國立清華大學

Electromagnetism

Introduction to Electrodynamics 4th David J. Griffiths

Chap.8 Conversation Laws

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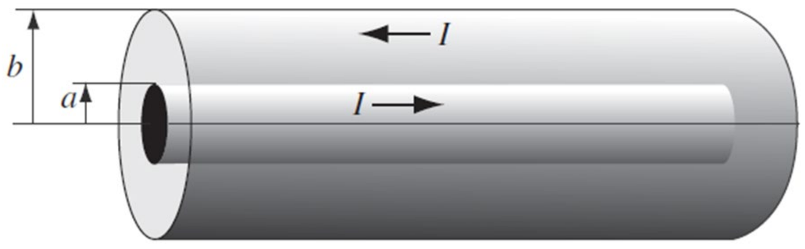
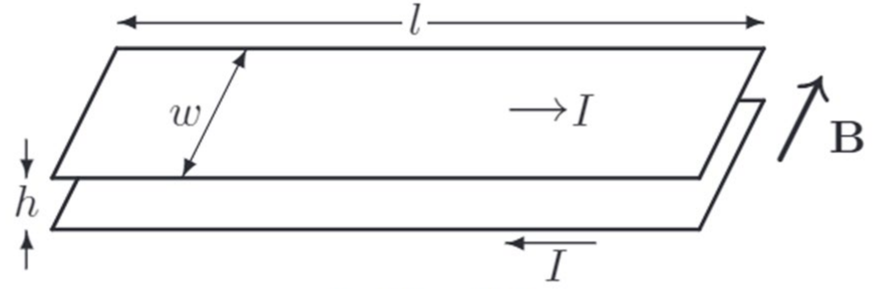
Exercise List

1, 4, 6, 16, 19, 23

Problem 8.1 Calculate the power (energy per unit time) transported down the cables of Ex. 7.13 and Prob. 7.62, assuming the two conductors are held at potential difference V, and carry current I (down one and back up the other).



Example 7.13.



Problem 7.62

$$\mathbf{E}=\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{s}\hat{\mathbf{s}} \qquad \mathbf{B}=\frac{\mu_0 I}{2\pi}\frac{1}{s}\hat{\phi}$$

$$\mathbf{S}=\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})=\frac{\lambda I}{4\pi^2\varepsilon_0}\frac{1}{s^2}\hat{\mathbf{z}}$$

$$P=\int\mathbf{S}\cdot d\mathbf{a}=\int_a^b S2\pi s ds=\frac{\lambda I}{2\pi\varepsilon_0}\int_a^b\frac{1}{s}ds=\frac{\lambda I}{2\pi\varepsilon_0}\ln\frac{b}{a}$$

$$V=\int\mathbf{E}\cdot d\mathbf{l}=\frac{\lambda}{2\pi\varepsilon_0}\int_a^b\frac{1}{s}ds=\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{b}{a}$$

$$P=IV$$

$$\mathbf{E}=\frac{\sigma}{\varepsilon_0}\hat{\mathbf{z}} \qquad \mathbf{B}=\mu_0 K\hat{\mathbf{x}}=\frac{\mu_0 I}{w}\hat{\mathbf{x}}$$

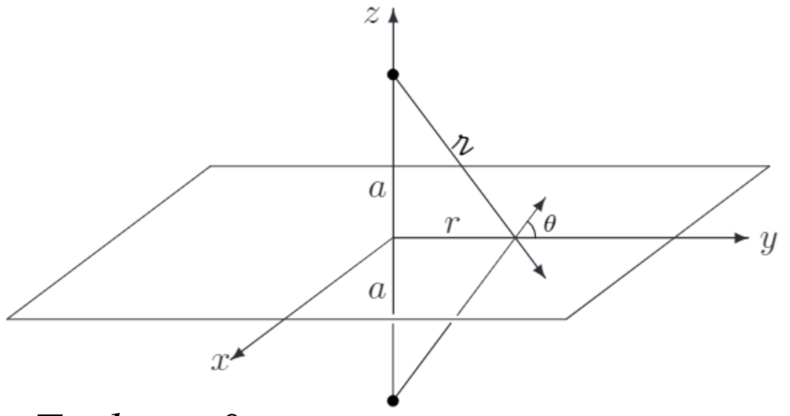
$$\mathbf{S}=\frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})=\frac{\sigma I}{\varepsilon_0 w}\hat{\mathbf{y}}$$

$$P=\int\mathbf{S}\cdot d\mathbf{a}=Swh=\frac{\sigma I h}{\varepsilon_0}$$

$$V=\int\mathbf{E}\cdot d\mathbf{l}=\frac{\sigma}{\varepsilon_0}h$$

$$P=IV$$

Problem 8.4



(a) Consider two equal point charges q, separated by a distance $2a$. Construct the plane equidistant from the two charges. By integrating Maxwell's stress tensor over this plane, determine the force of one charge on the other.

