國立清華大學

Electromagnetism

Introduction to Electrodynamics 4th David J. Griffiths

Chap.7

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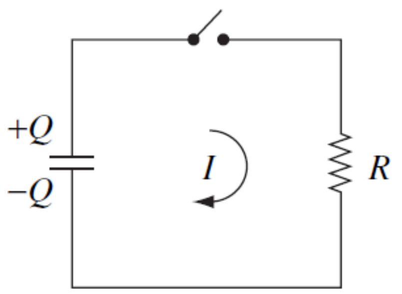
Exercise List

2, 6, 8, 11

Problem 7.2 A capacitor C has been charged up to potential V_0; at time $t = 0$, it is connected to a resistor R, and begins to discharge (Fig. 7.5a).



(a)

(a) Determine the charge on the capacitor as a function of time, $Q(t)$. What is the current through the resistor, $I(t)$?

$$V = IR = \frac{Q}{C},\ \frac{dQ}{dt} = -I = -\frac{V}{R} = -\frac{Q}{RC}$$

$$\Rightarrow Q(t) = Q(0)e^{-\frac{t}{RC}} = CV_0 e^{-\frac{t}{RC}} \Rightarrow I(t) = -\frac{dQ(t)}{dt} = \frac{V_0}{R}e^{-\frac{t}{RC}}$$

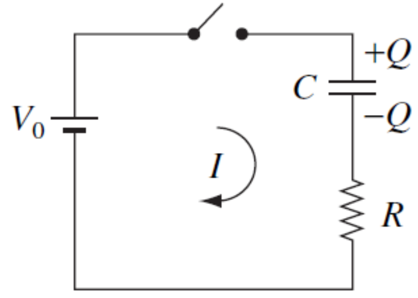
(b) What was the original energy stored in the capacitor (Eq. 2.55)? By integrating Eq. 7.7, confirm that the heat delivered to the resistor is equal to the energy lost by the capacitor.

$$(\text{Eq. } 2.55)\, W = \frac{1}{2}CV^2 = \frac{1}{2}CV_0^2$$

$$(\text{Eq. } 7.7)\, P = I^2 R \Rightarrow \int_0^\infty P dt = \int_0^\infty I^2 R dt = \frac{V_0^2}{R}\int_0^\infty e^{-\frac{2t}{RC}} dt = \frac{V_0^2}{R}\left(-\frac{RC}{2}\right)(-1) = \frac{1}{2}CV_0^2$$

Problem 7.2

Now imagine *charging up* the capacitor, by connecting it (and the resistor) to a battery of voltage V_0, at time $t = 0$ (Fig. 7.5b).



(b)

(c) Again, determine $Q(t)$ and $I(t)$.

$$V_0 = \frac{Q}{C} + IR,\ \frac{dQ}{dt} = I = \frac{1}{R}\left(V_0 - \frac{Q}{C}\right) = \frac{CV_0 - Q}{RC} \Rightarrow \frac{dQ}{CV_0 - Q} = \frac{dt}{RC}$$

$$\Rightarrow -\ln\left(CV_0 - Q\right) = \frac{t}{RC} + \underline{const.} \Rightarrow Q(t) = CV_0 - \underline{A}e^{-\frac{t}{RC}} = CV_0\left(1 - e^{-\frac{t}{RC}}\right) \Rightarrow I(t) = \frac{dQ(t)}{dt} = \frac{V_0}{R}e^{-\frac{t}{RC}}$$

(d) Find the total energy output of the battery ($\int V_0 I\, dt$). Determine the heat delivered to the resistor. What is the final energy stored in the capacitor? What fraction of the work done by the battery shows up as energy in the capacitor? [Notice that the answer is independent of R!]

$$\left(\text{Battery}\right) P = IV \Rightarrow \int_0^\infty IVdt = \frac{V_0^2}{R}\int_0^\infty e^{-\frac{t}{RC}}dt = \frac{V_0^2}{R}(-RC)(-1) = CV_0^2$$

Heat $= \frac{1}{2}CV_0^2$, since the current I keep the same.

Therefore the fianl energy stored in C is $CV_0^2 - \frac{1}{2}CV_0^2 = \frac{1}{2}CV_0^2$

Problem 7.6 A rectangular loop of wire is situated so that one end (height h) is between the plates of a parallel-plate capacitor (Fig. 7.9), oriented parallel to the field **E**. The other end is way outside, where the field is essentially zero. What is the emf in this loop? If the total resistance is R, what current flows? Explain. [*Warning:* This is a trick question, so be careful; if you have invented a perpetual motion machine, there's probably something wrong with it.]

