國立清華大學

Electromagnetism

Introduction to Electrodynamics 4th David J. Griffiths

Chap.9 Electromagnetic Waves

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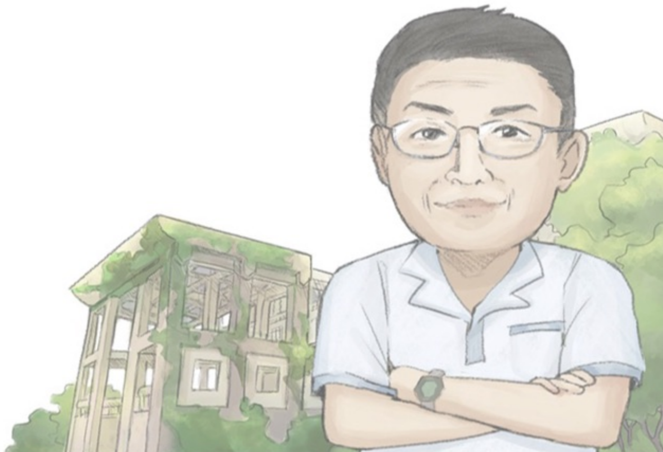
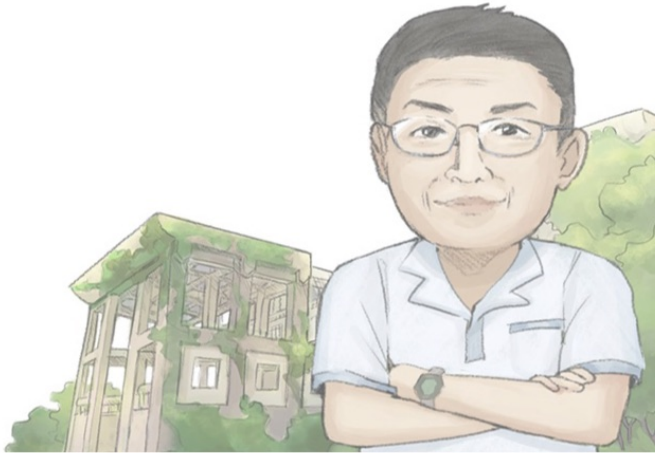


Exercise List

2, 6, 8, 10, 12, 17, 19, 20, 30, 31, 37, 40

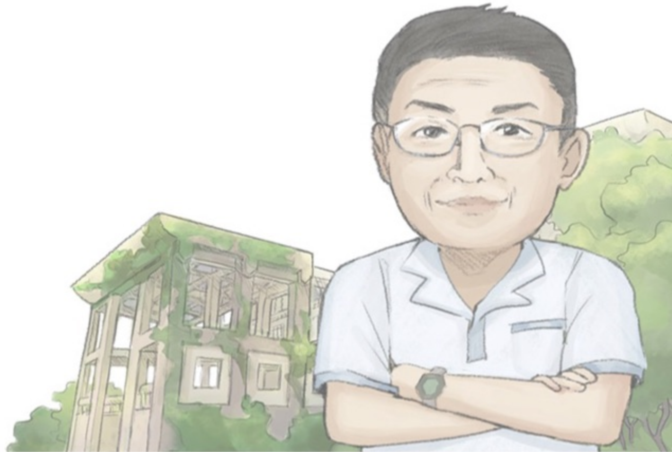


Problem 9.2 Show that the **standing wave** $f(z, t) = A\sin(kz)\cos(kvt)$ satisfies the wave equation, and express it as the sum of a wave traveling to the left and a wave traveling to the right (Eq. 9.6).

$$f(z,t) = A\sin(kz)\cos(kvt) = \frac{A}{2}\left[\sin(kz+kvt)+\sin(kz-kvt)\right] = \frac{A}{2}\sin\left[k(z+vt)\right]+\frac{A}{2}\sin\left[k(z-vt)\right]$$

$$\left\{\begin{aligned} \frac{\partial^2 f}{\partial z^2} &= -Ak^2\sin(kz)\cos(kvt) \\ \frac{\partial^2 f}{\partial t^2} &= -Ak^2v^2\sin(kz)\cos(kvt) \end{aligned}\right\} \Rightarrow \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

積化和差
$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$
$\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}$
$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{2}$
$\sin\alpha\sin\beta = -\frac{\cos(\alpha+\beta)-\cos(\alpha-\beta)}{2}$

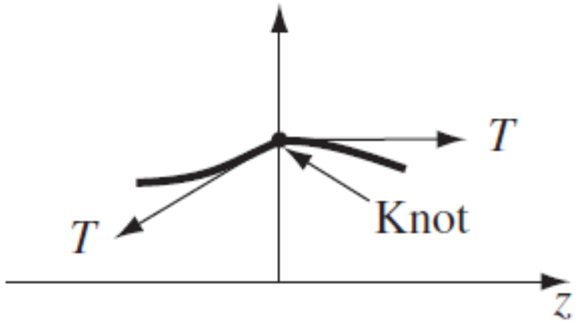


Problem 9.6

(a) Formulate an appropriate boundary condition, to replace Eq. 9.27, for the case of two strings under tension T joined by a knot of mass m.

Con. $\mathbf{T}_+ \sin_+ + \mathbf{T}_- \sin\theta_- = 0 \Rightarrow \left.\frac{\partial f}{\partial z}\right|_{0^+} - \left.\frac{\partial f}{\partial z}\right|_{0^-} = 0$

Discon. $\mathbf{T}_+ \sin_+ + \mathbf{T}_- \sin\theta_- = m\mathbf{a} \Rightarrow \left.\frac{\partial f}{\partial z}\right|_{0^+} - \left.\frac{\partial f}{\partial z}\right|_{0^-} = \frac{m}{T}\left.\frac{\partial^2 f}{\partial t^2}\right|_{z=0}$

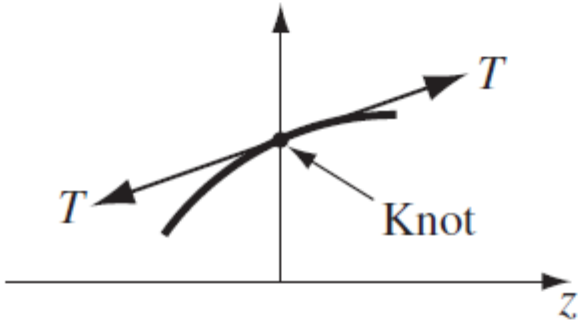


(a) Discontinuous slope; force on knot

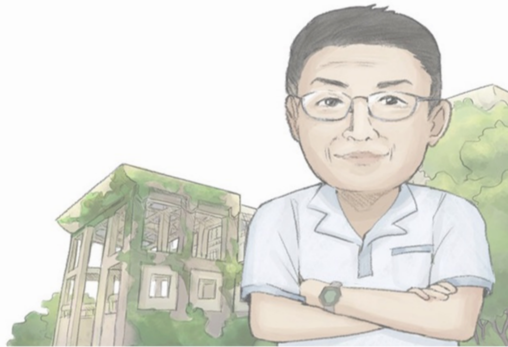
(b) Find the amplitude and phase of the reflected and transmitted waves for the case where the knot has a mass m and the second string is massless.

