-a}{s}\right] = \frac{\mu_0 I^2}{2\pi}\frac{a^2}{s\left(s+a\right)}\hat{\mathbf{y}}$$



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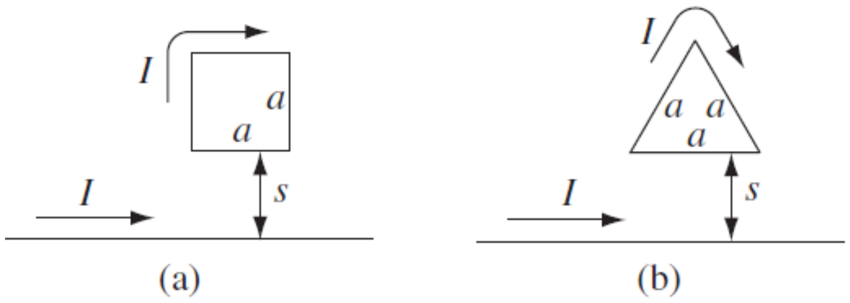


FIGURE 5.24

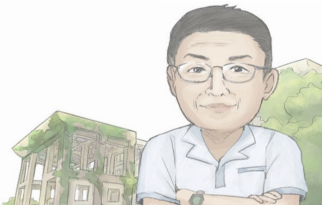
(b)

$$\mathbf{F} = \mathbf{F}_{\nearrow} + \mathbf{F}_{\searrow} + \mathbf{F}_{\leftarrow}$$

$$= \left[\int_0^{\frac{a}{2}} \frac{\mu_0 I^2}{2\pi y}(-dx\hat{\mathbf{y}}) + \int_s^{s+a} \frac{\mu_0 I^2}{2\pi y}(dy\hat{\mathbf{x}})\right] + \left[\int_{\frac{a}{2}}^{a} \frac{\mu_0 I^2}{2\pi y}(-dx\hat{\mathbf{y}}) + \int_{s+a}^{s} \frac{\mu_0 I^2}{2\pi y}(dy\hat{\mathbf{x}})\right] + \frac{\mu_0 I^2}{2\pi}\frac{a}{s}\hat{\mathbf{y}}$$

$$= \overbrace{-\hat{\mathbf{y}}\int_0^{\frac{a}{2}} \frac{\mu_0 I^2 dx}{2\pi\left(\sqrt{3}x + s\right)}}^{y=\sqrt{3}x+s} + 0 + \overbrace{-\hat{\mathbf{y}}\int_{\frac{a}{2}}^{a} \frac{\mu_0 I^2 dx}{2\pi\left(-\sqrt{3}x + \sqrt{3}a + s\right)}}^{y=-\sqrt{3}x+\sqrt{3}a+s} + 0 + \frac{\mu_0 I^2}{2\pi}\frac{a}{s}\hat{\mathbf{y}}$$

$$= \hat{\mathbf{y}}\frac{\mu_0 I^2}{2\pi}\left(\frac{a}{s} - \frac{1}{\sqrt{3}}\ln\frac{\frac{\sqrt{3}a}{2}+s}{s} - \frac{2}{\sqrt{3}}\ln\frac{s}{\frac{\sqrt{3}a}{2}+s}\right) = \hat{\mathbf{y}}\frac{\mu_0 I^2}{2\pi}\left[\frac{a}{s} - \frac{2}{\sqrt{3}}\ln\frac{\frac{\sqrt{3}a}{2}+s}{s}\right] = \hat{\mathbf{y}}\frac{\mu_0 I^2}{2\pi}\left[\frac{a}{s} - \frac{2}{\sqrt{3}}\ln\left(1 + \frac{\sqrt{3}}{2}\frac{a}{s}\right)\right]$$



Problem 5.11 Find the magnetic field at point P on the axis of a tightly wound **solenoid** (helical coil) consisting of n turns per unit length wrapped around a cylindrical tube of radius a and carrying current I (Fig. 5.25). Express your answer in terms of θ_1 and θ_2 (it's easiest that way). Consider the turns to be essentially circular, and use the result of Ex. 5.6. What is the field on the axis of an *infinite* solenoid (infinite in both directions)?