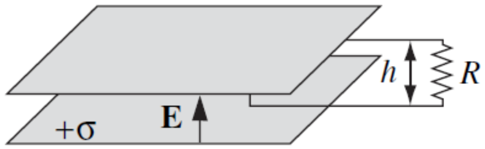


FIGURE 7.9

$$V = \begin{cases} \dfrac{\sigma}{\varepsilon_0} h, & \text{ideally} \\ 0, & \text{considering fringing field} \end{cases}$$

Problem 7.8 A square loop of wire (side a) lies on a table, a distance s from a very long straight wire, which carries a current I, as shown in Fig. 7.18.

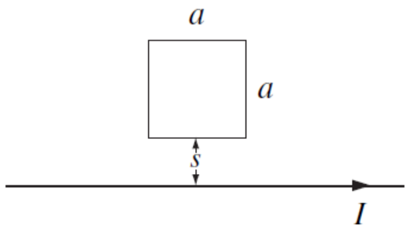


FIGURE 7.18

(a) Find the flux of **B** through the loop.

$$\text{Flux} = \int \mathbf{B}\cdot d\mathbf{a} = \int_0^a \int_s^{s+a} \frac{\mu_0 I}{2\pi s'} ds' dl' = \frac{\mu_0 I a}{2\pi} \ln \frac{s+a}{s}$$

(b) If someone now pulls the loop directly away from the wire, at speed v, what emf is generated? In what direction (clockwise or counterclockwise) does the current flow?

$$\frac{ds}{dt} = v$$

$$V = -\frac{d(\text{Flux})}{dt} = \frac{\mu_0 I a}{2\pi}\frac{d}{dt}\left(\ln\frac{s+a}{s}\right) = \frac{\mu_0 I a}{2\pi}\frac{v/s - v(s+a)/s^2}{s+a/s} = \frac{\mu_0 I a}{2\pi}\frac{sv - sv + av}{s^2+sa} = \frac{\mu_0 I a^2 v}{2\pi s(s+a)}$$

(c) What if the loop is pulled to the *right* at speed v?

$$V = 0$$

Problem 7.11 A square loop is cut out of a thick sheet of aluminum. It is then placed so that the top portion is in a uniform magnetic field **B**, and is allowed to fall under gravity (Fig. 7.20). (In the diagram, shading indicates the field region; **B** points into the page.) If the magnetic field is 1 T (a pretty standard laboratory field), find the terminal velocity of the loop (in m/s). Find the velocity of the loop as a function of time. How long does it take (in seconds) to reach, say, 90% of the terminal velocity? What would happen if you cut a tiny slit in the ring, breaking the circuit? [*Note:* The dimensions of the loop cancel out; determine the actual *numbers*, in the units indicated.]

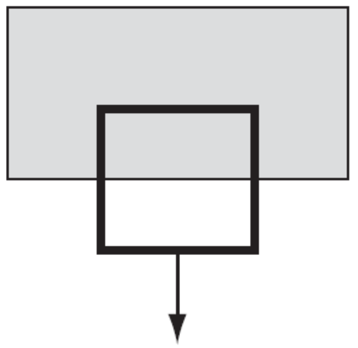


FIGURE 7.20

$$mg - F_B = ma, \quad |\mathbf{F_B}| = \int |\mathbf{I}\times\mathbf{B}|\,dl = IBL \quad V = \frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a} = BLv = IR$$

$$\Rightarrow mg - IBL = mg - \frac{BLv_t}{R}BL = 0 \Rightarrow \boxed{v_t = \frac{mgR}{B^2L^2}}$$

$$\Rightarrow mg - \frac{(BL)^2 v}{R} = m\frac{dv}{dt} \Rightarrow \frac{dv}{dt} = g - \frac{(BL)^2 v}{mR} \Rightarrow v(t) = \frac{g}{(BL)^2/mR}\left(1 - e^{\frac{-(BL)^2 t}{mR}}\right) = \boxed{v_t\left(1 - e^{\frac{-B^2L^2 t}{mR}}\right)}$$

$$t_{90\%} \Rightarrow \frac{v(t)}{v_t} = 0.9 = 1 - e^{\frac{-B^2L^2 t_{90\%}}{mR}} \Rightarrow t_{90\%} = \frac{-mR}{B^2L^2}\ln(0.1) = \boxed{\frac{mR}{B^2L^2}\ln(10)}$$