$$\left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+} - \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \frac{m}{T}\left.\frac{\partial^2 f}{\partial t^2}\right|_{z=0} \Rightarrow ik_2\tilde{A}_T - ik_1\left(\tilde{A}_I - \tilde{A}_R\right) = \frac{m}{T}\left(-\omega^2\tilde{A}_T\right)$$

$$\left\{\begin{array}{l} \left.\tilde{f}\right|_{0^+} = \left.\tilde{f}\right|_{0^-} \\ \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^+} - \left.\frac{\partial \tilde{f}}{\partial z}\right|_{0^-} = \frac{m}{T}\left.\frac{\partial^2 f}{\partial t^2}\right|_{z=0} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \tilde{A}_T = \tilde{A}_I + \tilde{A}_R \\ \left(k_2 - \frac{im\omega^2}{T}\right)\tilde{A}_T = k_1\left(\tilde{A}_I - \tilde{A}_R\right) \end{array}\right\}$$



(b) Continuous slope; no force on knot



Problem 9.6

$$\tilde{A}_T = \tilde{A}_I + \tilde{A}_R, \qquad \left(k_2 - \frac{im\omega^2}{T}\right)\tilde{A}_T = k_1\left(\tilde{A}_I - \tilde{A}_R\right) \;\Rightarrow \tilde{A}_R = \frac{k_1 - \left(k_2 - \frac{im\omega^2}{T}\right)}{k_1 + \left(k_2 - \frac{im\omega^2}{T}\right)}\tilde{A}_I, \quad \tilde{A}_T = \frac{2k_1}{k_1 + \left(k_2 - \frac{im\omega^2}{T}\right)}\tilde{A}_I$$

$$\frac{k_2}{k_1} = \sqrt{\frac{\mu_2}{\mu_1}} \cong 0 \text{ since } \mu_2 \text{ is massless}, \quad \beta = \frac{m\omega^2}{k_1 T} \Rightarrow \tilde{A}_R = \frac{1+i\beta}{1-i\beta}\tilde{A}_I, \qquad \tilde{A}_T = \frac{2}{1-i\beta}\tilde{A}_I,$$

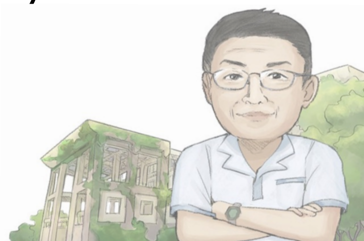
$$\left|\tilde{A}_R\right| = \sqrt{\left(\frac{1+i\beta}{1-i\beta}\tilde{A}_I\right)\left(\frac{1-i\beta}{1+i\beta}\tilde{A}_I^*\right)} = \left|\tilde{A}_I\right|, \qquad \left|\tilde{A}_T\right| = \sqrt{\left(\frac{2}{1-i\beta}\tilde{A}_I\right)\left(\frac{2}{1+i\beta}\tilde{A}_I^*\right)} = \frac{2}{\sqrt{1+\beta^2}}\left|\tilde{A}_I\right|$$

$$\Rightarrow \tilde{A}_R = A_R e^{i\delta_R} = \frac{1+i\beta}{1-i\beta}A_I e^{i\delta_I} = e^{i\phi_R} A_I e^{i\delta_I} \Rightarrow e^{i\phi_R} = \frac{1+i\beta}{1-i\beta} = \frac{\left(1-\beta^2\right)+2i\beta}{1+\beta^2} = \cos\phi_R + i\sin\phi_R$$

$$\Rightarrow \phi_R = \tan^{-1}\frac{2\beta}{1-\beta^2} \Rightarrow \delta_R = \delta_I + \tan^{-1}\frac{2\beta}{1+\beta^2}$$

$$\Rightarrow \tilde{A}_T = A_T e^{i\delta_T} = \frac{2}{1-i\beta}A_I e^{i\delta_I} = \frac{2}{\sqrt{1+\beta^2}}e^{i\phi_T} A_I e^{i\delta_I} \Rightarrow e^{i\phi_T} = \frac{\sqrt{1+\beta^2}}{2}\frac{2}{1-i\beta} = \frac{\sqrt{1+\beta^2}}{2}\frac{2+2i\beta}{1+\beta^2} = \frac{1+i\beta}{\sqrt{1+\beta^2}}$$

$$\Rightarrow \phi_T = \tan^{-1}\beta \Rightarrow \delta_T = \delta_I + \tan^{-1}\beta$$



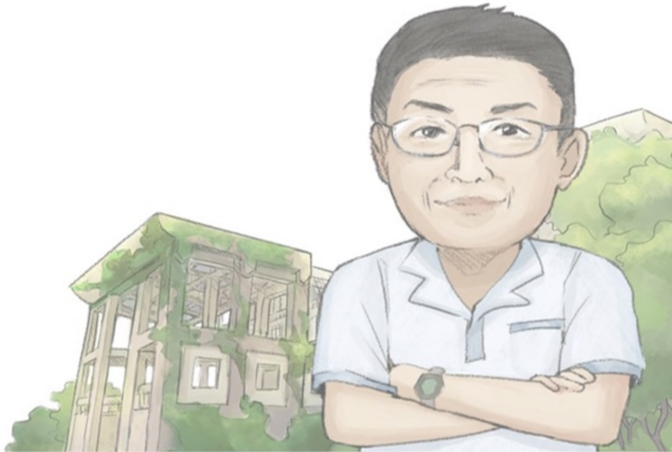
Problem 9.8 Equation 9.36 describes the most general **linearly** polarized wave on a string. Linear (or "plane") polarization (so called because the displacement is parallel to a fixed vector $\hat{\mathbf{n}}$) results from the combination of horizontally and vertically polarized waves of the *same phase* (Eq. 9.39). If the two components are of equal amplitude, but *out of phase* by 90° (say, $\delta_v = 0,\ \delta_h = 90°$), the result is a *circularly* polarized wave. In that case:

(a) At a fixed point z, show that the string moves in a circle about the z axis. Does it go *clockwise* or *counterclockwise*, as you look down the axis toward the origin? How would you construct a wave circling the *other* way? (In optics, the clockwise case is called **right circular polarization**, and the counterclockwise, **left circular polarization**.)[3]

Taking real part :

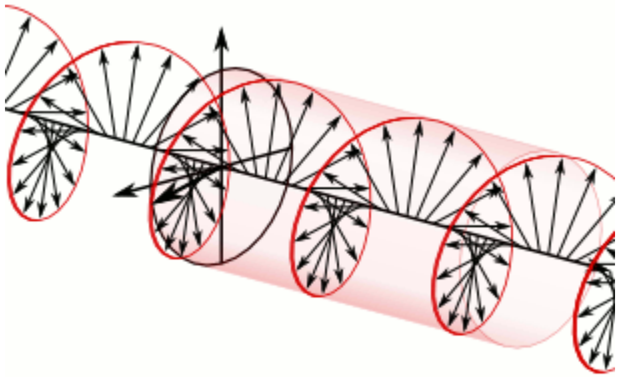
$$\mathbf{f}_{\perp} = A\cos(kz-\omega t)\,\hat{\mathbf{x}},\quad \mathbf{f}_{\|} = A\cos\left(kz-\omega t+90°\right)\hat{\mathbf{y}} = A\sin(kz-\omega t)\,\hat{\mathbf{y}}$$

$$\Rightarrow \mathbf{f} = A\cos(kz-\omega t)\,\hat{\mathbf{x}} + A\sin(kz-\omega t)\,\hat{\mathbf{y}} \quad \text{(clockwise)}$$