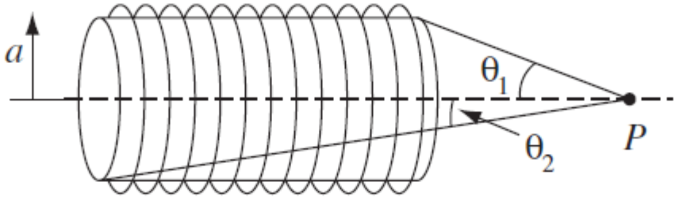


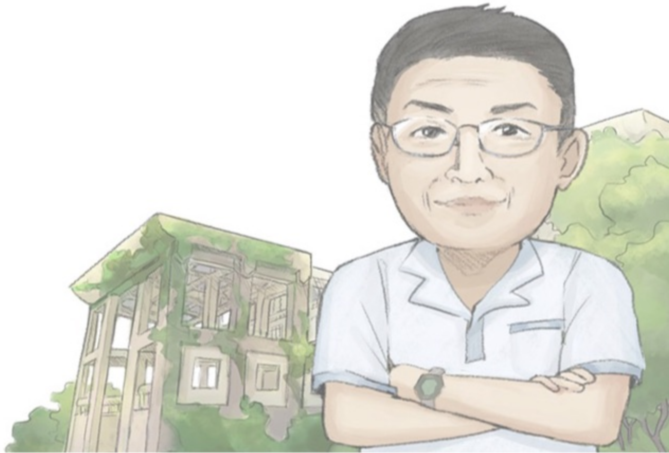
FIGURE 5.25

Eq(5.41): $\mathbf{B}(z)=\dfrac{\mu_0 I}{2}\dfrac{R^2}{\left(R^2+z^2\right)^{3/2}}\hat{\mathbf{z}}$

$$\mathbf{B}(x)=n\times\frac{\mu_0 I}{2}\int_{a\cot\theta 1}^{a\cot\theta_2}\frac{a^2dx}{\left(a^2+x^2\right)^{3/2}}\hat{\mathbf{x}}=\hat{\mathbf{x}}\frac{\mu_0 nI}{2}\int_{\theta 1}^{\theta_2}\frac{a^2\left(\dfrac{-a}{\sin^2\theta}d\theta\right)}{\left(a^2+a^2\cot^2\theta\right)^{3/2}}$$

$$=\hat{\mathbf{x}}\frac{\mu_0 nI}{2}\int_{\theta 1}^{\theta_2}\frac{a^2\left(\dfrac{-a}{\sin^2\theta}d\theta\right)}{\left(\dfrac{a^2}{\sin^2\theta}\right)^{3/2}}=\hat{\mathbf{x}}\frac{\mu_0 nI}{2}\int_{\theta 1}^{\theta_2}-\sin\theta d\theta=\frac{\mu_0 nI}{2}\left(\cos\theta_2-\cos\theta_1\right)\hat{\mathbf{x}}$$

$=\mu_0 nI\hat{\mathbf{x}}$ for an infinite solenoid, $\theta_2=0,\ \theta_1=\pi$, so $\cos\theta_2-\cos\theta_1=2$



Problem 5.41 A current I flows to the right through a rectangular bar of conducting material, in the presence of a uniform magnetic field $\mathbf{B}$ pointing out of the page (Fig. 5.56).

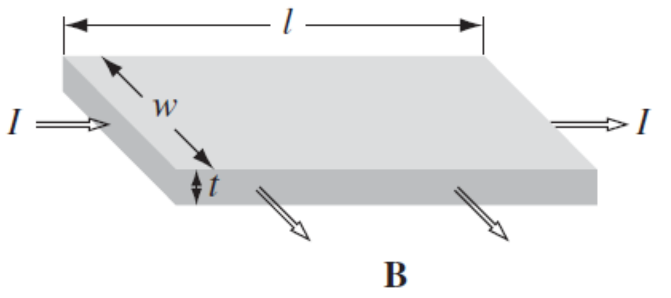


FIGURE 5.56

(a) If the moving charges are *positive*, in which direction are they deflected by the magnetic field? This deflection results in an accumulation of charge on the upper and lower surfaces of the bar, which in turn produces an electric force to counteract the magnetic one. Equilibrium occurs when the two exactly cancel. (This phenomenon is known as the **Hall effect**.)

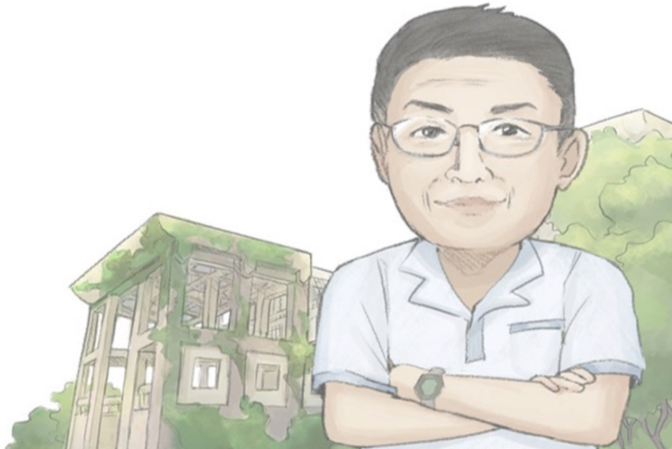
(a) $\mathbf{F} = q\mathbf{v} \times \mathbf{B} \Rightarrow$ Down, accumulating positive charge on lower surface

(b) Find the resulting potential difference (the **Hall voltage**) between the top and bottom of the bar, in terms of B, v (the speed of the charges), and the relevant dimensions of the bar.[23]