$$\left.\begin{cases}\text{For } xy \text{ plane } da_x = da_y = 0,\ da_z = -r\,dr\,d\phi \text{ (Force on the } \textit{upper} \text{ charge)} \\ \mathbf{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{\mathscr{r}^2}2\cos\theta\hat{\mathbf{r}} \text{ indept. of } \hat{\mathbf{z}} \to E_z = 0\end{cases}\right\} \Rightarrow \text{only } T_{zz}da_z \neq 0$$

$$\left(\overleftrightarrow{T}\cdot d\mathbf{a}\right)_z = T_{zx}da_x + T_{zy}da_y + T_{zz}da_z = \left[\varepsilon_0\left(E_zE_z - \frac{1}{2}E^2\right)\right](-r\,dr\,d\phi)$$

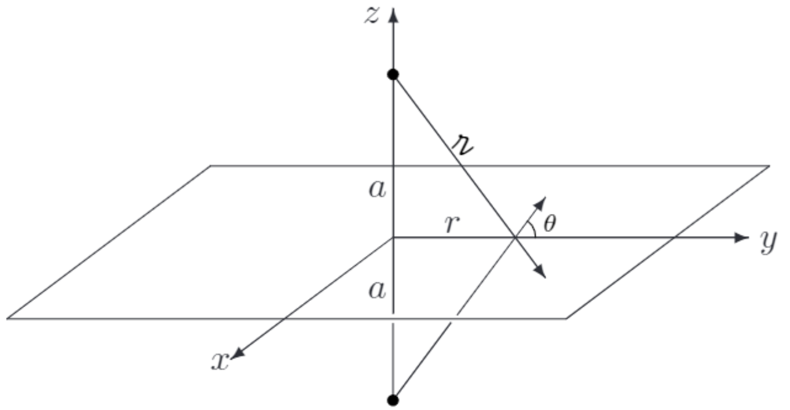
$$= \frac{\varepsilon_0}{2}E^2 r\,dr\,d\phi = \frac{\varepsilon_0}{2}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}^2}2\cos\theta\right)^2 r\,dr\,d\phi = \frac{\varepsilon_0}{2}\left(\frac{q}{2\pi\varepsilon_0}\right)^2\left(\frac{\cos\theta}{\mathscr{r}^2}\right)^2 r\,dr\,d\phi$$

$$\mathbf{F}_z = \int\left(\overleftrightarrow{T}\cdot d\mathbf{a}\right)_z = \frac{\varepsilon_0}{2}\left(\frac{q}{2\pi\varepsilon_0}\right)^2\int\left(\frac{\cos\theta}{\mathscr{r}^2}\right)^2 r\,dr\,d\phi = \frac{q^2}{8\pi^2\varepsilon_0}2\pi\int\left(\frac{r}{\mathscr{r}^3}\right)^2 r\,dr = \frac{q^2}{4\pi\varepsilon_0}\int\frac{r^3}{\left(r^2+a^3\right)^3}dr$$

$$= \frac{q^2}{4\pi\varepsilon_0}\frac{1}{2}\int\frac{u}{\left(u+a^2\right)^3}du = \frac{q^2}{4\pi\varepsilon_0}\frac{1}{2}\left[-\frac{1}{\left(u+a^2\right)} + \frac{a^2}{2\left(u+a^2\right)^3}\right]_0^\infty = \frac{q^2}{4\pi\varepsilon_0}\frac{1}{2}\left[0+\frac{1}{a^2}-\frac{a^2}{2a^4}\right] = \frac{q^2}{4\pi\varepsilon_0}\frac{1}{(2a)^2}$$

Problem 8.4

(b) Do the same for charges that are opposite in sign.