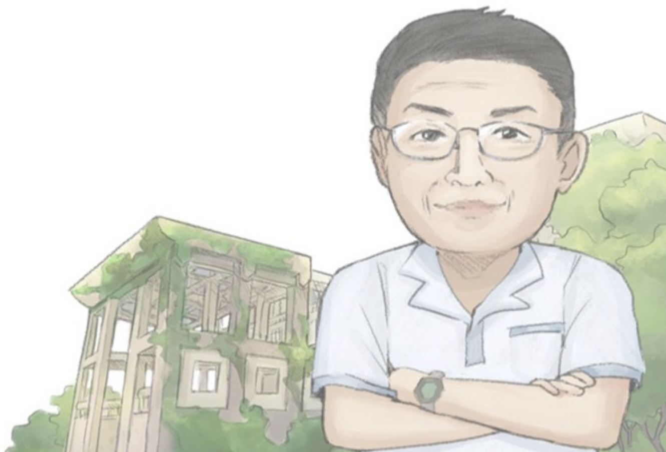
Problem 9.8

(b) Sketch the string at time $t = 0$.



(c) How would you shake the string in order to produce a circularly polarized wave?

Shake the string around in a circle.



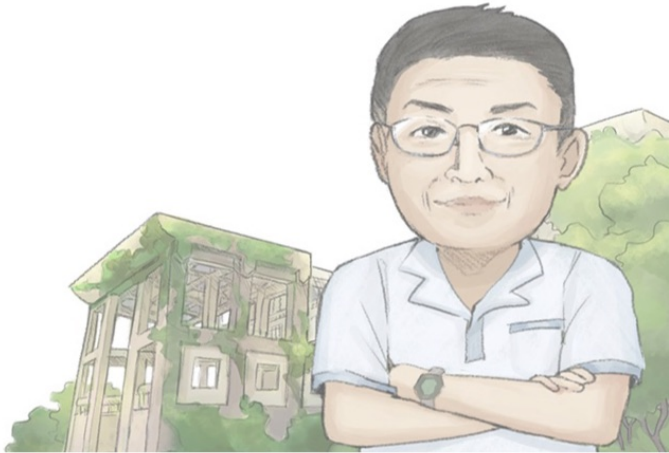
Problem 9.10 The intensity of sunlight hitting the earth is about 1300 W/m^2. If sunlight strikes a perfect absorber, what pressure does it exert? How about a perfect reflector? What fraction of atmospheric pressure does this amount to?

Perfect absorber $\quad P_{abs} = \frac{I}{C} = \frac{1300\,\mathrm{W/m^2}}{3.0\times10^8\,\mathrm{m/s}} = 4.3\times10^{-6}\,\mathrm{N/m^2}$

Perfect reflector $\quad P_{ref} = 2\times P_a = 8.6\times10^{-6}\,\mathrm{N/m^2}$

Atmospheric pressure $\quad P_{atm} = 1.03\times10^5\,\mathrm{N/m^2}$

$$\frac{P_{ref}}{P_{atm}} = \frac{8.6\times10^{-6}}{1.03\times10^5} = 8.3\times10^{-11}$$



Problem 9.12 In the complex notation there is a clever device for finding the time average of a product. Suppose $f(\mathbf{r}, t) = A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_a)$ and $g(\mathbf{r}, t) = B\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_b)$. Show that $\langle fg\rangle = (1/2)\mathrm{Re}(\tilde{f}\tilde{g}^*)$, where the star denotes complex conjugation. [Note that this only works if the two waves have the same $\mathbf{k}$ and ω, but they need not have the same amplitude or phase.] For example,

$$\langle u\rangle = \frac{1}{4}\mathrm{Re}\left(\epsilon_0\tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right) \quad \text{and} \quad \langle\mathbf{S}\rangle = \frac{1}{2\mu_0}\mathrm{Re}\left(\tilde{\mathbf{E}}\times\tilde{\mathbf{B}}^*\right).$$

$$\langle fg\rangle = \frac{1}{T}\int_0^T A\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_a)B\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta_b)\,dt$$

$$= \frac{AB}{2T}\int_0^T \left[\cos(2\mathbf{k}\cdot\mathbf{r} - 2\omega t + \delta_a + \delta_b) + \cos(\delta_a - \delta_b)\right]dt$$

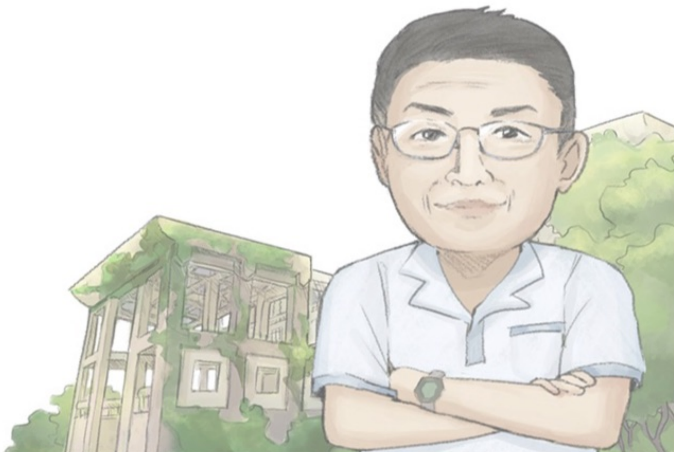
$$= \frac{AB}{2T}\cos(\delta_a - \delta_b)T = \frac{1}{2}ab\cos(\delta_a - \delta_b)$$

$\tilde{f} = \tilde{A}e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}, \tilde{g} = \tilde{B}e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$, where $\tilde{A} = Ae^{i\delta_a}, \tilde{B} = Be^{i\delta_b}$

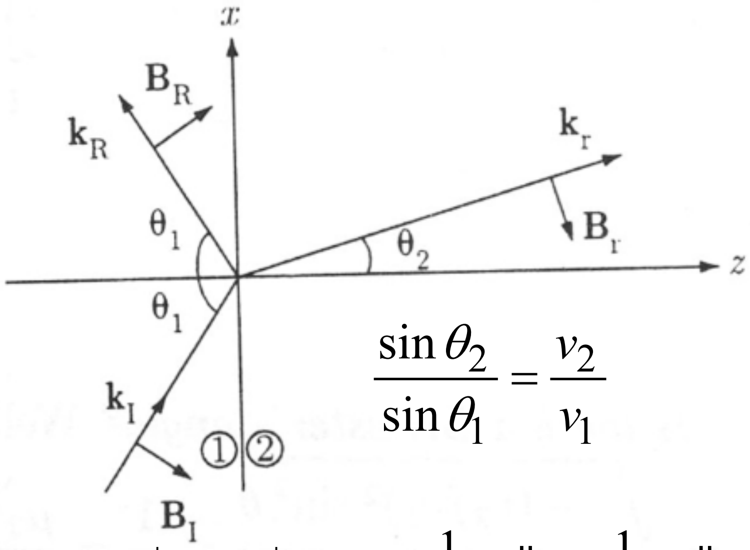
$$\Rightarrow \frac{1}{2}\tilde{f}\tilde{g}^* = \frac{1}{2}\tilde{A}e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\tilde{B}^*e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega t)} = \frac{1}{2}ABe^{i(\delta_a - \delta_b)}$$

$$\mathrm{Re}\left\{\frac{1}{2}\tilde{f}\tilde{g}^*\right\} = \frac{1}{2}AB\cos(\delta_a - \delta_b) = \langle fg\rangle$$