(b) $\mathbf{F_B} = q\mathbf{v} \times \mathbf{B} = \mathbf{F_E} = q\mathbf{E} \Rightarrow E = vB \Rightarrow V_{\text{lower}} - V_{\text{upper}} = Et = vBt$

(c) How would your analysis change if the moving charges were *negative*? [The Hall effect is the classic way of determining the sign of the mobile charge carriers in a material.]

$$\text{(c)} \begin{cases} \text{(a)} \rightarrow \text{Down, accumulating negative charge on lower surface} \\ \text{(b)} \rightarrow V_{\text{upper}} - V_{\text{lower}} = vBt \end{cases}$$



Problem 5.50 Magnetostatics treats the "source current" (the one that sets up the field) and the "recipient current" (the one that experiences the force) so asymmetrically that it is by no means obvious that the magnetic force between two current loops is consistent with Newton's third law. Show, starting with the Biot-Savart law (Eq. 5.34) and the Lorentz force law (Eq. 5.16), that the force on loop 2 due to loop 1 (Fig. 5.61) can be written as

$$\mathbf{F}_2 = -\frac{\mu_0}{4\pi} I_1 I_2 \oint \oint \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\mathbf{l}_1 \cdot d\mathbf{l}_2. \tag{5.91}$$

In this form, it is clear that $\mathbf{F}_2 = -\mathbf{F}_1$, since $\hat{\boldsymbol{\mathscr{r}}}$ changes direction when the roles of 1 and 2 are interchanged. (If you seem to be getting an "extra" term, it will help to note that $d\mathbf{l}_2 \cdot \hat{\boldsymbol{\mathscr{r}}} = d\mathscr{r}$.)

$$\mathbf{F}_{\text{mag}} = \int I\,(d\mathbf{l} \times \mathbf{B}). \tag{5.16}$$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}. \tag{5.34}$$

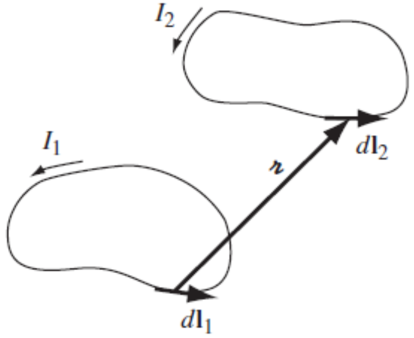
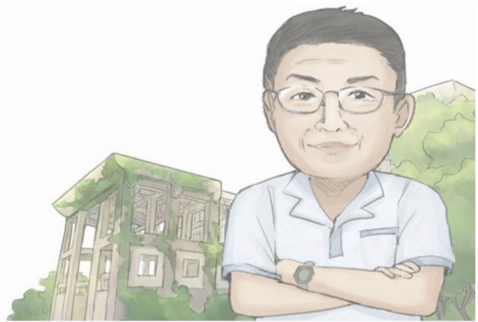


FIGURE 5.61

$$\boldsymbol{\mathscr{r}} = (x_2 - x_1)\hat{\mathbf{x}} + (y_2 - y_1)\hat{\mathbf{y}} + (z_2 - z_1)\hat{\mathbf{z}}$$

$$\nabla_1\left(\frac{1}{\mathscr{r}}\right) = \nabla_1\left[\frac{1}{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}}\right] = -\frac{1}{2}\frac{-2\left[(x_2-x_1)\hat{\mathbf{x}} + (y_2-y_1)\hat{\mathbf{y}} + (z_2-z_1)\hat{\mathbf{z}}\right]}{\left[(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2\right]^{3/2}} = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = -\nabla_2\left(\frac{1}{\mathscr{r}}\right)$$

Biot-Savart Law, the field from loop 1: $\mathbf{B} = \dfrac{\mu_0 I_1}{4\pi} \oint_1 \dfrac{d\mathbf{l}_1 \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$



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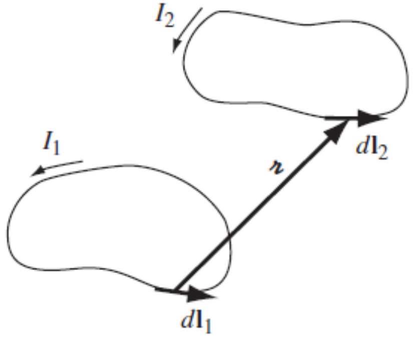