$$\left.\begin{cases}\text{For } xy \text{ plane } da_x = da_y = 0,\ da_z = -r\,dr\,d\phi \text{ (Force on the } \textit{upper} \text{ charge)} \\ \mathbf{E} = -\dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{\mathscr{r}^2}2\sin\theta\hat{\mathbf{z}} = \mathbf{E}_z \quad \text{indept. of } \hat{\mathbf{x}}, \hat{\mathbf{y}} \to E_x = E_y = 0\end{cases}\right\} \Rightarrow \text{only } T_{zz}da_z \neq 0$$

$$\left(\overleftrightarrow{T}\cdot d\mathbf{a}\right)_z = \left[\varepsilon_0\left(E_zE_z - \frac{1}{2}E^2\right)\right](-r\,dr\,d\phi) = \left(\frac{\varepsilon_0}{2}E_z^2\right)(-r\,dr\,d\phi) = -\frac{\varepsilon_0}{2}\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}^2}2\sin\theta\right)^2 r\,dr\,d\phi$$

$$= -\frac{\varepsilon_0}{2}\left(\frac{q}{2\pi\varepsilon_0}\right)^2\left(\frac{\sin\theta}{\mathscr{r}^2}\right)^2 r\,dr\,d\phi$$

$$\mathbf{F}_z = \int\left(\overleftrightarrow{T}\cdot d\mathbf{a}\right)_z = -\frac{\varepsilon_0}{2}\left(\frac{q}{2\pi\varepsilon_0}\right)^2\int\left(\frac{\sin\theta}{\mathscr{r}^2}\right)^2 r\,dr\,d\phi = -\frac{q^2}{8\pi^2\varepsilon_0}2\pi\int\left(\frac{a}{\mathscr{r}^3}\right)^2 r\,dr = -\frac{q^2}{4\pi\varepsilon_0}\int\frac{a^2r}{\left(r^2+a^3\right)^3}dr$$

$$= -\frac{q^2a^2}{4\pi\varepsilon_0}\left[-\frac{1}{4}\frac{1}{\left(r^2+a^2\right)^2}\right]_0^\infty = -\frac{q^2a^2}{4\pi\varepsilon_0}\left[0+\frac{1}{4a^4}\right] = -\frac{q^2}{4\pi\varepsilon_0}\frac{1}{(2a)^2} \quad v.s \quad \frac{q^2}{4\pi\varepsilon_0}\frac{1}{(2a)^2}$$

Problem 8.6 A charged parallel-plate capacitor (with uniform electric field $\mathbf{E} = E\,\hat{\mathbf{z}}$) is placed in a uniform magnetic field $\mathbf{B} = B\,\hat{\mathbf{x}}$, as shown in Fig. 8.6.

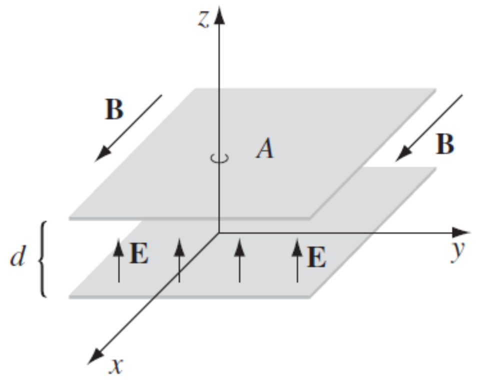


FIGURE 8.6

(a) Find the electromagnetic momentum in the space between the plates.

$$\mathbf{g}_{em} = \varepsilon_0 (\mathbf{E} \times \mathbf{B}) = \varepsilon_0 EB\hat{\mathbf{y}} \Rightarrow \mathbf{p}_{em} = \varepsilon_0 EBAd\hat{\mathbf{y}}$$

(b) Now a resistive wire is connected between the plates, along the z axis, so that the capacitor slowly discharges. The current through the wire will experience a magnetic force; what is the total impulse delivered to the system, during the discharge?[7]

$$\mathbf{I} = \int_0^\infty \mathbf{F}dt = \int_0^\infty I(\mathbf{l} \times \mathbf{B})dt = \int_0^\infty IBd(\hat{\mathbf{z}} \times \hat{\mathbf{x}})dt = Bd\hat{\mathbf{y}}\int_0^\infty Idt = Bd\hat{\mathbf{y}}\int_0^\infty \left(-\frac{dQ}{dt}\right)dt$$

$$= Bd\hat{\mathbf{y}}\int_0^\infty (-dQ) = Bd\hat{\mathbf{y}}\left[-Q(t=\infty) + Q(t=0)\right] = BdQ_0\hat{\mathbf{y}} = Bd(\varepsilon_0 EA)\hat{\mathbf{y}}$$