積化和差
$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}$
$\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2}$
$\cos\alpha\cos\beta = \frac{\cos(\alpha+\beta) + \cos(\alpha-\beta)}{2}$
$\sin\alpha\sin\beta = -\frac{\cos(\alpha+\beta) - \cos(\alpha-\beta)}{2}$



Problem 9.17 Analyze the case of polarization *perpendicular* to the plane of incidence (i.e. electric fields in the y direction, in Fig. 9.15). Impose the boundary conditions (Eq. 9.101), and obtain the Fresnel equations for $\tilde{E}_{0_R}$ and $\tilde{E}_{0_T}$. Sketch $(\tilde{E}_{0_R}/\tilde{E}_{0_I})$ and $(\tilde{E}_{0_T}/\tilde{E}_{0_I})$ as functions of θ_I, for the case $\beta = n_2/n_1 = 1.5$. (Note that for this β the reflected wave is *always* 180° out of phase.) Show that there is no Brewster's angle for *any* n_1 and n_2: $\tilde{E}_{0_R}$ is *never* zero (unless, of course, $n_1 = n_2$ and $\mu_1 = \mu_2$, in which case the two media are optically indistinguishable). Confirm that your Fresnel equations reduce to the proper forms at normal incidence. Compute the reflection and transmission coefficients, and check that they add up to 1.



$$\frac{\sin\theta_2}{\sin\theta_1} = \frac{v_2}{v_1}$$

$$\begin{cases} \tilde{\mathbf{E}}^I = \tilde{E}_0^I e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}\hat{\mathbf{y}} \\ \tilde{\mathbf{B}}^I = \frac{1}{v_1}\tilde{E}_0^I e^{i(\mathbf{k}_I\cdot\mathbf{r}-\omega t)}\left(-\cos\theta_1\hat{\mathbf{x}}+\sin\theta_1\hat{\mathbf{z}}\right) \\ \tilde{\mathbf{E}}^R = \tilde{E}_0^R e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}\hat{\mathbf{y}} \\ \tilde{\mathbf{B}}^R = \frac{1}{v_1}\tilde{E}_0^R e^{i(\mathbf{k}_R\cdot\mathbf{r}-\omega t)}\left(\cos\theta_1\hat{\mathbf{x}}+\sin\theta_1\hat{\mathbf{z}}\right) \\ \tilde{\mathbf{E}}^T = \tilde{E}_0^T e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\hat{\mathbf{y}} \\ \tilde{\mathbf{B}}^T = \frac{1}{v_2}\tilde{E}_0^T e^{i(\mathbf{k}_T\cdot\mathbf{r}-\omega t)}\left(-\cos\theta_2\hat{\mathbf{x}}+\sin\theta_2\hat{\mathbf{z}}\right) \end{cases}$$

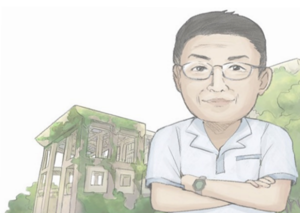
$B.C.$ $\text{(i)}\ \varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$ $\text{(ii)}\ \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}$ $\text{(iii)}\ B_1^{\perp} = B_2^{\perp}$ $\text{(iv)}\ \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} = \frac{1}{\mu_2}\mathbf{B}_2^{\parallel}$

at $z = 0 \Rightarrow \mathbf{k}_I\cdot\mathbf{r}-\omega t = \mathbf{k}_R\cdot\mathbf{r}-\omega t = \mathbf{k}_T\cdot\mathbf{r}-\omega t$

$\Rightarrow$ dropping the exponential factors in B.C

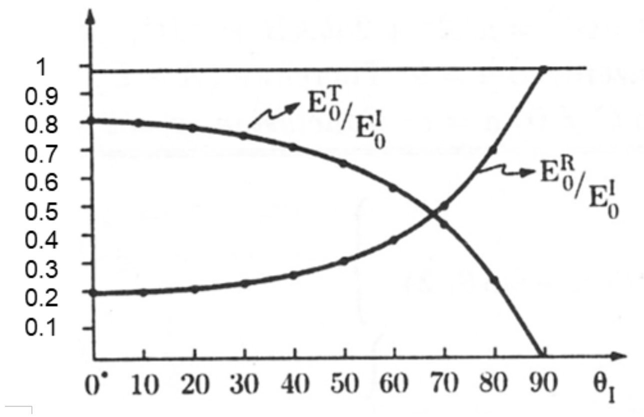
$$\begin{cases} \text{(i)}: 0 = 0\left(\mathbf{E}\text{ in the } y \text{ direction}\right) \\ \text{(ii)}: \tilde{E}_0^I + \tilde{E}_0^R = \tilde{E}_0^T \\ \text{(iii)}: \frac{1}{v_1}\tilde{E}_0^I\sin\theta + \frac{1}{v_1}\tilde{E}_0^R\sin\theta = \frac{1}{v_2}\tilde{E}_0^T\sin\theta_2 \\ \text{(iv)}\ \frac{1}{\mu_1}\left[\frac{1}{v_1}\tilde{E}_0^I\left(-\cos\theta_1\right)+\frac{1}{v_1}\tilde{E}_0^R\cos\theta_1\right] = \frac{1}{\mu_2}\frac{1}{v_2}\tilde{E}_0^T\left(-\cos\theta_2\right) \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{E}_0^I + \tilde{E}_0^R = \tilde{E}_0^T \\ \tilde{E}_0^I + \tilde{E}_0^R = \left(\frac{v_1\sin\theta_2}{v_2\sin\theta_1}\right)\tilde{E}_0^T = \tilde{E}_0^T \\ \tilde{E}_0^I - \tilde{E}_0^R = \left(\frac{\mu_1 v_1\cos\theta_2}{\mu_2 v_2\cos\theta_1}\right)\tilde{E}_0^T \end{cases}$$



Problem 9.17

$$\begin{cases}\tilde{E}_0^I+\tilde{E}_0^R=\tilde{E}_0^T\\ \tilde{E}_0^I-\tilde{E}_0^R=\underbrace{\frac{\mu_1 v_1}{\mu_2 v_2}}_{\beta}\underbrace{\frac{\cos\theta_2}{\cos\theta_1}}_{\alpha}\tilde{E}_0^T=\alpha\beta\tilde{E}_0^T\end{cases}\Rightarrow\begin{cases}\tilde{E}_0^T=\frac{2}{1+\alpha\beta}\tilde{E}_0^I\\ \tilde{E}_0^R=\frac{1-\alpha\beta}{1+\alpha\beta}\tilde{E}_0^I\end{cases}$$