FIGURE 5.61

The force on loop2: $\mathbf{F} = I_2\oint_2 d\mathbf{l}_2\times\mathbf{B} = \frac{\mu_0 I_1 I_2}{4\pi}\oint_1\oint_2 \frac{d\mathbf{l}_2\times\left(d\mathbf{l}_1\times\hat{\boldsymbol{\mathscr{r}}}\right)}{\mathscr{r}^2} = \frac{\mu_0 I_1 I_2}{4\pi}\oint_1\oint_2 \frac{d\mathbf{l}_1\left(d\mathbf{l}_2\cdot\hat{\boldsymbol{\mathscr{r}}}\right)-\hat{\boldsymbol{\mathscr{r}}}\left(d\mathbf{l}_1\cdot d\mathbf{l}_2\right)}{\mathscr{r}^2}$

$$= \frac{\mu_0 I_1 I_2}{4\pi}\oint_1 d\mathbf{l}_1\oint_2 \frac{d\mathbf{l}_2\cdot\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} - \frac{\mu_0}{4\pi} I_1 I_2 \oint_1\oint_2 \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\left(d\mathbf{l}_1\cdot d\mathbf{l}_2\right)$$

$$= \frac{\mu_0 I_1 I_2}{4\pi}\oint_1 d\mathbf{l}_1 \underbrace{\oint_2 \nabla_2\left(\frac{-1}{\mathscr{r}}\right)\cdot d\mathbf{l}_2}_{\int_A^B(\nabla T)\cdot d\mathbf{l} = T(\mathbf{B})-T(\mathbf{A})} - \frac{\mu_0}{4\pi} I_1 I_2 \oint_1\oint_2 \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\left(d\mathbf{l}_1\cdot d\mathbf{l}_2\right) = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1\oint_2 \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\left(d\mathbf{l}_1\cdot d\mathbf{l}_2\right)$$

Problem 5.16 Two long coaxial solenoids each carry current I, but in opposite directions, as shown in Fig. 5.42. The inner solenoid (radius a) has n_1 turns per unit length, and the outer one (radius b) has n_2. Find $\mathbf{B}$ in each of the three regions: (i) inside the inner solenoid, (ii) between them, and (iii) outside both.

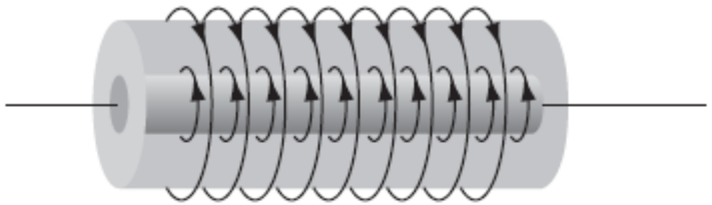
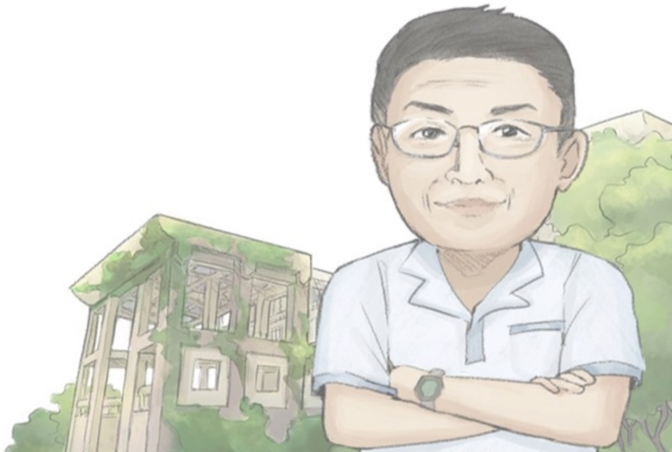


FIGURE 5.42

For a long solenoid, consisting of n closely wound turns per length of radius R :

$$\mathbf{B} = \begin{cases} \mu_0 n I \hat{\mathbf{z}}, & \text{inside the solenoid} \\ \mathbf{0}, & \text{outside the solenoid} \end{cases}$$

(i) $\mu_0 \left(n_2 - n_1\right) I \hat{\mathbf{x}}$ (ii) $\mu_0 n_2 I \hat{\mathbf{x}}$ (iii) $\mathbf{0}$



Problem 5.17 A large parallel-plate capacitor with uniform surface charge σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v, as shown in Fig. 5.43.

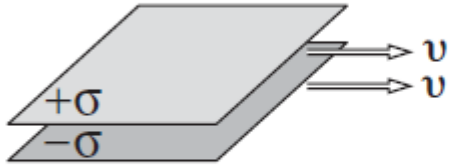


FIGURE 5.43

(a) Find the magnetic field between the plates and also above and below them.

(b) Find the magnetic force per unit area on the upper plate, including its direction.