Problem 8.16[17] A sphere of radius R carries a uniform polarization $\mathbf{P}$ and a uniform magnetization $\mathbf{M}$ (not necessarily in the same direction). Find the electromagnetic momentum of this configuration. [*Answer:* $(4/9)\pi\mu_0 R^3(\mathbf{M}\times\mathbf{P})$]

$$\mathbf{E}=\begin{cases}-\dfrac{1}{3\varepsilon_0}\mathbf{P} & r<R\\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}\right] & r>R\end{cases}\qquad \hat{\mathbf{B}}=\begin{cases}\dfrac{2}{3}\mu_0\mathbf{M} & r<R\\ \dfrac{\mu_0}{4\pi}\dfrac{m}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}\right] & r>R\end{cases}$$

$$\mathbf{p}=\frac{4}{3}\pi R^3\mathbf{P}\qquad \mathbf{m}=\frac{4}{3}\pi R^3\mathbf{M}$$

$$\mathbf{p}_{mom}=\varepsilon_0\int\mathbf{E}\times\mathbf{B}d\tau=\begin{cases}\varepsilon_0\displaystyle\int\left(-\frac{1}{3\varepsilon_0}\mathbf{P}\right)\times\left(\frac{2}{3}\mu_0\mathbf{M}\right)d\tau & r<R\\ \varepsilon_0\displaystyle\int\left(\frac{1}{4\pi\varepsilon_0}\frac{1}{r^3}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}\right]\right)\times\left(\frac{\mu_0}{4\pi}\frac{m}{r^3}\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}\right]\right)d\tau & r>R\end{cases}$$

Problem 8.16[17] A sphere of radius R carries a uniform polarization $\mathbf{P}$ and a uniform magnetization $\mathbf{M}$ (not necessarily in the same direction). Find the electromagnetic momentum of this configuration. [*Answer:* $(4/9)\pi\mu_0 R^3(\mathbf{M}\times\mathbf{P})$]

$$\mathbf{p}_{mom}^{in}=-\frac{2}{9}\mu_0(\mathbf{P}\times\mathbf{M})\int d\tau=-\frac{2}{9}\mu_0(\mathbf{P}\times\mathbf{M})\left(\frac{4}{3}\pi R^3\right)=-\frac{8}{27}\mu_0\pi R^3(\mathbf{P}\times\mathbf{M})=\frac{8}{27}\mu_0\pi R^3(\mathbf{M}\times\mathbf{P})$$

$$\mathbf{p}_{mom}^{out}=\frac{\mu_0}{16\pi^2}\int\frac{1}{r^6}\left[3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{p}\right]\times\left[3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}-\mathbf{m}\right]d\tau$$

$$=\frac{\mu_0}{16\pi^2}\int\frac{1}{r^6}\left\{-2(\mathbf{p}\times\mathbf{m})+3\hat{\mathbf{r}}\left[\hat{\mathbf{r}}\cdot(\mathbf{p}\times\mathbf{m})\right]\right\}r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

setting $\mathbf{p}\times\mathbf{m}$ along z axis $\Rightarrow\hat{\mathbf{r}}\cdot(\mathbf{p}\times\mathbf{m})=|\mathbf{p}\times\mathbf{m}|\cos\theta$

$$\hat{\mathbf{r}}=\underbrace{\sin\theta\cos\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}}_{\int d\phi=0}+\cos\theta\hat{\mathbf{z}}$$

$$=\frac{\mu_0}{16\pi^2}\left(\int\frac{1}{r^4}dr\right)\left\{-2(\mathbf{p}\times\mathbf{m})\int\sin\theta\,d\theta\,d\phi+3\underbrace{|\mathbf{p}\times\mathbf{m}|\hat{\mathbf{z}}}_{\mathbf{p}\times\mathbf{m}}\int\cos^2\theta\sin\theta\,d\theta\,d\phi\right\}$$

$$=-\frac{4}{27}\mu_0\pi R^3(\mathbf{P}\times\mathbf{M})=\frac{4}{27}\mu_0\pi R^3(\mathbf{M}\times\mathbf{P})$$

$$\mathbf{p}_{mom}^{tot}=\left(\frac{8}{27}+\frac{4}{27}\right)\mu_0\pi R^3(\mathbf{M}\times\mathbf{P})=\frac{4}{9}\mu_0\pi R^3(\mathbf{M}\times\mathbf{P})$$