If we cut the loop: free falling

Exercise List

18, 24, 27, 28

Problem 7.18 A square loop, side a, resistance R, lies a distance s from an infinite straight wire that carries current I (Fig. 7.29). Now someone cuts the wire, so I drops to zero. In what direction does the induced current in the square loop flow, and what total charge passes a given point in the loop during the time this current flows? If you don't like the scissors model, turn the current down *gradually*:

$$I(t) = \begin{cases} (1-\alpha t)I, & \text{for } 0 \le t \le 1/\alpha, \\ 0, & \text{for } t > 1/\alpha. \end{cases}$$

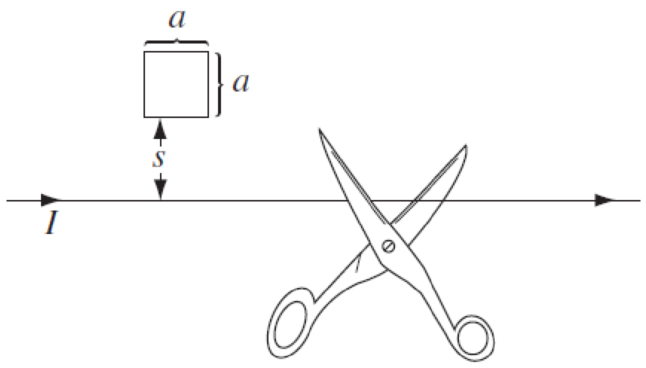


FIGURE 7.29

counterclockwise

$$V = I_{loop}R = \frac{dQ}{dt}R$$

$$= -\frac{d\Phi}{dt} = -\frac{1}{dt}\int \mathbf{B}\cdot d\mathbf{a} = -\frac{1}{dt}\int\left(\frac{\mu_0 I}{2\pi s'}\hat{\boldsymbol{\varphi}}\right)\cdot d\mathbf{a} = -\frac{d}{dt}\int_s^{s+a}\frac{\mu_0 I}{2\pi s'}(a)\,ds' = -\frac{d}{dt}\left(\frac{\mu_0 a I}{2\pi}\ln\frac{s+a}{s}\right)$$

$$= -\frac{\mu_0 a}{2\pi}\ln\frac{s+a}{s}\frac{dI}{dt}$$

$$\Rightarrow Q = \frac{1}{R}\int_I^0 -\frac{\mu_0 a}{2\pi}\ln\frac{s+a}{s}\,dI = \frac{\mu_0 a I}{2\pi R}\ln\frac{s+a}{s}$$

Problem 7.24 Find the self-inductance per unit length of a long solenoid, of radius R, carrying n turns per unit length.

$$\Phi_{\text{per turm}} = \int \mathbf{B} \cdot d\mathbf{a} = (\mu_0 n I)(\pi R^2) \Rightarrow \Phi_{\text{per length}} = (\mu_0 n I)(\pi R^2)(n) = LI \Rightarrow L = \mu_0 n^2 \pi R^2$$

Problem 7.27 A capacitor C is charged up to a voltage V and connected to an inductor L, as shown schematically in Fig. 7.39. At time $t = 0$, the switch S is closed. Find the current in the circuit as a function of time. How does your answer change if a resistor R is included in series with C and L?

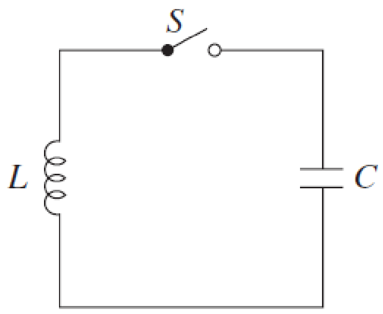


FIGURE 7.39

$$V_C = \frac{Q}{C},\ V_L = -L\frac{dI}{dt} = -L\frac{d}{dt}\left(-\frac{dQ}{dt}\right)$$

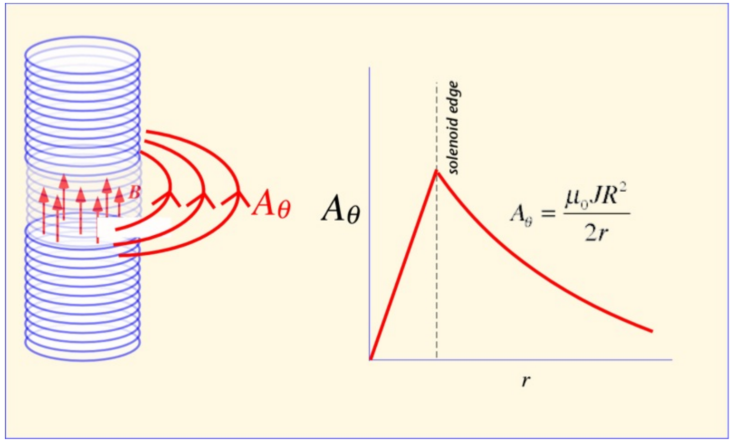
$$V_C + V_L = 0 = \frac{Q}{C} + L\frac{d}{dt}\frac{dQ}{dt} \Rightarrow \frac{dQ}{dt^2} + \frac{1}{LC}Q = 0 \Rightarrow Q(t) = \frac{1}{2}VC\left(e^{\frac{it}{\sqrt{LC}}} + e^{\frac{-it}{\sqrt{LC}}}\right) = VC\cos\frac{t}{\sqrt{LC}} \Rightarrow I = -V\sqrt{\frac{C}{L}}\sin\frac{t}{\sqrt{LC}}$$