(c) At what speed v would the magnetic force balance the electrical force?[15]

For an infinite uniform surface current $\mathbf{K} = K\hat{\mathbf{x}}$, flowing on the xy plane:

$$\mathbf{B} = \begin{cases} +\dfrac{\mu_0 K}{2}\hat{\mathbf{y}}, & \text{for } z < 0 \\ -\dfrac{\mu_0 K}{2}\hat{\mathbf{y}}, & \text{for } z > 0 \end{cases}$$

$$\begin{cases} \text{(i) Between: } \mathbf{B} = \dfrac{\mu_0 \sigma v}{2}\hat{\mathbf{y}} - \dfrac{\mu_0(-\sigma v)}{2}\hat{\mathbf{y}} = \mu_0 \sigma v \hat{\mathbf{y}}, \text{ elsewhere: } \mathbf{B} = \mathbf{0} \\ \text{(ii) Upper: } \mathbf{F} = \mathbf{K} \times \mathbf{B} = \sigma v \hat{\mathbf{x}} \times \left[-\dfrac{\mu_0(-\sigma v)}{2}\hat{\mathbf{y}} \right] = \dfrac{\mu_0 \sigma^2 v^2}{2}\hat{\mathbf{z}} \\ \text{(iii) } \mathbf{F_E} = q\mathbf{E} = \sigma \times \left(\dfrac{-\sigma}{2\varepsilon_0}\hat{\mathbf{z}} \right) = -\dfrac{\sigma^2}{2\varepsilon_0}\hat{\mathbf{z}} = -\dfrac{\mu_0 \sigma^2 v^2}{2}\hat{\mathbf{z}} \Rightarrow v = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}} = c \end{cases}$$

Problem 5.25 If $\mathbf{B}$ is *uniform*, show that $\mathbf{A}(\mathbf{r}) = -\frac{1}{2}(\mathbf{r} \times \mathbf{B})$ works. That is, check that $\nabla \cdot \mathbf{A} = 0$ and $\nabla \times \mathbf{A} = \mathbf{B}$. Is this result unique, or are there other functions with the same divergence and curl?

$\mathbf{B}$ is uniform: $\nabla \times \mathbf{B} = 0$

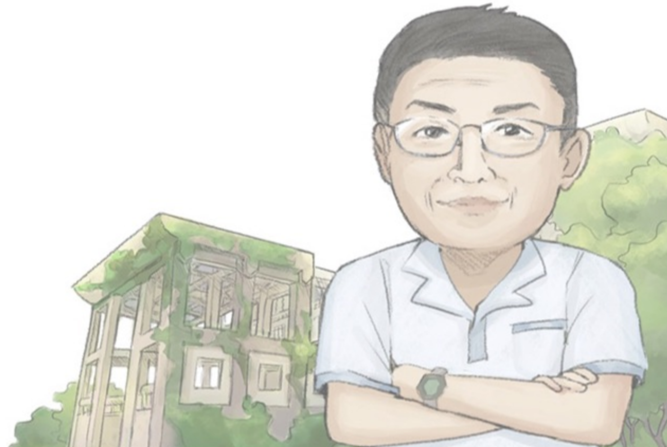
$$\nabla \cdot \mathbf{A} = -\frac{1}{2}\nabla \cdot (\mathbf{r} \times \mathbf{B}) = -\frac{1}{2}\left[\mathbf{B} \cdot (\nabla \times \mathbf{r}) - \mathbf{r} \cdot (\nabla \times \mathbf{B})\right] = 0$$

$$\nabla \times \mathbf{A} = -\frac{1}{2}\nabla \times (\mathbf{r} \times \mathbf{B}) = -\frac{1}{2}\left[(\mathbf{B} \cdot \nabla)\mathbf{r} - (\mathbf{r} \cdot \nabla)\mathbf{B} + \mathbf{r}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{r})\right]$$

$$= -\frac{1}{2}\left[\mathbf{B} - 0 + 0 - 3\mathbf{B}\right] = \mathbf{B}$$

$$\nabla \times \mathbf{r} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{vmatrix} = 0$$

$$\nabla \cdot \mathbf{r} = 3$$



Problem 5.47 The magnetic field on the axis of a circular current loop (Eq. 5.41) is far from uniform (it falls off sharply with increasing z). You can produce a more nearly uniform field by using *two* such loops a distance d apart (Fig. 5.59).

(a) Find the field (B) as a function of z, and show that $\partial B/\partial z$ is zero at the point midway between them ($z = 0$).

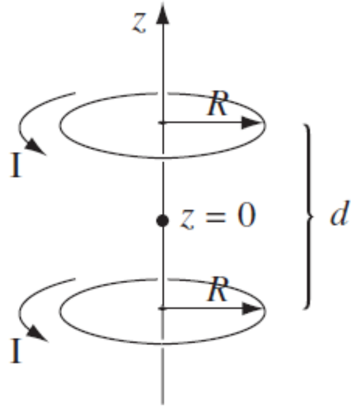


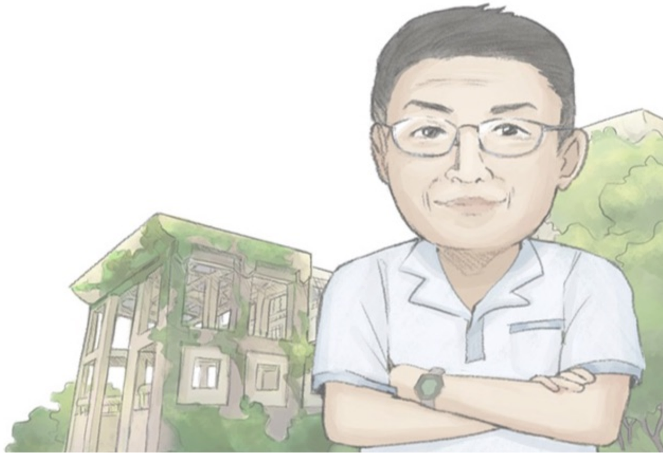
FIGURE 5.59

$$B(z) = \frac{\mu_0 I}{4\pi}\left(\frac{\cos\theta}{\mathscr{r}^2}\right)2\pi R = \frac{\mu_0 I}{2}\frac{R^2}{(R^2+z^2)^{3/2}}. \tag{5.41}$$