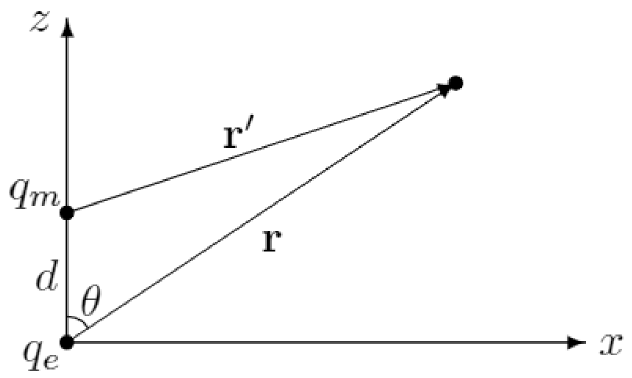
Problem 8.19[19] Suppose you had an electric charge q_e and a magnetic monopole q_m. The field of the electric charge is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q_e}{\mathscr{r}^2}\hat{\mathscr{r}}$$

(of course), and the field of the magnetic monopole is

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q_m}{\mathscr{r}^2}\hat{\mathscr{r}}.$$

Find the total angular momentum stored in the fields, if the two charges are separated by a distance d. [*Answer:* $(\mu_0/4\pi)q_eq_m$.][20]



Angular momentum density (Eq. 8.33)

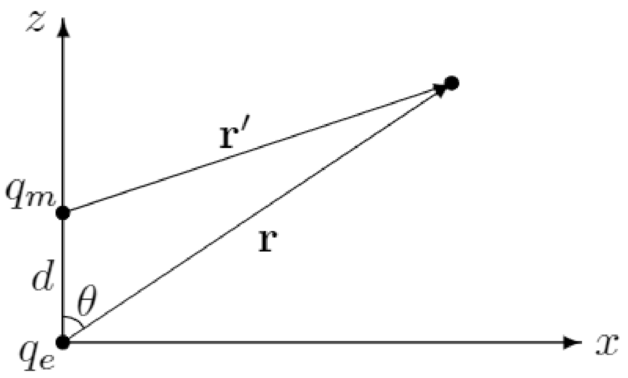
$$\mathbf{l} = (\mathbf{r}\times\mathbf{g})$$

Momentum density (Eq. 8.32)

$$\mathbf{g} = \varepsilon_0(\mathbf{E}\times\mathbf{B}) = \varepsilon_0\left[\left(\frac{q_e}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}\right)\times\left(\frac{\mu_0 q_m}{4\pi}\frac{\mathbf{r}'}{r'^3}\right)\right] = \varepsilon_0\left[\left(\frac{q_e}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}\right)\times\left(\frac{\mu_0 q_m}{4\pi}\frac{\mathbf{r}-d\hat{\mathbf{z}}}{\left(r^2+d^2-2dr\cos\theta\right)^{3/2}}\right)\right]$$

$$= \frac{\mu_0 q_e q_m}{(4\pi)2}\frac{-d(\mathbf{r}\times\hat{\mathbf{z}})}{r^3\left(r^2+d^2-2dr\cos\theta\right)^{3/2}}$$

Problem 8.19[19]



$$\mathbf{l}=(\mathbf{r}\times\mathbf{g})=-\frac{\mu_0 q_e q_m d}{(4\pi)^2}\frac{\mathbf{r}\times(\mathbf{r}\times\hat{\mathbf{z}})}{r^3\left(r^2+d^2-2dr\cos\theta\right)^{3/2}}$$

$$=-\frac{\mu_0 q_e q_m d}{(4\pi)^2}\frac{\mathbf{r}(\mathbf{r}\cdot\mathbf{z})-r^2\hat{\mathbf{z}}}{r^3\left(r^2+d^2-2dr\cos\theta\right)^{3/2}}=-\frac{\mu_0 q_e q_m d}{(4\pi)^2}\frac{r^2\cos\theta\hat{\mathbf{r}}-r^2\hat{\mathbf{z}}}{r^3\left(r^2+d^2-2dr\cos\theta\right)^{3/2}}$$

$$\mathbf{L}=\int\mathbf{l}d\tau=-\frac{\mu_0 q_e q_m d}{(4\pi)^2}\int\frac{r^2\cos\theta(\cos\theta\hat{\mathbf{z}})-r^2\hat{\mathbf{z}}}{r^3\left(r^2+d^2-2dr\cos\theta\right)^{3/2}}r^2\sin\theta drd\theta d\phi$$

$$=-\frac{\mu_0 q_e q_m d}{(4\pi)^2}\hat{\mathbf{z}}\int\frac{r^2\left(\cos^2\theta-1\right)}{r^3\left(r^2+d^2-2dr\cos\theta\right)^{3/2}}r^2\sin\theta drd\theta d\phi$$

$$=-\frac{\mu_0 q_e q_m d}{(4\pi)^2}\hat{\mathbf{z}}\int_{-1}^{1}\int_0^\infty\frac{r\left(u^2-1\right)}{\left(r^2+d^2-2dru\right)^{3/2}}dr(-du)(2\pi)=\cdots=\frac{\mu_0 q_e q_m}{4\pi}\hat{\mathbf{z}}$$