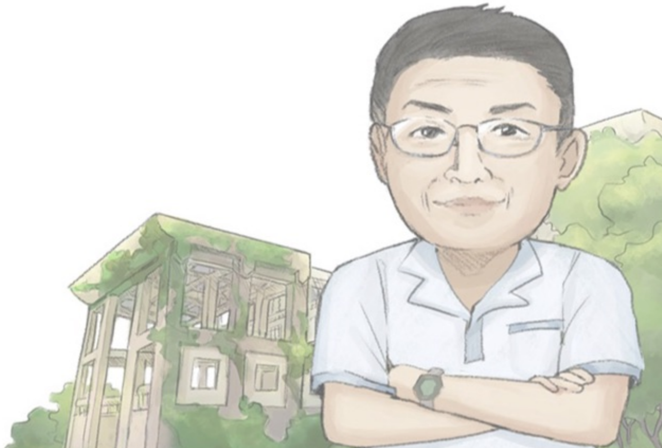


$$\alpha\beta=\frac{\cos\theta_2}{\cos\theta_1}\beta=\frac{\beta\sqrt{1-\sin^2\theta_2}}{\cos\theta_1}=\frac{\beta\sqrt{1-\left(\frac{v_2}{v_1}\sin\theta_1\right)^2}}{\cos\theta_1}=\frac{\beta\sqrt{1-\left(\frac{n_1}{n_2}\sin\theta_1\right)^2}}{\cos\theta_1}=\frac{\sqrt{2.25-\sin^2\theta_I}}{\cos\theta_I}$$

$$\text{If Brewster's angle}: E_0^R=0\Rightarrow\alpha\beta=1\Rightarrow\alpha=\frac{\sqrt{1-\left(\frac{v_2}{v_1}\sin\theta_1\right)^2}}{\cos\theta_1}=\frac{1}{\beta}=\frac{\mu_2 v_2}{\mu_1 v_1}$$

$$\Rightarrow 1=\frac{v_2^2}{v_1^2}\left(\sin^2\theta_1+\frac{\mu_2^2}{\mu_1^2}\cos^2\theta_1\right)=\frac{n_1^2}{n_2^2}\left(\sin^2\theta_1+\frac{\mu_2^2}{\mu_1^2}\cos^2\theta_1\right)=\frac{n_1^2}{n_2^2}\left[1+\left(\frac{\mu_2^2}{\mu_1^2}-1\right)\cos^2\theta_1\right]$$

$$\Rightarrow\cos\theta_1=\frac{(n_2/n_1)^2-1}{(\mu_2/\mu_1)^2-1}=\frac{(\mu_2\varepsilon_2/\mu_1\varepsilon_1)^2-1}{(\mu_2/\mu_1)^2-1}=\frac{\varepsilon_2/\varepsilon_1-\mu_1/\mu_2}{\mu_2/\mu_1-\mu_1/\mu_2}$$



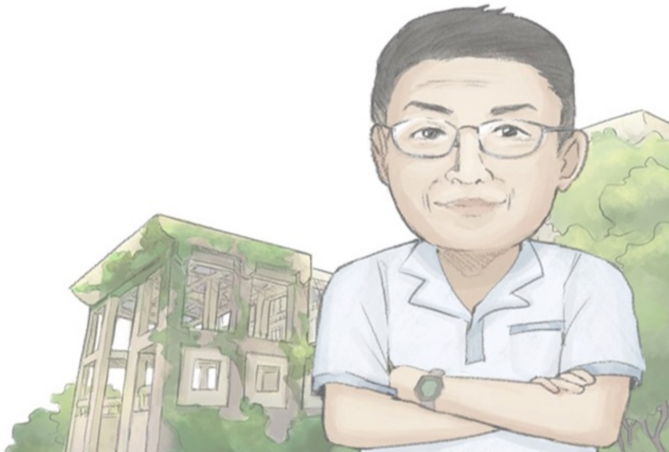
Problem 9.17

At normal incidence : $\alpha = \dfrac{\cos\theta_2}{\cos\theta_1} = 1$

$$\Rightarrow \begin{cases} \tilde{E}_0^T = \dfrac{2}{1+\beta}\tilde{E}_0^I \\ \tilde{E}_0^R = \dfrac{1-\beta}{1+\beta}\tilde{E}_0^I \end{cases} \text{consistent with Eq.9.82}$$

$$R = \left(\frac{E_0^R}{E_0^I}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2, \quad T = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\alpha\left(\frac{E_0^T}{E_0^I}\right)^2 = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2$$

$$R + T = 1$$



Problem 9.19

(a) Suppose you imbedded some free charge in a piece of glass. About how long would it take for the charge to flow to the surface?

(b) Silver is an excellent conductor, but it's expensive. Suppose you were designing a microwave experiment to operate at a frequency of 10^{10} Hz. How thick would you make the silver coatings?

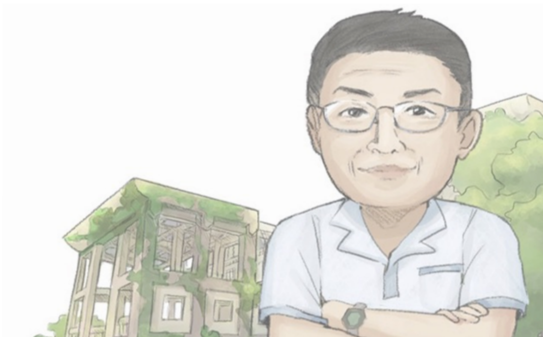
(c) Find the wavelength and propagation speed in copper for radio waves at 1 MHz. Compare the corresponding values in air (or vacuum).

$$\text{(a) Eq.9.120} \Rightarrow \tau = \frac{\varepsilon}{\sigma} = \frac{\varepsilon_0 \varepsilon_r}{\sigma} \cong \frac{\varepsilon_0 n^2}{\sigma} = \frac{\varepsilon_0 \times 2.25}{\sigma} = \frac{\left(8.85\times10^{-12}\right)\times 2.25}{10^{-12}} = 20\,\text{sec}$$

$$\text{(b) } d = \delta = \sqrt{\frac{2}{\omega\sigma\mu}} = \sqrt{\frac{2}{(2\pi f)(1/\rho)\mu_0}} = \sqrt{\frac{2}{\left(2\pi\times10^{10}\right)\left(1/1.59\times10^{-8}\right)\left(4\pi\times10^{-7}\right)}} = 6.4\times10^{-7}\,\text{m} = 0.00064\text{mm}$$

$$\text{(c) } \lambda = \frac{2\pi}{k} \cong 2\pi d = 2\pi\sqrt{\frac{2}{(2\pi f)(1/\rho)\mu_0}} = 2\pi\sqrt{\frac{2}{\left(2\pi\times10^{6}\right)\left(1/1.68\times10^{-8}\right)\left(4\pi\times10^{-7}\right)}} = 4\times10^{-4}\,\text{m} = 0.4\text{mm}$$

$$v = \lambda f = \left(4\times10^{-4}\right)\times10^{6} = 400\,\text{m/s} \qquad \text{In vacuum}: \lambda = \frac{c}{f} = 300\text{m}, \quad v = c$$



Problem 9.20

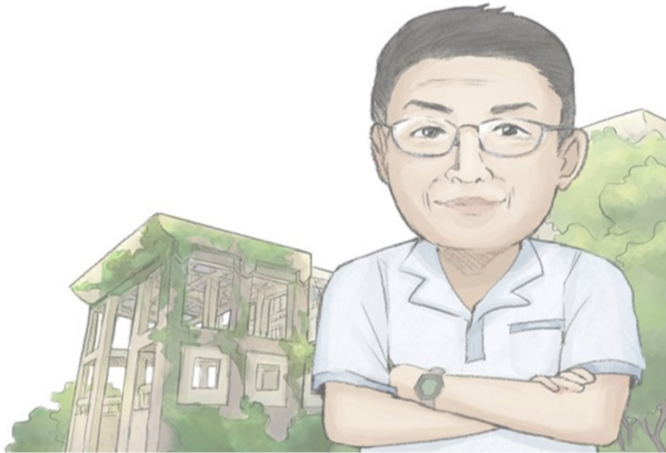
(a) Show that the skin depth in a poor conductor ($\sigma \ll \omega\epsilon$) is $(2/\sigma)\sqrt{\epsilon/\mu}$ (independent of frequency). Find the skin depth (in meters) for (pure) water. (Use the static values of ϵ, μ, and σ; your answers will be valid, then, only at relatively low frequencies.)