$$\mathbf{B}(z) = \hat{\mathbf{z}}\frac{\mu_0 I R^2}{2}\left\{\frac{1}{\left[R^2+(d/2+z)^2\right]^{3/2}}+\frac{1}{\left[R^2+(d/2-z)^2\right]^{3/2}}\right\}$$

$$\partial_z \mathbf{B}(z)\Big|_{z=0} = \hat{\mathbf{z}}\frac{-3\mu_0 I R^2}{2}\left\{\frac{d/2+z}{\left[R^2+(d/2+z)^2\right]^{5/2}}-\frac{d/2-z}{\left[R^2+(d/2-z)^2\right]^{5/2}}\right\}_{z=0}$$

$$= \hat{\mathbf{z}}\frac{-3\mu_0 I R^2}{2}\left\{\frac{d/2}{\left[R^2+(d/2)^2\right]^{5/2}}-\frac{d/2}{\left[R^2+(d/2)^2\right]^{5/2}}\right\} = 0$$



Problem 5.47 The magnetic field on the axis of a circular current loop (Eq. 5.41) is far from uniform (it falls off sharply with increasing z). You can produce a more nearly uniform field by using *two* such loops a distance d apart (Fig. 5.59).

(b) If you pick d just right, the *second* derivative of B will *also* vanish at the midpoint. This arrangement is known as a **Helmholtz coil**; it's a convenient way of producing relatively uniform fields in the laboratory. Determine d such that $\partial^2 B/\partial z^2 = 0$ at the midpoint, and find the resulting magnetic field at the center. [*Answer:* $8\mu_0 I/5\sqrt{5}R$]

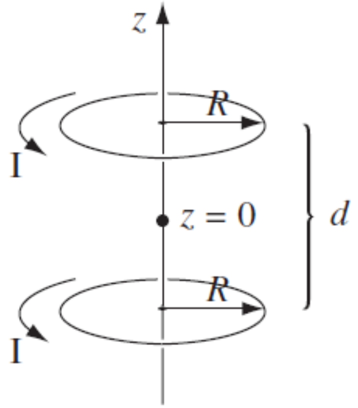


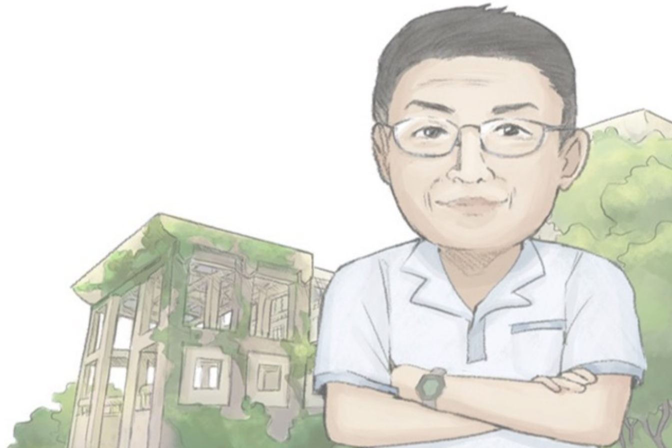
FIGURE 5.59

$$\partial_z^2 \mathbf{B}(z)\Big|_{z=0} = \hat{\mathbf{z}}\frac{-3\mu_0 I R^2}{2}\left\{\frac{(-5/2)2(d/2+z)^2}{\left[R^2+(d/2+z)^2\right]^{7/2}}+\frac{1}{\left[R^2+(d/2+z)^2\right]^{5/2}}+\frac{(-5/2)2(d/2-z)^2}{\left[R^2+(d/2-z)^2\right]^{7/2}}+\frac{1}{\left[R^2+(d/2-z)^2\right]^{5/2}}\right\}_{z=0}$$

$$=\hat{\mathbf{z}}\frac{3\mu_0 I R^2}{2}\left\{\frac{10(d/2)^2}{\left[R^2+(d/2)^2\right]^{7/2}}-\frac{2}{\left[R^2+(d/2)^2\right]^{5/2}}\right\}=\hat{\mathbf{z}}\frac{3\mu_0 I R^2}{8\left[R^2+(d/2)^2\right]^{7/2}}\left[10d^2-2\left(4R^2+d^2\right)\right]$$

$$=\hat{\mathbf{z}}\frac{3\mu_0 I R^2}{\left[R^2+(d/2)^2\right]^{7/2}}\left(d^2-R^2\right)\Rightarrow d=R$$

$$\Rightarrow \mathbf{B}(0)=\hat{\mathbf{z}}\frac{\mu_0 I R^2}{2}\left\{\frac{1}{\left[R^2+(R/2)^2\right]^{3/2}}+\frac{1}{\left[R^2+(R/2)^2\right]^{3/2}}\right\}=\frac{8\mu_0 I}{5^{3/2}R}\hat{\mathbf{z}}$$



Problem 5.60 A uniformly charged solid sphere of radius R carries a total charge Q, and is set spinning with angular velocity ω about the z axis.