$$V_R = -IR = R\frac{dQ}{dt}$$

$$V_C + V_L + V_R = 0 = \frac{Q}{C} + L\frac{d}{dt}\frac{dQ}{dt} + R\frac{dQ}{dt} \Rightarrow \frac{dQ}{dt^2} + \frac{R}{L}\frac{dQ}{dt} + \frac{1}{LC}Q = \frac{dQ}{dt^2} + 2\alpha\frac{dQ}{dt} + \omega_0^2 Q = 0$$

$$\Rightarrow Q(t) = Q_0 e^{-\alpha t}\cos\left(t\sqrt{\omega_0^2 - \alpha^2}\right) = VCe^{-\frac{Rt}{2L}}\cos\left(t\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\right)$$

Problem 7.28 Find the energy stored in a section of length l of a long solenoid (radius R, current I, n turns per unit length), (a) using Eq. 7.30 (you found L in Prob. 7.24); (b) using Eq. 7.31 (we worked out $\mathbf{A}$ in Ex. 5.12); (c) using Eq. 7.35; (d) using Eq. 7.34 (take as your volume the cylindrical tube from radius $a < R$ out to radius $b > R$).



$$\text{(a)}(7.30)\ W = \frac{1}{2}LI^2 = \frac{1}{2}\left(\mu_0 n^2 \pi R^2\right) I^2 \times l$$

$$\text{(b)}(7.31)\ W = \frac{1}{2} I \oint \mathbf{A}\cdot d\mathbf{l} = \frac{1}{2}\oint \mathbf{A}\cdot \mathbf{I} dl = \frac{1}{2}\oint \left(\frac{\mu_0 n I R}{2}\hat{\boldsymbol{\varphi}}\right)\cdot\left(I\hat{\boldsymbol{\varphi}}\right) dl' \times (nl) = \frac{1}{2}\frac{\mu_0 n I R}{2}(2\pi R)\times(nl) = \frac{1}{2}\mu_0 \pi l (nIR)^2$$

$$\text{(c)}(7.35)\ W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 d\tau == \frac{1}{2\mu_0}\int_0^l \int_0^R \left(\mu_0 n I\right)^2 2\pi s ds dz = \frac{1}{2\mu_0}\left(\mu_0 n I\right)^2\left(\pi R^2 l\right) = \frac{1}{2}\mu_0 \pi l (nIR)^2$$

$$\text{(d)}(7.34)\ W = \frac{1}{2\mu_0}\left[\int B^2 d\tau - \int \nabla\cdot(\mathbf{A}\times\mathbf{B}) d\tau\right] = \frac{1}{2\mu_0}\left[\int_V B^2 d\tau - \oint_S (\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}\right]$$

$$= \frac{1}{2\mu_0}\left[\int_0^l \int_a^R \left(\mu_0 n I\right)^2 2\pi s ds dz - \int_0^l \int_0^{2\pi}\left(\frac{\mu_0 n I a}{2}\mu_0 n I \underbrace{\hat{\boldsymbol{\varphi}}\times\hat{\mathbf{z}}}_{\hat{\mathbf{s}}}\right)\cdot\left(-a d\varphi dz \hat{\mathbf{s}}\right)\right]$$

$$= \frac{1}{2\mu_0}\left[\left(\mu_0 n I\right)^2 \pi\left(R^2 - a^2\right) l + \frac{\left(\mu_0 n I\right)^2 a^2}{2} 2\pi l\right]$$

$$= \frac{1}{2}\mu_0 \pi l (nIR)^2$$

Exercise List

31, 40, 42, 53, 57

Problem 7.31 Suppose the circuit in Fig. 7.41 has been connected for a long time when suddenly, at time $t = 0$, switch S is thrown from A to B, bypassing the battery.
(a) What is the current at any subsequent time t?

(b) What is the total energy delivered to the resistor?

(c) Show that this is equal to the energy originally stored in the inductor.