Poor conductor

$$\kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{\frac{1}{2}} \cong \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[1+\frac{1}{2}\left(\frac{\sigma}{\varepsilon\omega}\right)^2-1\right]^{\frac{1}{2}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \Rightarrow d=\delta=\frac{1}{\kappa}=\frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$$

Water

$$d=\frac{2}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}=\frac{2}{1/2.5\times10^5}\sqrt{\frac{80.1\times\varepsilon_0}{\mu_0}}=1.19\times10^4\,\text{m}$$



Problem 9.20

(b) Show that the skin depth in a good conductor ($\sigma \gg \omega\epsilon$) is $\lambda/2\pi$ (where λ is the wavelength *in the conductor*). Find the skin depth (in nanometers) for a typical metal ($\sigma \approx 10^7 (\Omega\,\text{m})^{-1}$) in the visible range ($\omega \approx 10^{15}/\text{s}$), assuming $\epsilon \approx \epsilon_0$ and $\mu \approx \mu_0$. Why are metals opaque?

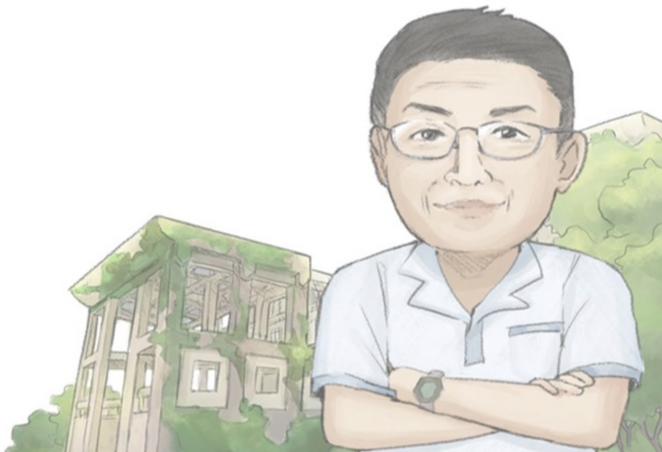
Good conductor

$$k \cong \kappa \Rightarrow \lambda = \frac{2\pi}{k} \cong \frac{2\pi}{\kappa} = 2\pi d \Rightarrow d = \frac{\lambda}{2\pi}$$

Metal

$$\kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{\frac{1}{2}} \cong \omega\sqrt{\frac{\varepsilon\mu}{2}}\sqrt{\frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\frac{\left(10^{15}\right)\left(4\pi\times10^{-7}\right)(10^7}{2}} = 8\times10^7$$

$$\Rightarrow d = \frac{1}{\kappa} = 13\text{nm}$$



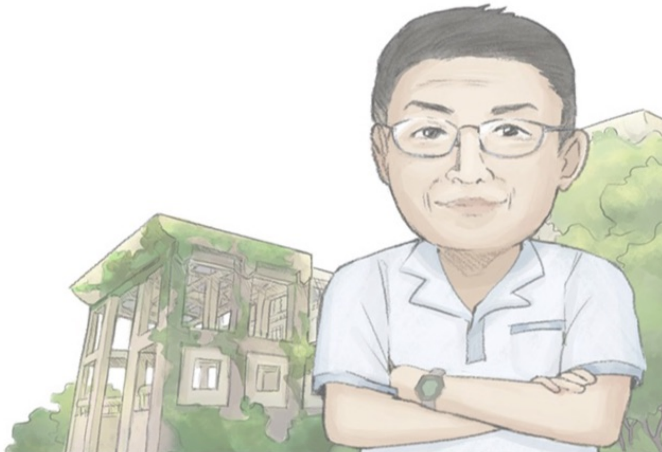
Problem 9.20

(c) Show that in a good conductor the magnetic field lags the electric field by 45°, and find the ratio of their amplitudes. For a numerical example, use the "typical metal" in part (b).

$$k \cong \kappa \Rightarrow \text{Eq.9.134}: \phi = \tan^{-1}\left(\frac{k}{\kappa}\right) \cong \tan^{-1}(1) = 45^\circ$$

$$\text{Eq.9.137}: \frac{B_0}{E_0} = \frac{K}{\omega} = \sqrt{\omega\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}} \cong \sqrt{\omega\mu\frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\mu\sigma}{\omega}} = \sqrt{\frac{(4\pi\times10^{-7})(10^7)}{10^{15}}} = 10^{-7}\ \text{sec/m}$$

$$\text{In vacuum } \frac{B_0}{E_0} = \frac{1}{c} = 3.33\times10^{-9}\ \text{sec/m}$$

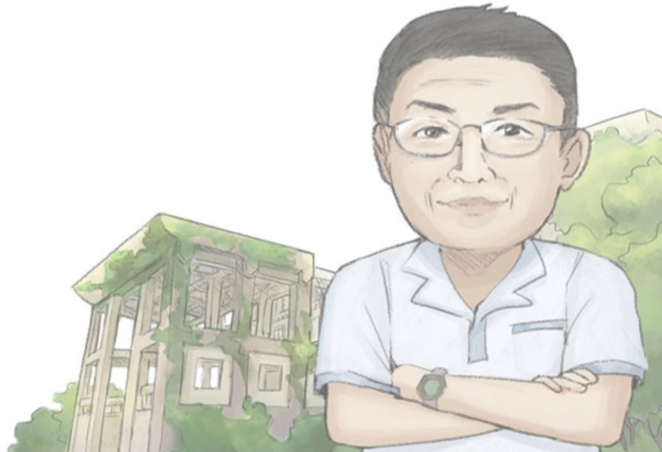


Problem 9.30 Confirm that the energy in the TE_{mn} mode travels at the group velocity. [*Hint:* Find the time averaged Poynting vector $\langle \mathbf{S} \rangle$ and the energy density $\langle u \rangle$ (use Prob. 9.12 if you wish). Integrate over the cross section of the wave guide to get the energy per unit time and per unit length carried by the wave, and take their ratio.]

