國立清華大學

Electromagnetism

Introduction to Electrodynamics 4th David J. Griffiths

Chap.6

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Exercise List

4, 10, 13, 15, 17, 21, 25, 27//

Problem 6.4 Derive Eq. 6.3. [Here's one way to do it: Assume the dipole is an infinitesimal square, of side ϵ (if it's not, chop it up into squares, and apply the argument to each one). Choose axes as shown in Fig. 6.8, and calculate $\mathbf{F} = I \int (d\mathbf{l} \times \mathbf{B})$ along each of the four sides. Expand $\mathbf{B}$ in a Taylor series—on the right side, for instance,

$$\mathbf{B} = \mathbf{B}(0, \epsilon, z) \cong \mathbf{B}(0, 0, z) + \epsilon \left.\frac{\partial \mathbf{B}}{\partial y}\right|_{(0,0,z)}.$$

For a more sophisticated method, see Prob. 6.22.]

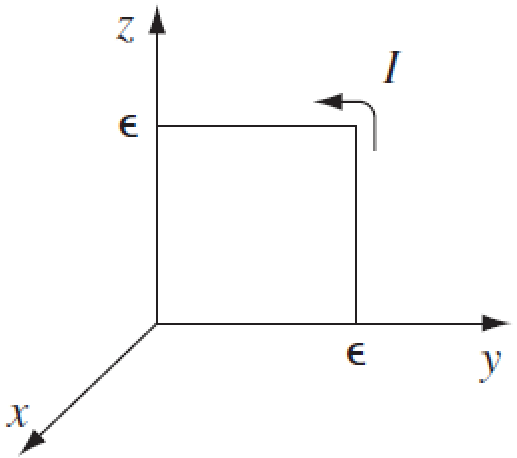


FIGURE 6.8

$$\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B}) \text{...eq.(6.3)}$$

$$= I\int (d\mathbf{l}\times\mathbf{B}) \qquad d\mathbf{l} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}} \qquad \mathbf{m} = \varepsilon^2\hat{\mathbf{x}}$$

$$= I\left[\int_0^{\varepsilon}(dy\hat{\mathbf{y}})\times\mathbf{B}(0,y,0) + \int_0^{\varepsilon}(dz\hat{\mathbf{z}})\times\mathbf{B}(0,\varepsilon,\mathrm{z}) + \int_{\varepsilon}^{0}(dy\hat{\mathbf{y}})\times\mathbf{B}(0,y,\varepsilon) + \int_{\varepsilon}^{0}(dy\hat{\mathbf{y}})\times\mathbf{B}(0,0,z)\right]$$

$$= I\left\{\int_0^{\varepsilon}(dy\hat{\mathbf{y}})\times\left[\mathbf{B}(0,y,0)\underline{-\mathbf{B}(0,y,\varepsilon)}\right] + \int_0^{\varepsilon}(dz\hat{\mathbf{z}})\times\left[\underline{\mathbf{B}(0,\varepsilon,\mathrm{z})}-\mathbf{B}(0,0,z)\right]\right\}$$

$$= I\left\{\int_0^{\varepsilon}(dy\hat{\mathbf{y}})\times\left[\mathbf{B}(0,y,0)\underline{-\mathbf{B}(0,y,0)-\varepsilon\left.\frac{\partial\mathbf{B}}{\partial z}\right|_{(0,y,0)}}\right] + \int_0^{\varepsilon}(dz\hat{\mathbf{z}})\times\left[\underline{\mathbf{B}(0,0,\mathrm{z})+\varepsilon\left.\frac{\partial\mathbf{B}}{\partial y}\right|_{(0,0,z)}} - \mathbf{B}(0,0,z)\right]\right\}$$

Problem 6.4

$$\mathbf{F} = I\varepsilon\left[-\int_0^{\varepsilon}(dy\hat{\mathbf{y}})\times\left.\frac{\partial\mathbf{B}}{\partial z}\right|_{(0,y,0)} + \int_0^{\varepsilon}(dz\hat{\mathbf{z}})\times\left.\frac{\partial\mathbf{B}}{\partial y}\right|_{(0,0,z)}\right]$$

$$= I\varepsilon\left\{-\int_0^{\varepsilon}(dy\hat{\mathbf{y}})\times\left[\left.\frac{\partial\mathbf{B}}{\partial z}\right|_{(0,0,0)} + y\left.\frac{\partial\mathbf{B}}{\partial y\partial z}\right|_{(0,0,0)}\right] + \int_0^{\varepsilon}(dz\hat{\mathbf{z}})\times\left[\left.\frac{\partial\mathbf{B}}{\partial y}\right|_{(0,0,0)} + z\left.\frac{\partial\mathbf{B}}{\partial y\partial z}\right|_{(0,0,0)}\right]\right\}$$