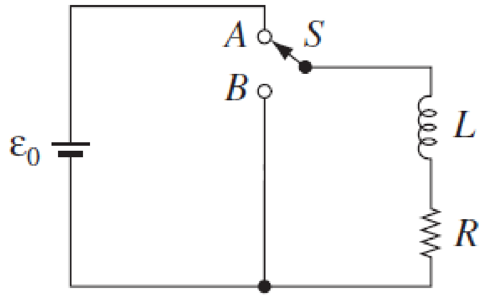


FIGURE 7.41

(a) $V_L + V_R = 0 = -L\frac{dI}{dt} - IR \Rightarrow I = \frac{V_0}{R}e^{-\frac{Rt}{L}}$

(b) $W_R(t=\infty) = \int P dt = \int I^2 R dt = \frac{V_0^2}{R}\int_0^\infty e^{-\frac{2Rt}{L}} dt = \frac{V_0^2}{R}\frac{L}{2R} = \frac{L}{2}\left(\frac{V_0}{R}\right)^2$

(c) $W_L(t=0) = \frac{1}{2}LI^2 = \frac{1}{2}L\left(\frac{V_0}{R}\right)^2$

Problem 7.40 Sea water at frequency $\nu = 4 \times 10^8$ Hz has permittivity $\epsilon = 81\epsilon_0$, permeability $\mu = \mu_0$, and resistivity $\rho = 0.23\ \Omega \cdot \text{m}$. What is the ratio of conduction current to displacement current? [*Hint:* Consider a parallel-plate capacitor immersed in sea water and driven by a voltage $V_0 \cos(2\pi \nu t)$.]

$$\begin{cases} \mathbf{J}_c = \sigma \mathbf{E} = \dfrac{1}{\rho}\mathbf{E} \Rightarrow J_c = \dfrac{1}{\rho}E = \dfrac{1}{\rho}\dfrac{V}{d} \\ \mathbf{J}_d = \dfrac{\partial \mathbf{D}}{\partial t} = \dfrac{\partial(\varepsilon \mathbf{E})}{\partial t} \Rightarrow J_d = \dfrac{\partial(\varepsilon E)}{\partial t} = \dfrac{\varepsilon}{d}\dfrac{\partial V}{\partial t} = -2\pi\nu\varepsilon \dfrac{V_0 \sin(2\pi\nu t)}{d} \end{cases} \Rightarrow \frac{\text{Amplitude}\{J_c\}}{\text{Amplitude}\{J_d\}} = \frac{1}{\rho}\frac{1}{2\pi\nu\varepsilon} \approx 2.41$$

Problem 7.42 A rare case in which the electrostatic field **E** for a circuit can actually be *calculated* is the following:[28] Imagine an infinitely long cylindrical sheet, of uniform resistivity and radius a. A slot (corresponding to the battery) is maintained at $\pm V_0/2$, at $\phi = \pm\pi$, and a steady current flows over the surface, as indicated in Fig. 7.51. According to Ohm's law, then,

$$V(a,\phi) = \frac{V_0\phi}{2\pi}, \quad (-\pi < \phi < +\pi).$$

(a) Use separation of variables in cylindrical coordinates to determine $V(s,\phi)$ inside and outside the cylinder. [*Answer:* $(V_0/\pi)\tan^{-1}[(s\sin\phi)/(a+s\cos\phi)]$, $(s<a)$; $(V_0/\pi)\tan^{-1}[(a\sin\phi)/(s+a\cos\phi)]$, $(s>a)$]

(b) Find the surface charge density on the cylinder. [*Answer:* $(\epsilon_0 V_0/\pi a)\tan(\phi/2)$]

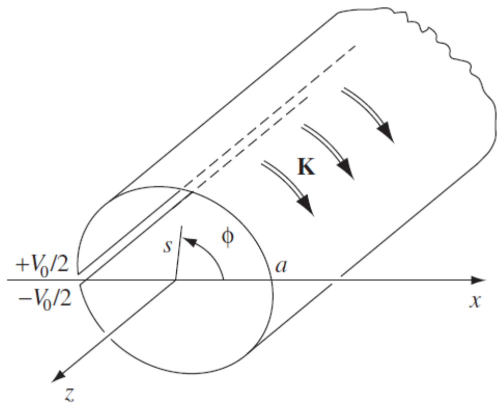


FIGURE 7.51

Prob. 3.24
$$\begin{cases} V_{in}(s,\varphi) = \sum_{k=1}^{\infty} s^k b_k \sin(k\varphi) & s<a \\ V_{out}(s,\varphi) = \sum_{k=1}^{\infty} s^{-k} d_k \sin(k\varphi) & s>a \end{cases}$$