(a) What is the magnetic dipole moment of the sphere?

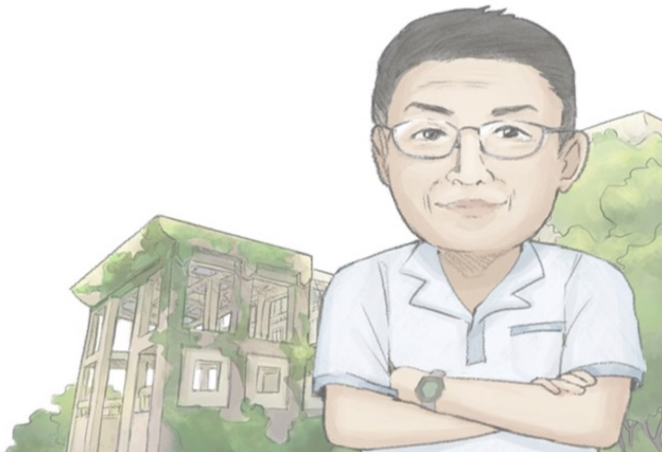
$$\rho = Q \Big/ \frac{4}{3}\pi R^3 ,$$

$$\mathbf{J} = \rho \mathbf{v} = \rho\left(r \sin\theta \omega \hat{\mathbf{r}} \times \hat{\mathbf{z}}\right) = \rho r \omega \sin\theta \hat{\boldsymbol{\varphi}}$$

$$\mathbf{m} = \frac{1}{2}\int (\mathbf{r}\times\mathbf{J})\, d\tau \text{...eq(5.90)} \qquad = \frac{1}{2}\int \left[r\hat{\mathbf{r}} \times \rho\left(r\sin\theta\omega\hat{\boldsymbol{\varphi}}\right)\right] r^2 \sin\theta dr d\theta d\varphi$$

$$= \frac{1}{2}\rho\omega\int \left(-\hat{\boldsymbol{\theta}}\right) r^4 \sin^2\theta dr d\theta d\varphi \qquad = \frac{1}{2}\rho\omega\int \left[-\begin{pmatrix} \cos\theta\cos\varphi\hat{\mathbf{x}} + \cos\theta\sin\varphi\hat{\mathbf{y}} \\ -\sin\theta\hat{\mathbf{z}} \end{pmatrix} \right] r^4 \sin^2\theta dr d\theta d\varphi$$

$$= \frac{1}{2} Q \left(\frac{4}{3}\pi R^3\right)^{-1} \omega \left(\frac{1}{5} R^5\right)\left(\frac{4}{3}\right)(2\pi)\hat{\mathbf{z}} \qquad = \frac{1}{5} Q\omega R^2 \hat{\mathbf{z}}$$



Problem 5.60 A uniformly charged solid sphere of radius R carries a total charge Q, and is set spinning with angular velocity ω about the z axis.

(b) Find the average magnetic field within the sphere (see Prob. 5.59).

(c) Find the approximate vector potential at a point (r, θ) where $r \gg R$.

(d) Find the *exact* potential at a point (r, θ) outside the sphere, and check that it is consistent with (c). [*Hint:* refer to Ex. 5.11.]

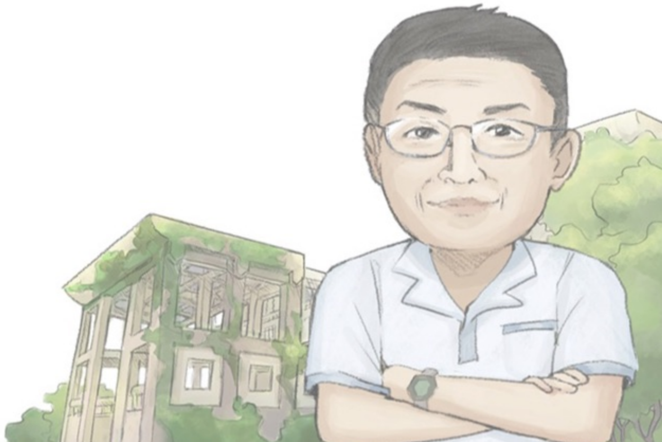
(b) $\mathbf{B}_{\text{ave}} = \dfrac{\mu_0}{4\pi}\dfrac{2\mathbf{m}}{R^3} = \dfrac{\mu_0}{4\pi}\dfrac{2Q\omega}{5R}\hat{\mathbf{z}}$...by eq(5.93)