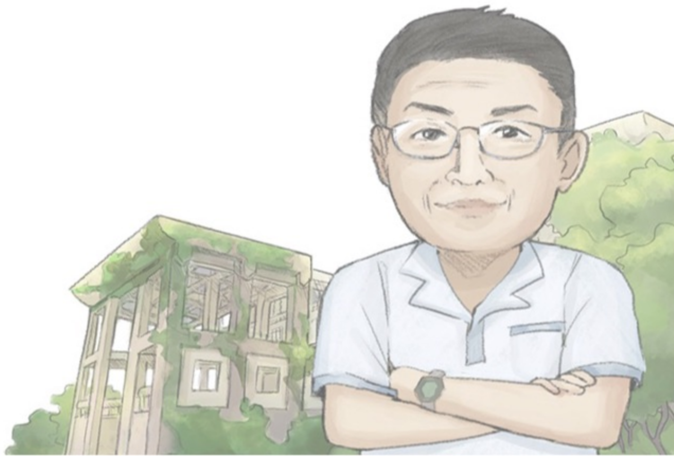
$$\left.\begin{aligned}
&\text{(i)}\ \ E_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right),\\
&\text{(ii)}\ \ E_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial E_z}{\partial y} - \omega\frac{\partial B_z}{\partial x}\right),\\
&\text{(iii)}\ \ B_x = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right),\\
&\text{(iv)}\ \ B_y = \frac{i}{(\omega/c)^2 - k^2}\left(k\frac{\partial B_z}{\partial y} + \frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right).
\end{aligned}\right\} \tag{9.180}$$

$$\text{TE}_{mn} \qquad E_z = 0 \quad B_z = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\Rightarrow \begin{cases}
E_x = \dfrac{i\omega}{(\omega/c)^2 - k^2}\left(\dfrac{-n\pi}{b}\right)B_0 \cos\left(\dfrac{m\pi x}{a}\right)\sin\left(\dfrac{n\pi y}{b}\right) & B_x = \dfrac{ik}{(\omega/c)^2 - k^2}\left(\dfrac{-m\pi}{a}\right)B_0 \sin\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi y}{b}\right)\\[2ex]
E_y = \dfrac{-i\omega}{(\omega/c)^2 - k^2}\left(\dfrac{-m\pi}{a}\right)B_0 \sin\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi y}{b}\right) & B_y = \dfrac{ik}{(\omega/c)^2 - k^2}\left(\dfrac{-n\pi}{b}\right)B_0 \cos\left(\dfrac{m\pi x}{a}\right)\sin\left(\dfrac{n\pi y}{b}\right)\\[2ex]
E_z = 0 & B_z = B_0 \cos\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi y}{b}\right)
\end{cases}$$



Problem 9.30 Confirm that the energy in the TE_{mn} mode travels at the group velocity. [*Hint:* Find the time averaged Poynting vector $\langle \mathbf{S} \rangle$ and the energy density $\langle u \rangle$ (use Prob. 9.12 if you wish). Integrate over the cross section of the wave guide to get the energy per unit time and per unit length carried by the wave, and take their ratio.]



Same $k, \omega \Rightarrow \langle \mathbf{S} \rangle = \mathrm{Re}\left\{ \frac{1}{2\mu_0} \tilde{\mathbf{E}} \times \tilde{\mathbf{B}}^* \right\}$ easily know that only $\hat{\mathbf{z}}$ component survived

$$\langle \mathbf{S} \rangle = \frac{1}{2\mu_0} \frac{i\omega}{(\omega/c)^2 - k^2} \frac{-ik}{(\omega/c)^2 - k^2} B_0^2 \left[\left(\frac{-n\pi}{b} \right)^2 \cos^2\left(\frac{m\pi x}{a} \right) \sin^2\left(\frac{n\pi y}{b} \right) + \left(\frac{-m\pi}{a} \right)^2 \sin^2\left(\frac{m\pi x}{a} \right) \cos^2\left(\frac{n\pi y}{b} \right) \right] \hat{\mathbf{z}}$$

$$= \frac{1}{2\mu_0} \frac{\omega k \pi^2 B_0^2}{\left[(\omega/c)^2 - k^2 \right]^2} \left[\left(\frac{n}{b} \right)^2 \cos^2\left(\frac{m\pi x}{a} \right) \sin^2\left(\frac{n\pi y}{b} \right) + \left(\frac{m}{a} \right)^2 \sin^2\left(\frac{m\pi x}{a} \right) \cos^2\left(\frac{n\pi y}{b} \right) \right] \hat{\mathbf{z}}$$

$$\int \langle \mathbf{S} \rangle \cdot d\mathbf{a} = \int_0^b \int_0^a \frac{\pi^2}{2\mu_0} \frac{\omega k B_0^2}{\left[(\omega/c)^2 - k^2 \right]^2} \left[\left(\frac{n}{b} \right)^2 \cos^2\left(\frac{m\pi x}{a} \right) \sin^2\left(\frac{n\pi y}{b} \right) + \left(\frac{m}{a} \right)^2 \sin^2\left(\frac{m\pi x}{a} \right) \cos^2\left(\frac{n\pi y}{b} \right) \right] dx dy$$

$$= \frac{ab}{8\mu_0} \frac{\omega k \pi^2 B_0^2}{\left[(\omega/c)^2 - k^2 \right]^2} \left[\left(\frac{n}{b} \right)^2 + \left(\frac{m}{a} \right)^2 \right]$$

$$\int_0^a \cos^2\left(\frac{m\pi x}{a} \right) dx = \int_0^a \sin^2\left(\frac{m\pi x}{a} \right) dx = \frac{a}{2}$$

$$\int_0^b \cos^2\left(\frac{n\pi y}{b} \right) dy = \int_0^b \sin^2\left(\frac{n\pi y}{b} \right) dy = \frac{b}{2}$$

Problem 9.30 Confirm that the energy in the TE_{mn} mode travels at the group velocity. [*Hint:* Find the time averaged Poynting vector $\langle \mathbf{S} \rangle$ and the energy density $\langle u \rangle$ (use Prob. 9.12 if you wish). Integrate over the cross section of the wave guide to get the energy per unit time and per unit length carried by the wave, and take their ratio.]

$$\text{Same } k,\omega \Rightarrow \langle u \rangle = \frac{1}{4}\left(\varepsilon_0 \tilde{\mathbf{E}}\cdot\tilde{\mathbf{E}}^* + \frac{1}{\mu_0}\tilde{\mathbf{B}}\cdot\tilde{\mathbf{B}}^*\right)$$

$$\int \langle u \rangle da = \frac{ab}{16}B_0^2 \frac{\pi^2}{\left[(\omega/c)^2 - k^2\right]^2}\left(\varepsilon_0\omega^2 + \frac{1}{\mu_0}k^2\right)\left[\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right] + \frac{ab}{16}B_0^2\frac{1}{\mu_0}$$

$$= \frac{ab}{16}B_0^2\frac{1}{\mu_0}\frac{\pi^2}{\left[(\omega/c)^2 - k^2\right]^2}\left(\frac{\omega^2}{c^2} + k^2\right)\left[\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right] + \frac{ab}{16}B_0^2\frac{1}{\mu_0}$$

$$|E_x|^2 = \left[\frac{\omega}{(\omega/c)^2 - k^2}\right]^2\left(\frac{n\pi}{b}\right)^2 B_0^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)$$

$$|E_y|^2 = \left[\frac{\omega}{(\omega/c)^2 - k^2}\right]^2\left(\frac{m\pi}{a}\right)^2 B_0^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)$$