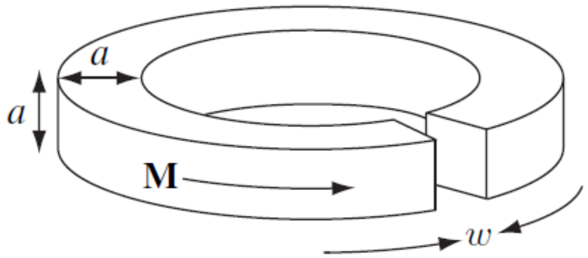
$$= I\varepsilon\left[-\left(\varepsilon\hat{\mathbf{y}}\times\left.\frac{\partial\mathbf{B}}{\partial z}\right|_{(0,0,0)} + \frac{1}{2}\varepsilon^2\hat{\mathbf{y}}\times\left.\frac{\partial\mathbf{B}}{\partial y\partial z}\right|_{(0,0,0)}\right) + \left(\varepsilon\hat{\mathbf{z}}\times\left.\frac{\partial\mathbf{B}}{\partial y}\right|_{(0,0,0)} + \frac{1}{2}\varepsilon^2\hat{\mathbf{z}}\times\left.\frac{\partial\mathbf{B}}{\partial y\partial z}\right|_{(0,0,0)}\right)\right]$$

$$\approx I\varepsilon^2\left(-\hat{\mathbf{y}}\times\left.\frac{\partial\mathbf{B}}{\partial z}\right|_{(0,0,0)} + \hat{\mathbf{z}}\times\left.\frac{\partial\mathbf{B}}{\partial y}\right|_{(0,0,0)}\right) \qquad = I\varepsilon^2\left(-\frac{\partial B_z}{\partial z}\hat{\mathbf{x}} + \frac{\partial B_x}{\partial z}\hat{\mathbf{z}} - \frac{\partial B_y}{\partial y}\hat{\mathbf{x}} + \frac{\partial B_x}{\partial y}\hat{\mathbf{y}}\right)$$

$$= I\varepsilon^2\left(\underline{-\frac{\partial B_z}{\partial z}\hat{\mathbf{x}} - \frac{\partial B_y}{\partial y}\hat{\mathbf{x}}} + \frac{\partial B_x}{\partial y}\hat{\mathbf{y}} + \frac{\partial B_x}{\partial z}\hat{\mathbf{z}}\right) \qquad = I\varepsilon^2\left(\underline{\frac{\partial B_x}{\partial x}\hat{\mathbf{x}}} + \frac{\partial B_x}{\partial y}\hat{\mathbf{y}} + \frac{\partial B_x}{\partial z}\hat{\mathbf{z}}\right) \text{ since } \nabla\cdot\mathbf{B} = 0 = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}$$

$$= \nabla(\mathbf{m}\cdot\mathbf{B}) \qquad = \nabla\left(I\varepsilon^2\hat{\mathbf{x}}\cdot\mathbf{B}\right) \qquad = \nabla\left(I\varepsilon^2 B_x\right) \qquad = I\varepsilon^2\left(\frac{\partial B_x}{\partial x}\hat{\mathbf{x}} + \frac{\partial B_x}{\partial y}\hat{\mathbf{y}} + \frac{\partial B_x}{\partial z}\hat{\mathbf{z}}\right) \text{ QED.}$$

Problem 6.10 An iron rod of length L and square cross section (side a) is given a uniform longitudinal magnetization $\mathbf{M}$, and then bent around into a circle with a narrow gap (width w), as shown in Fig. 6.14. Find the magnetic field at the center of the gap, assuming $w \ll a \ll L$. [*Hint:* treat it as the superposition of a *complete* torus plus a square loop with reversed current.]