Problem 8.23

(a) Carry through the argument in Sect. 8.1.2, starting with Eq. 8.6, but using $\mathbf{J}_f$ in place of $\mathbf{J}$. Show that the Poynting vector becomes

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}, \tag{8.46}$$

and the rate of change of the energy density in the fields is

$$\frac{\partial u}{\partial t} = \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}.$$

For *linear* media, show that[24]

$$u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{H}). \tag{8.47}$$

Problem 8.23

$$\frac{dW}{dt} = \int_V (\mathbf{E}\cdot\mathbf{J})\,d\tau$$

$$\Rightarrow \frac{dW}{dt} = \int_V \left(\mathbf{E}\cdot\mathbf{J}_f\right)d\tau = \int_V \left[\mathbf{E}\cdot\left(\nabla\times\mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}\right)\right]d\tau$$

$$= \int_V \left[\mathbf{E}\cdot(\nabla\times\mathbf{H}) - \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}\right]d\tau$$

$$= \int_V \left[\mathbf{H}\cdot(\nabla\times\mathbf{E}) - \nabla\cdot(\mathbf{E}\times\mathbf{H}) - \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}\right]d\tau \qquad \nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$= -\int_V \left(\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}\right)d\tau - \int_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{a} \quad \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\frac{\partial u_{em}}{\partial t} = \mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t} = \frac{1}{\mu}\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t} + \mathbf{E}\cdot\frac{\varepsilon\partial \mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{1}{\mu}\mathbf{B}\cdot\mathbf{B} + \varepsilon\mathbf{E}\cdot\mathbf{E}\right) = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{H} + \mathbf{E}\cdot\mathbf{D})$$

$$\Rightarrow u_{em} = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$$

Problem 8.23

(b) In the same spirit, reproduce the argument in Sect. 8.2.2, starting with Eq. 8.15, with ρ_f and $\mathbf{J}_f$ in place of ρ and $\mathbf{J}$. Don't bother to construct the Maxwell stress tensor, but do show that the momentum density is[25]

$$\mathbf{g} = \mathbf{D} \times \mathbf{B}. \tag{8.48}$$

$$\left(\nabla \cdot \overleftrightarrow{\mathbf{T}}\right)_j = \epsilon_0 \left[(\nabla \cdot \mathbf{E})E_j + (\mathbf{E} \cdot \nabla)E_j - \frac{1}{2}\nabla_j E^2\right] + \frac{1}{\mu_0}\left[(\nabla \cdot \mathbf{B})B_j + (\mathbf{B} \cdot \nabla)B_j - \frac{1}{2}\nabla_j B^2\right].$$

$$\mathbf{f} = \nabla \cdot \overleftrightarrow{\mathbf{T}} - \epsilon_0\mu_0 \frac{\partial \mathbf{S}}{\partial t}, \tag{8.19}$$

$$\boxed{\mathbf{g} = \mu_0\epsilon_0 \mathbf{S} = \epsilon_0(\mathbf{E} \times \mathbf{B}),} \tag{8.29}$$

$$\mathbf{f} = \varepsilon_0\left[(\nabla\cdot\mathbf{E})\mathbf{E} - \mathbf{E}\times(\nabla\times\mathbf{E})\right] - \frac{1}{\mu_0}\left[\mathbf{B}\times(\nabla\times\mathbf{B})\right] - \varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\times\mathbf{B})$$

$$\Rightarrow \mathbf{f} = \left[(\nabla\cdot\mathbf{D})\mathbf{E} - \mathbf{D}\times(\nabla\times\mathbf{E})\right] - \left[\mathbf{H}\times(\nabla\times\mathbf{B})\right] - \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B})$$

$$= \left\{\left[(\nabla\cdot\mathbf{D})\mathbf{E} - \mathbf{D}\times(\nabla\times\mathbf{E})\right] + \left[(\nabla\cdot\mathbf{B})\mathbf{H} - \mathbf{H}\times(\nabla\times\mathbf{B})\right]\right\} - \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) \qquad \nabla\cdot\mathbf{B} = 0$$

$$\Rightarrow \mathbf{g} = \mathbf{D}\times\mathbf{B}$$