Problem 7.42

(a) Use separation of variables in cylindrical coordinates to determine $V(s,\phi)$ inside and outside the cylinder. [*Answer:* $(V_0/\pi)\tan^{-1}[(s\sin\phi)/(a+s\cos\phi)]$, $(s<a)$; $(V_0/\pi)\tan^{-1}[(a\sin\phi)/(s+a\cos\phi)]$, $(s>a)$]

(b) Find the surface charge density on the cylinder. [*Answer:* $(\epsilon_0 V_0/\pi a)\tan(\phi/2)$]

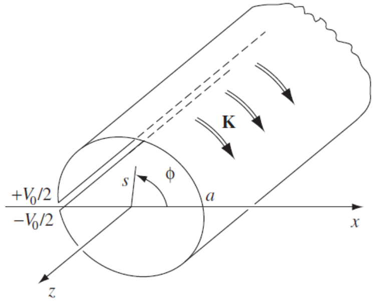


FIGURE 7.51

$$s=a\Rightarrow V_{out}=\frac{V_0\varphi}{2\pi}=V_{in}=\sum_{k=1}^{\infty}a^k b_k\sin(k\varphi)\overset{\text{Fourier's Trick}}{\Rightarrow}a^k b_k\delta_{kk'}=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{V_0\varphi}{2\pi}\sin(k'\varphi)\,d\varphi$$

$$\Rightarrow a^k b_k=\frac{1}{\pi}\frac{V_0}{2\pi}\left[\frac{-\varphi}{k}\cos(k\varphi)\Big|_{-\pi}^{\pi}+\int_{-\pi}^{\pi}\frac{1}{k}\cos(k\varphi)\,d\varphi\right]\Rightarrow a^k b_k=\frac{1}{\pi}\frac{V_0}{2\pi}\left[\frac{-\varphi}{k}\cos(k'\varphi)\Big|_{-\pi}^{\pi}+\frac{1}{k^2}\sin(k\varphi)\Big|_{-\pi}^{\pi}\right]$$

$$\Rightarrow a^k b_k=\frac{1}{\pi}\frac{V_0}{2\pi}\frac{-2\pi}{k}(-1)^k=-\frac{V_0}{\pi k}(-1)^k\Rightarrow b_k=-\frac{V_0}{\pi k}\left(-\frac{1}{a}\right)^k$$

$$\Rightarrow\begin{cases}V_{in}(s,\varphi)=-\dfrac{V_0}{\pi}\displaystyle\sum_{k=1}^{\infty}\left[\frac{1}{k}\left(-\frac{s}{a}\right)^k\sin(k\varphi)\right] & s<a\\ V_{out}(s,\varphi)=-\dfrac{V_0}{\pi}\displaystyle\sum_{k=1}^{\infty}\left[\frac{1}{k}\left(-\frac{a}{s}\right)^k\sin(k\varphi)\right] & s>a\end{cases}$$

Problem 7.42

(a) Use separation of variables in cylindrical coordinates to determine $V(s, \phi)$ inside and outside the cylinder. [*Answer:* $(V_0/\pi)\tan^{-1}[(s\sin\phi)/(a + s\cos\phi)]$, $(s < a)$; $(V_0/\pi)\tan^{-1}[(a\sin\phi)/(s + a\cos\phi)]$, $(s > a)$]

(b) Find the surface charge density on the cylinder. [*Answer:* $(\epsilon_0 V_0/\pi a)\tan(\phi/2)$]

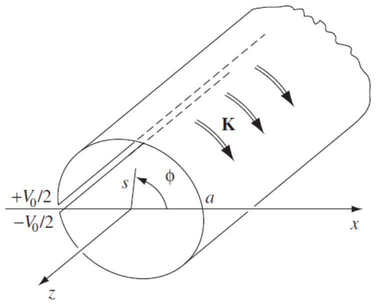


FIGURE 7.51

$$V_{in}(s,\varphi) = -\frac{V_0}{\pi}\sum_{k=1}^{\infty}\left[\frac{1}{k}\left(-\frac{s}{a}\right)^k \sin(k\varphi)\right] = -\frac{V_0}{\pi}\,\mathrm{Im}\left\{\sum_{k=1}^{\infty}\left[\frac{1}{k}\left(-\frac{s}{a}\right)^k e^{ik\varphi}\right]\right\} = -\frac{V_0}{\pi}\,\mathrm{Im}\left\{\sum_{k=1}^{\infty}\left[\frac{1}{k}\left(-\frac{s}{a}e^{i\varphi}\right)^k\right]\right\}$$