$$\mathbf{K} = \mathbf{M} \times \hat{\mathbf{n}} \Rightarrow K = M$$

$$B_{torus} = \frac{\mu_0 n I}{2\pi s} = \frac{\mu_0 K}{L} = \frac{\mu_0 M}{L}$$

$$B_{square\ loop} = \frac{\sqrt{2}\mu_0 I}{\pi R} = \frac{\sqrt{2}\mu_0 K w}{\pi (a/2)} = \frac{\sqrt{2}\mu_0 M w}{\pi (a/2)}$$

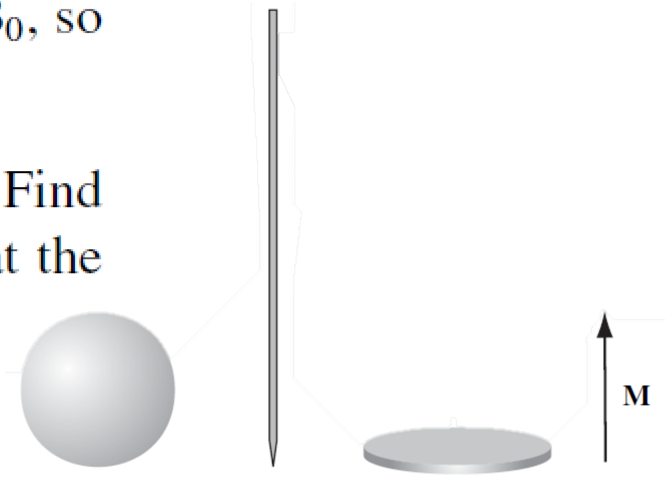
$$\mathbf{B} = \frac{\mu_0 \mathbf{M}}{L} - \frac{\sqrt{2}\mu_0 \mathbf{M} w}{\pi (a/2)} = \mu_0 \mathbf{M}\left(\frac{1}{L} - \frac{2\sqrt{2} w}{\pi a}\right)$$

Problem 6.13 Suppose the field inside a large piece of magnetic material is $\mathbf{B}_0$, so that $\mathbf{H}_0 = (1/\mu_0)\mathbf{B}_0 - \mathbf{M}$, where $\mathbf{M}$ is a "frozen-in" magnetization.

(a) Now a small spherical cavity is hollowed out of the material (Fig. 6.21). Find the field at the center of the cavity, in terms of $\mathbf{B}_0$ and $\mathbf{M}$. Also find $\mathbf{H}$ at the center of the cavity, in terms of $\mathbf{H}_0$ and $\mathbf{M}$.

(b) Do the same for a long needle-shaped cavity running parallel to $\mathbf{M}$.

(c) Do the same for a thin wafer-shaped cavity perpendicular to $\mathbf{M}$.



(a) Sphere (b) Needle (c) Wafer

Assume the cavities are small enough so $\mathbf{M}$, $\mathbf{B}_0$, and $\mathbf{H}_0$ are essentially constant. Compare Prob. 4.16. [*Hint:* Carving out a cavity is the same as superimposing an object of the same shape but opposite magnetization.]

(a) Eq 6.16: $\mathbf{B}_{sphere} = \frac{2}{3}\mu_0\mathbf{M} \Rightarrow \mathbf{B} = \underline{\mathbf{B}_0 - \frac{2}{3}\mu_0\mathbf{M}} \Rightarrow \mathbf{H} = \frac{1}{\mu_0}\mathbf{B} = \frac{1}{\mu_0}\mathbf{B}_0 - \frac{2}{3}\mathbf{M} = \mathbf{H}_0 + \mathbf{M} - \frac{2}{3}\mathbf{M} = \underline{\mathbf{H}_0 + \frac{1}{3}\mathbf{M}}$

(b) $B_{solenoid} = \mu_0 nI = \mu_0 K = \mu_0 M \Rightarrow \mathbf{B}_{solenoid} = \mu_0\mathbf{M} \Rightarrow \mathbf{B} = \underline{\mathbf{B}_0 - \mu_0\mathbf{M}} \Rightarrow \mathbf{H} = \underline{\mathbf{H}_0}$

(c) $B_{loop} = \frac{\mu_0 I}{2R} = 0$ for $R \to$ large enough $\Rightarrow \mathbf{B} = \underline{\mathbf{B}_0} \Rightarrow \mathbf{H} = \underline{\mathbf{H}_0 + \mathbf{M}}$