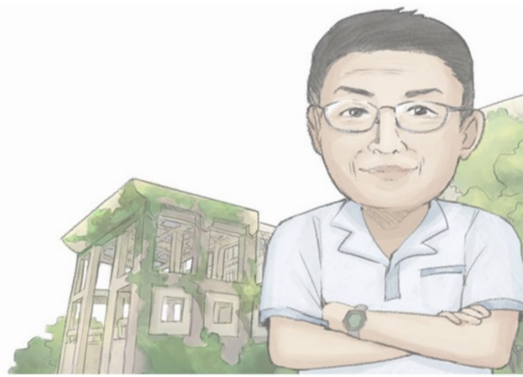
(c) $\mathbf{A} \simeq \dfrac{\mu_0}{4\pi}\dfrac{m\sin\theta}{r^2}\hat{\boldsymbol{\varphi}} = \dfrac{\mu_0}{4\pi}\dfrac{Q\omega R^2}{5}\dfrac{\sin\theta}{r^2}\hat{\boldsymbol{\varphi}}$

(d) $\mathbf{A} = \dfrac{\mu_0 R^4 \omega\sigma}{3}\dfrac{\sin\theta}{r^2}\hat{\boldsymbol{\varphi}}$...from eq(5.69)

$$\mathbf{A} = \int_0^R \frac{\mu_0 r'^4 \omega(\rho dr')}{3}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\varphi}} = \frac{\mu_0 R^5 \omega\rho}{15}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\varphi}}$$

$$= \frac{\mu_0 R^5 \omega}{15}\frac{Q}{\frac{4}{3}\pi R^3}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\varphi}} = \frac{\mu_0}{4\pi}\frac{Q\omega R^2}{5}\frac{\sin\theta}{r^2}\hat{\boldsymbol{\varphi}} \text{ ...same}$$





Problem 5.60 A uniformly charged solid sphere of radius R carries a total charge Q, and is set spinning with angular velocity ω about the z axis.

(e) Find the magnetic field at a point (r, θ) *inside* the sphere (Prob. 5.30), and check that it is consistent with (b).

From Prob.5.30 $\mathbf{B}=\frac{\mu_0\omega Q}{4\pi R}\left[\left(1-\frac{3r^2}{5R^2}\right)\cos\theta\hat{\mathbf{r}}-\left(1-\frac{6r^2}{5R^2}\right)\sin\theta\hat{\boldsymbol{\theta}}\right]$

$$\Rightarrow \mathbf{B}_{\text{ave}}=\frac{1}{\text{Vol}}\int\frac{\mu_0\omega Q}{4\pi R}\left[\left(1-\frac{3r^2}{5R^2}\right)\cos\theta\hat{\mathbf{r}}-\left(1-\frac{6r^2}{5R^2}\right)\sin\theta\hat{\boldsymbol{\theta}}\right]r^2\sin\theta\, dr\, d\theta\, d\varphi$$

$$=\frac{3}{4\pi R^3}\frac{\mu_0\omega Q}{4\pi R}\int\left[\begin{array}{l}\left(1-\frac{3r^2}{5R^2}\right)\cos\theta\left(\sin\theta\cos\varphi\hat{\mathbf{x}}+\sin\theta\sin\varphi\hat{\mathbf{y}}+\cos\theta\hat{\mathbf{z}}\right)\\-\left(1-\frac{6r^2}{5R^2}\right)\sin\theta\left(\cos\theta\cos\varphi\hat{\mathbf{x}}+\cos\theta\sin\varphi\hat{\mathbf{y}}-\sin\theta\hat{\mathbf{z}}\right)\end{array}\right]r^2\sin\theta\, dr\, d\theta\, d\varphi$$

$$=\frac{3\mu_0\omega Q}{8\pi R^4}\int\left[\left(1-\frac{3r^2}{5R^2}\right)\cos^2\theta\hat{\mathbf{z}}+\left(1-\frac{6r^2}{5R^2}\right)\sin^2\theta\hat{\mathbf{z}}\right]r^2\sin\theta\, dr\, d\theta$$

$$=\hat{\mathbf{z}}\frac{3\mu_0\omega Q}{8\pi R^4}\left[\left(\frac{1}{3}R^3-\frac{3R^5}{25R^2}\right)\left(\frac{2}{3}\right)+\left(\frac{1}{3}R^3-\frac{6R^5}{25R^2}\right)\left(\frac{4}{3}\right)\right]=\frac{3\mu_0\omega Q}{8\pi R^4}\left(\frac{4}{15}R^3\right)\hat{\mathbf{z}}=\frac{\mu_0}{4\pi}\frac{2Q\omega}{5R}\hat{\mathbf{z}}\ \text{...same}$$