$$\equiv -\frac{V_0}{\pi}\,\mathrm{Im}\left\{\sum_{n=1}^{\infty}\left[\frac{1}{n}(-x)^n\right]\right\} = \frac{V_0}{\pi}\,\mathrm{Im}\left\{\sum_{n=1}^{\infty}\left[(-1)^{n+1}\frac{x^n}{n}\right]\right\} = \frac{V_0}{\pi}\,\mathrm{Im}\{\ln(1+x)\} = \frac{V_0}{\pi}\,\mathrm{Im}\left\{\boxed{\ln\left(1+\frac{s}{a}e^{i\varphi}\right)}\right\}$$

$$= \frac{V_0}{\pi}\tan^{-1}\frac{\mathrm{Im}\left\{1+\frac{s}{a}e^{i\varphi}\right\}}{\mathrm{Re}\left\{1+\frac{s}{a}e^{i\varphi}\right\}} = \frac{V_0}{\pi}\tan^{-1}\frac{\frac{1}{2i}\left[\left(1+\frac{s}{a}e^{i\varphi}\right)-\left(1+\frac{s}{a}e^{-i\varphi}\right)\right]}{\frac{1}{2}\left[\left(1+\frac{s}{a}e^{i\varphi}\right)+\left(1+\frac{s}{a}e^{-i\varphi}\right)\right]} = \frac{V_0}{\pi}\tan^{-1}\frac{\frac{1}{2i}\left(\frac{s}{a}e^{i\varphi}-\frac{s}{a}e^{-i\varphi}\right)}{\frac{1}{2}\left(2+\frac{s}{a}e^{i\varphi}+\frac{s}{a}e^{i\varphi}\right)}$$

$$= \frac{V_0}{\pi}\tan^{-1}\frac{\frac{s}{a}\sin\varphi}{1+\frac{s}{a}\cos\varphi} = \frac{V_0}{\pi}\tan^{-1}\frac{s\sin\varphi}{a+s\cos\varphi} \Rightarrow \begin{cases} V_{in}(s,\varphi) = \dfrac{V_0}{\pi}\tan^{-1}\dfrac{s\sin\varphi}{a+s\cos\varphi} & s<a \\ V_{out}(s,\varphi) = \dfrac{V_0}{\pi}\tan^{-1}\dfrac{a\sin\varphi}{s+a\cos\varphi} & s>a \end{cases}$$

Problem 7.42

(a) Use separation of variables in cylindrical coordinates to determine $V(s,\phi)$ inside and outside the cylinder. [*Answer:* $(V_0/\pi)\tan^{-1}[(s\sin\phi)/(a+s\cos\phi)]$, $(s<a)$; $(V_0/\pi)\tan^{-1}[(a\sin\phi)/(s+a\cos\phi)]$, $(s>a)$]

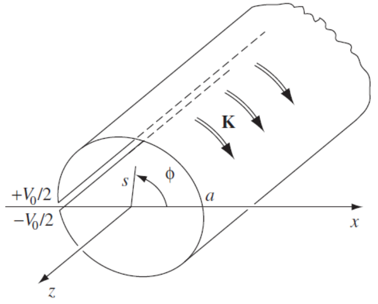


FIGURE 7.51

(b) Find the surface charge density on the cylinder. [*Answer:* $(\epsilon_0 V_0/\pi a)\tan(\phi/2)$]

$$\sigma(\varphi) = -\varepsilon_0\left(\left.\frac{\partial V_{out}}{\partial s}\right|_{s=a} - \left.\frac{\partial V_{in}}{\partial s}\right|_{s=a}\right) \qquad \frac{\partial \tan^{-1} x}{\partial x} = \frac{1}{1+x^2}$$

$$= \frac{-\varepsilon_0 V_0}{\pi}\left\{\left.\frac{1}{1+\left(\frac{a\sin\varphi}{s+a\cos\varphi}\right)^2}\left[-\frac{a\sin\varphi}{(s+a\cos\varphi)^2}\right]\right|_{s=a} - \left.\frac{1}{1+\left(\frac{s\sin\varphi}{a+s\cos\varphi}\right)^2}\left[\frac{\sin\varphi}{a+s\cos\varphi} - \frac{s\sin\varphi\cos\varphi}{(a+s\cos\varphi)^2}\right]\right|_{s=a}\right\}$$

$$= \frac{-\varepsilon_0 V_0}{\pi}\left\{\frac{1}{1+\left(\frac{a\sin\varphi}{a+a\cos\varphi}\right)^2}\left[-\frac{a\sin\varphi}{(a+a\cos\varphi)^2} - \frac{a\sin\varphi}{(a+a\cos\varphi)^2}\right]\right\}$$

$$= \frac{\varepsilon_0 V_0}{\pi}\frac{2a\sin\varphi}{(a+a\cos\varphi)^2+(a\sin\varphi)^2} = \frac{\varepsilon_0 V_0}{\pi}\frac{2a\sin\varphi}{2a^2+2a^2\cos\varphi} = \frac{\varepsilon_0 V_0}{\pi a}\frac{\sin\varphi}{1+\cos\varphi} = \frac{\varepsilon_0 V_0}{\pi a}\tan\frac{\varphi}{2}$$