Problem 6.15 If $\mathbf{J}_f = \mathbf{0}$ everywhere, the curl of $\mathbf{H}$ vanishes (Eq. 6.19), and we can express $\mathbf{H}$ as the gradient of a scalar potential W:

$$\mathbf{H} = -\nabla W.$$

According to Eq. 6.23, then,

$$\nabla^2 W = (\nabla \cdot \mathbf{M}),$$

so W obeys Poisson's equation, with $\nabla \cdot \mathbf{M}$ as the "source." This opens up all the machinery of Chapter 3. As an example, find the field inside a uniformly magnetized sphere (Ex. 6.1) by separation of variables. [*Hint:* $\nabla \cdot \mathbf{M} = 0$ everywhere except at the surface ($r = R$), so W satisfies Laplace's equation in the regions $r < R$ and $r > R$; use Eq. 3.65, and from Eq. 6.24 figure out the appropriate boundary condition on W.]

Basically, all the math part is the same as "*Uniformly Polarized Sphere*"

$E \to H$

$P \to \mu_0 M$

$\varepsilon_0 \to \mu_0$

"Potentials":

$$\begin{cases} W_{\text{in}}(r,\theta) = \sum A_l r^l P_l(\cos\theta), & (r < R); \\ W_{\text{out}}(r,\theta) = \sum \frac{B_l}{r^{l+1}} P_l(\cos\theta), & (r > R). \end{cases}$$

Boundary Conditions:

$$\begin{cases} \text{(i)} & W_{\text{in}}(R,\theta) = W_{\text{out}}(R,\theta), \\ \text{(ii)} & -\left.\frac{\partial W_{\text{out}}}{\partial r}\right|_R + \left.\frac{\partial W_{\text{in}}}{\partial r}\right|_R = M^{\perp} = M\,\hat{\mathbf{z}}\cdot\hat{\mathbf{r}} = M\cos\theta. \end{cases}$$

$W_{\text{in}}(r,\theta) = \frac{M}{3} r\cos\theta = \frac{M}{3} z$, and hence $\mathbf{H}_{\text{in}} = -\nabla W_{\text{in}} = -\frac{M}{3}\hat{\mathbf{z}} = -\frac{1}{3}\mathbf{M}$, so

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0\left(-\frac{1}{3}\mathbf{M} + \mathbf{M}\right) = \boxed{\frac{2}{3}\mu_0 \mathbf{M}.} \checkmark$$

Problem 6.17 A current I flows down a long straight wire of radius a. If the wire is made of linear material (copper, say, or aluminum) with susceptibility χ_m, and the current is distributed uniformly, what is the magnetic field a distance s from the axis? Find all the bound currents. What is the *net* bound current flowing down the wire?

Copper is weakly diamagnetic, so the dipoles will line up opposite to the field. This results in a bound current running antiparallel to I, within the wire, and parallel to I along the surface (Fig. 6.20).

Ampere's law:

$$\oint \mathbf{H}\cdot d\mathbf{l} = H(2\pi s) = I_{f,enc} = \begin{cases} \dfrac{\pi s^2}{\pi a^2} I, & s < a \\ I, & s > a \end{cases}$$

$$\Rightarrow \mathbf{H} = \begin{cases} \dfrac{Is}{2\pi a^2}\hat{\phi}, & s < a \\ \dfrac{I}{2\pi s}\hat{\phi}, & s > a \end{cases} \Rightarrow \mathbf{B} = \mu \mathbf{H} = \begin{cases} \dfrac{\mu_0 (1+\chi_m) Is}{2\pi a^2}\hat{\phi}, & s < a \\ \dfrac{\mu_0 I}{2\pi s}\hat{\phi}, & s > a \end{cases}$$

$$\Rightarrow \begin{cases} \mathbf{J}_b = \nabla\times\mathbf{M} = \nabla\times(\chi_m \mathbf{H}) = \dfrac{\chi_m I}{\pi a^2}\hat{\mathbf{z}} = -\chi_m \mathbf{J}_f \\ \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = \chi_m \mathbf{H}\times\hat{\boldsymbol{\rho}} = -\dfrac{\chi_m I}{2\pi a}\hat{\mathbf{z}} = \dfrac{a}{2}\chi_m \mathbf{J}_f \end{cases} \Rightarrow \mathbf{I}_b = \mathbf{J}_b(\pi a^2) + \mathbf{K}_b(2\pi a) = 0$$

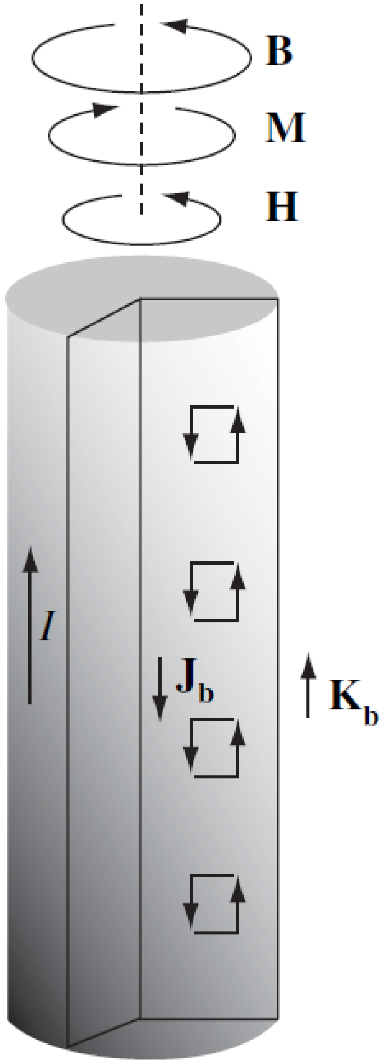


FIGURE 6.20

Problem 6.21

(a) Show that the energy of a magnetic dipole in a magnetic field **B** is

$$\boxed{U = -\mathbf{m} \cdot \mathbf{B}.} \qquad (6.34)$$

[Assume that the *magnitude* of the dipole moment is fixed, and all you have to do is move it into place and rotate it into its final orientation. The energy required to keep the current flowing is a different problem, which we will confront in Chapter 7.] Compare Eq. 4.6.

$$U = -\int_{\infty}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{l} = -\int_{\infty}^{\mathbf{r}} \nabla(\mathbf{m} \cdot \mathbf{B}) \cdot d\mathbf{l} = -\mathbf{m} \cdot \mathbf{B}(\mathbf{r}) + \mathbf{m} \cdot \mathbf{B}(\infty) = -\mathbf{m} \cdot \mathbf{B}(\mathbf{r})$$

$$= -\int_{\infty}^{\mathbf{r}} \nabla\left(\mathbf{m}_{\perp \mathbf{B}} \cdot \mathbf{B}\right) \cdot d\mathbf{l} + \int_{\frac{\pi}{2}}^{\theta} N d\theta'$$

$$= 0 + \int_{\frac{\pi}{2}}^{\theta} \left(|\mathbf{m} \times \mathbf{B}| = mB \sin\theta'\right) d\theta' \qquad = \left(-mB\cos\theta'\right)_{\frac{\pi}{2}}^{\theta} = -mB\cos\theta = -\mathbf{m} \cdot \mathbf{B}$$

Problem 6.21

(b) Show that the interaction energy of two magnetic dipoles separated by a displacement **r** is given by

$$U = \frac{\mu_0}{4\pi}\frac{1}{r^3}[\mathbf{m}_1 \cdot \mathbf{m}_2 - 3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}})]. \quad (6.35)$$

Compare Eq. 4.7.

Putting $\mathbf{m}_1$ at origin, $\mathbf{B}_1 = \frac{\mu_0}{4\pi r^3}\left[3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1\right]$

$$U = -\mathbf{m}_2 \cdot \mathbf{B}_1 = -\mathbf{m}_2 \cdot \frac{\mu_0}{4\pi r^3}\left[3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1\right] = \frac{\mu_0}{4\pi r^3}\left[\mathbf{m}_2 \cdot \mathbf{m}_1 - 3(\mathbf{m}_1 \cdot \hat{\mathbf{r}})(\mathbf{m}_2 \cdot \hat{\mathbf{r}})\right]$$

Problem 6.21

(c) Express your answer to (b) in terms of the angles θ_1 and θ_2 in Fig. 6.30, and use the result to find the stable configuration two dipoles would adopt if held a fixed distance apart, but left free to rotate.