Problem 7.53 The current in a long solenoid is increasing linearly with time, so the flux is proportional to t: $\Phi = \alpha t$. Two voltmeters are connected to diametrically opposite points (A and B), together with resistors (R_1 and R_2), as shown in Fig. 7.55. What is the reading on each voltmeter? Assume that these are *ideal* voltmeters that draw negligible current (they have huge internal resistance), and that a voltmeter registers $-\int_a^b \mathbf{E} \cdot d\mathbf{l}$ between the terminals and through the meter. [*Answer:* $V_1 = \alpha R_1/(R_1 + R_2)$; $V_2 = -\alpha R_2/(R_1 + R_2)$. Notice that $V_1 \neq V_2$, even though they are connected to the same points![32]]

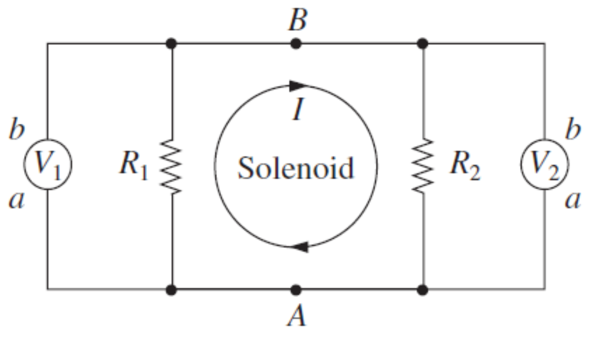


FIGURE 7.55

$$V = \frac{d\Phi}{dt} = \alpha \Rightarrow I = \frac{V}{R} = \frac{\alpha}{R_1 + R_2} \text{ counterclockwise}$$

$$\Rightarrow \begin{cases} V_1 = IR_1 = \dfrac{\alpha R_1}{R_1 + R_2} \\ V_2 = -IR_2 = \dfrac{\alpha R_2}{R_1 + R_2} \end{cases}$$

Problem 7.57 Two coils are wrapped around a cylindrical form in such a way that the *same flux passes through every turn of both coils*. (In practice this is achieved by inserting an iron core through the cylinder; this has the effect of concentrating the flux.) The **primary** coil has N_1 turns and the **secondary** has N_2 (Fig. 7.57). If the current I in the primary is changing, show that the emf in the secondary is given by

$$\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}, \tag{7.67}$$

where $\mathcal{E}_1$ is the (back) emf of the primary. [This is a primitive **transformer**—a device for raising or lowering the emf of an alternating current source. By choosing the appropriate number of turns, any desired secondary emf can be obtained. If you think this violates the conservation of energy, study Prob. 7.58.]

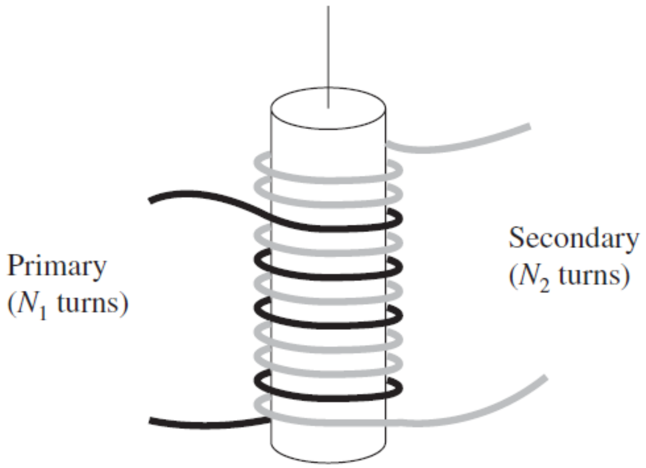


FIGURE 7.57

$$\frac{d\Phi_1}{dt} = \frac{d\Phi_2}{dt} \Rightarrow \frac{V_1}{V_2} = \frac{N_1\, d\Phi_1/dt}{N_2\, d\Phi_2/dt} = \frac{N_1}{N_2}$$