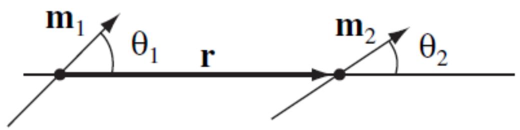


FIGURE 6.30

$$U=\frac{\mu_0}{4\pi r^3}\left[\mathbf{m}_2\cdot\mathbf{m}_1-3(\mathbf{m}_1\cdot\hat{\mathbf{r}})(\mathbf{m}_2\cdot\hat{\mathbf{r}})\right]$$

$$=\frac{\mu_0}{4\pi r^3}\left[m_1m_2\cos(\theta_1-\theta_2)-3(m_1\cos\theta_1)(m_2\cos\theta_2)\right]$$

$$=\frac{\mu_0 m_1m_2}{4\pi r^3}\left[\cos(\theta_1-\theta_2)-3\cos\theta_1\cos\theta_2\right]$$

$$=\frac{\mu_0 m_1m_2}{4\pi r^3}(\sin\theta_1\sin\theta_2-2\cos\theta_1\cos\theta_2)$$

$$=\frac{\mu_0 m_1m_2}{4\pi r^3}(1)\quad(\uparrow\uparrow)\qquad=\frac{\mu_0 m_1m_2}{4\pi r^3}(2)\quad(\rightarrow\leftarrow)$$

$$=\frac{\mu_0 m_1m_2}{4\pi r^3}(-1)\quad(\uparrow\downarrow)\qquad\boxed{=\frac{\mu_0 m_1m_2}{4\pi r^3}(-2)\quad(\rightarrow\rightarrow)}$$

Stable

$$\frac{\partial U}{\partial\theta_1}=\frac{\mu_0 m_1m_2}{4\pi r^3}(\cos\theta_1\sin\theta_2+2\sin\theta_1\cos\theta_2)=0$$

$$\Rightarrow\cos\theta_1\sin\theta_2=-2\sin\theta_1\cos\theta_2$$

$$\frac{\partial U}{\partial\theta_2}=0\Rightarrow\cos\theta_1\sin\theta_2=-\frac{1}{2}\sin\theta_1\cos\theta_2$$

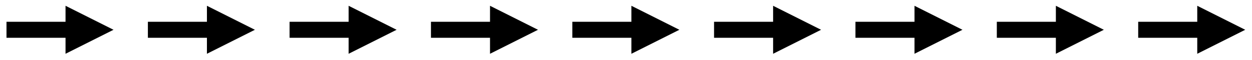
means $\cos\theta_1\sin\theta_2=\sin\theta_1\cos\theta_2=0$

$\Rightarrow\sin\theta_1=\sin\theta_2=0$ or

$\cos\theta_1=\cos\theta_2=0$

Problem 6.21

(d) Suppose you had a large collection of compass needles, mounted on pins at regular intervals along a straight line. How would they point (assuming the earth's magnetic field can be neglected)? [A rectangular array of compass needles aligns itself spontaneously, and this is sometimes used as a demonstration of "ferromagnetic" behavior on a large scale. It's a bit of a fraud, however, since the mechanism here is purely classical, and much weaker than the quantum mechanical **exchange forces** that are actually responsible for ferromagnetism.[13]]



Problem 6.25 Notice the following parallel:

$$\begin{cases} \nabla \cdot \mathbf{D} = 0, & \nabla \times \mathbf{E} = \mathbf{0}, & \epsilon_0 \mathbf{E} = \mathbf{D} - \mathbf{P}, & \text{(no free charge);} \\ \nabla \cdot \mathbf{B} = 0, & \nabla \times \mathbf{H} = \mathbf{0}, & \mu_0 \mathbf{H} = \mathbf{B} - \mu_0 \mathbf{M}, & \text{(no free current).} \end{cases}$$

Thus, the transcription $\mathbf{D} \to \mathbf{B}$, $\mathbf{E} \to \mathbf{H}$, $\mathbf{P} \to \mu_0 \mathbf{M}$, $\epsilon_0 \to \mu_0$ turns an electrostatic problem into an analogous magnetostatic one. Use this, together with your knowledge of the electrostatic results, to rederive

(a) the magnetic field inside a uniformly magnetized sphere (Eq. 6.16);

The electric field inside a uniformly polarized sphere:

$$\mathbf{E} = -\frac{\mathbf{P}}{3\varepsilon_0} \Rightarrow \mathbf{H} = -\frac{(\mu_0 \mathbf{M})}{3\mu_0} = -\frac{\mathbf{M}}{3} \Rightarrow \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \frac{2}{3} \mu_0 \mathbf{M}$$

Problem 6.25

(b) the magnetic field inside a sphere of linear magnetic material in an otherwise uniform magnetic field (Prob. 6.18);

The electric field $\mathbf{E}_0$ applied on a sphere of linear dielectric and the field after polarized inside the sphere is a uniform field $\mathbf{E}$:

$$\mathbf{E} = \frac{\mathbf{E}_0}{1+\chi_e/3} \Rightarrow \mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \Rightarrow \mu_0 \mathbf{M} = \mu_0 \chi_m \mathbf{H}$$

$$\Rightarrow \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 (1+\chi_m) \mathbf{H} = \frac{\mu_0 (1+\chi_m)}{1+\chi_m/3} \mathbf{H}_0$$

But $\mathbf{B}_0 = \mu_0 \mathbf{H}_0 \Rightarrow \mathbf{B} = \dfrac{1+\chi_m}{1+\chi_m/3} \mathbf{B}_0$

(c) the average magnetic field over a sphere, due to steady currents within the sphere (Eq. 5.93).

The average electric field over a sphere, due to charges within the sphere:

$$\mathbf{E}_{ave} = -\frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}}{R^3} = -\frac{1}{4\pi\varepsilon_0}\frac{1}{R^3}\int \mathbf{P}\, d\tau \Rightarrow \mathbf{H}_{ave} = -\frac{1}{4\pi\mu_0}\frac{1}{R^3}\int \mu_0 \mathbf{M}\, d\tau = -\frac{\mathbf{m}}{4\pi R^3}$$

$$\Rightarrow \mathbf{B}_{ave} = \mu_0 (\mathbf{H}_{ave} + \mathbf{M}_{ave}) = \mu_0 \left(-\frac{\mathbf{m}}{4\pi R^3} + \frac{\mathbf{m}}{\frac{4}{3}\pi R^3} \right) = \frac{2\mu_0 \mathbf{m}}{4\pi R^3}$$

Problem 6.27 At the interface between one linear magnetic material and another, the magnetic field lines bend (Fig. 6.32). Show that $\tan\theta_2/\tan\theta_1 = \mu_2/\mu_1$, assuming there is no free current at the boundary. Compare Eq. 4.68.

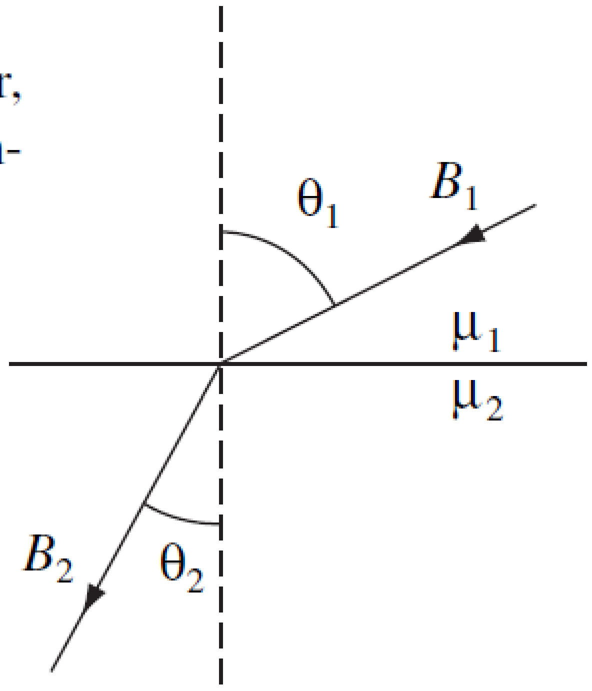


FIGURE 6.32

$B.C:$

$$\begin{cases} B_1^{\perp} = B_2^{\perp} \\ \mathbf{K}_f = 0 \Rightarrow \mathbf{H}_1^{\parallel} = \dfrac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \mathbf{H}_2^{\parallel} = \dfrac{1}{\mu_2}\mathbf{B}_2^{\parallel} \end{cases}$$

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{B_2^{\parallel}/B_2^{\perp}}{B_1^{\parallel}/B_1^{\perp}} = \frac{B_2^{\parallel}}{B_1^{\parallel}} = \frac{\mu_2}{\mu_1}$$