$$|E_z|^2 = 0$$

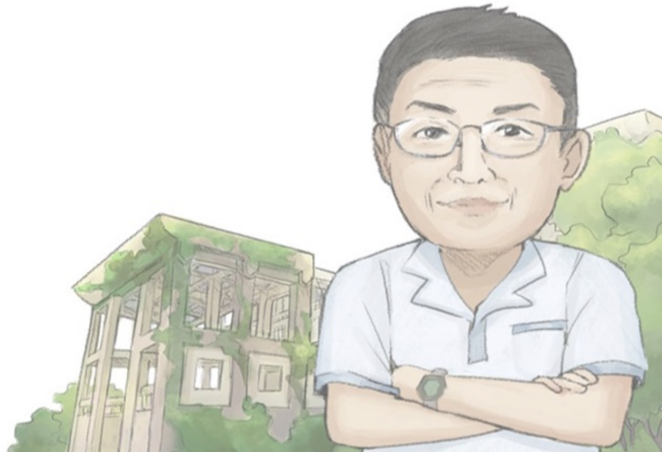
$$|B_x|^2 = \left[\frac{k}{(\omega/c)^2 - k^2}\right]^2\left(\frac{m\pi}{a}\right)^2 B_0^2\sin^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)$$

$$|B_y|^2 = \left[\frac{k}{(\omega/c)^2 - k^2}\right]^2\left(\frac{n\pi}{b}\right)^2 B_0^2\cos^2\left(\frac{m\pi x}{a}\right)\sin^2\left(\frac{n\pi y}{b}\right)$$

$$|B_z|^2 = B_0^2\cos^2\left(\frac{m\pi x}{a}\right)\cos^2\left(\frac{n\pi y}{b}\right)$$

$$\int_0^a \cos^2\left(\frac{m\pi x}{a}\right)dx = \int_0^a \sin^2\left(\frac{m\pi x}{a}\right)dx = \frac{a}{2}$$

$$\int_0^b \cos^2\left(\frac{n\pi y}{b}\right)dy = \int_0^b \sin^2\left(\frac{n\pi y}{b}\right)dy = \frac{b}{2}$$



Problem 9.30 Confirm that the energy in the TE$_{mn}$ mode travels at the group velocity. [*Hint:* Find the time averaged Poynting vector $\langle \mathbf{S} \rangle$ and the energy density $\langle u \rangle$ (use Prob. 9.12 if you wish). Integrate over the cross section of the wave guide to get the energy per unit time and per unit length carried by the wave, and take their ratio.]

$$k = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]}. \qquad (9.187)$$

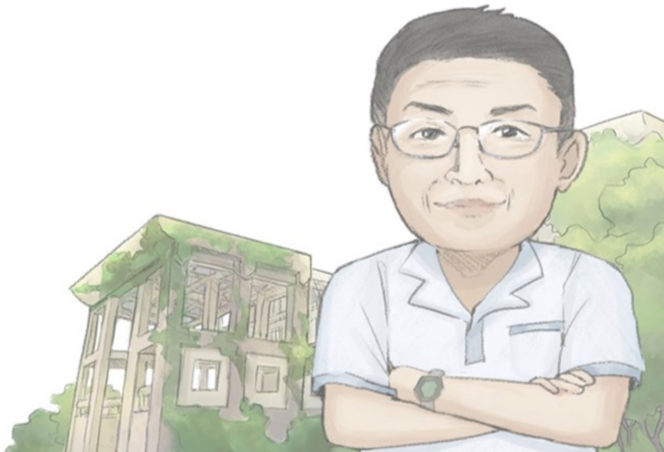
$$c\pi\sqrt{(m/a)^2 + (n/b)^2} \equiv \omega_{mn}, \qquad (9.188)$$

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}. \qquad (9.190)$$

$$\frac{\overbrace{\int \langle \mathbf{S} \rangle \cdot d\mathbf{a}}^{\text{energy per time}}}{\underbrace{\int \langle u \rangle da}_{\text{energy per length}}} = \frac{\frac{ab}{8}\frac{1}{\mu_0}\frac{\omega k \pi^2 B_0^2}{\left[(\omega/c)^2 - k^2\right]^2}\left[\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right]}{\frac{ab}{16}\frac{1}{\mu_0}B_0^2\frac{\pi^2}{\left[(\omega/c)^2 - k^2\right]^2}\left(\frac{\omega^2}{c^2} + k^2\right)\left[\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right] + \frac{ab}{16}\frac{1}{\mu_0}B_0^2}$$

$$= \frac{2\frac{\omega k \pi^2}{\left[(\omega_{mn}/c)^2\right]^2}\frac{\omega_{mn}^2}{\pi^2 c^2}}{\frac{\pi^2}{\left[(\omega_{mn}/c)^2\right]^2}\left(\frac{\omega^2}{c^2} + k^2\right)\frac{\omega_{mn}^2}{\pi^2 c^2} + 1} = \frac{2\frac{\omega k}{(\omega_{mn}/c)^2}}{\frac{1}{(\omega_{mn}/c)^2}\left(\frac{\omega^2}{c^2} + \frac{\omega^2}{c^2} - \frac{\omega_{mn}^2}{c^2}\right) + 1}$$

$$= \frac{\omega k}{(\omega/c)^2} = \frac{c^2 k}{\omega} = c\sqrt{1 - (\omega_{mn}/\omega)^2} = v_g$$



Problem 9.31 Work out the theory of TM modes for a rectangular wave guide. In particular, find the longitudinal electric field, the cutoff frequencies, and the wave and group velocities. Find the ratio of the lowest TM cutoff frequency to the lowest TE cutoff frequency, for a given wave guide. [*Caution*: What is the lowest TM mode?]

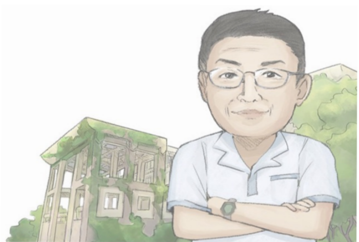
$$\left.\begin{aligned}
&\text{(i)}\quad E_x = \frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial x}+\omega\frac{\partial B_z}{\partial y}\right),\\
&\text{(ii)}\quad E_y = \frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial E_z}{\partial y}-\omega\frac{\partial B_z}{\partial x}\right),\\
&\text{(iii)}\quad B_x = \frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial B_z}{\partial x}-\frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right),\\
&\text{(iv)}\quad B_y = \frac{i}{(\omega/c)^2-k^2}\left(k\frac{\partial B_z}{\partial y}+\frac{\omega}{c^2}\frac{\partial E_z}{\partial x}\right).
\end{aligned}\right\}\qquad (9.180)$$

$$\text{TM}_{mn}\qquad E_z = E_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\qquad B_z = 0$$

$$\Rightarrow\begin{cases}
E_x = \dfrac{ik}{(\omega/c)^2-k^2}\left(\dfrac{-m\pi}{a}\right)E_0\sin\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi y}{b}\right) & B_x = \dfrac{-i\,\omega/c^2}{(\omega/c)^2-k^2}\left(\dfrac{-n\pi}{b}\right)E_0\cos\left(\dfrac{m\pi x}{a}\right)\sin\left(\dfrac{n\pi y}{b}\right)\\
E_y = \dfrac{ik}{(\omega/c)^2-k^2}\left(\dfrac{-n\pi}{b}\right)E_0\cos\left(\dfrac{m\pi x}{a}\right)\sin\left(\dfrac{n\pi y}{b}\right) & B_y = \dfrac{i\,\omega/c^2}{(\omega/c)^2-k^2}\left(\dfrac{-m\pi}{a}\right)E_0\sin\left(\dfrac{m\pi x}{a}\right)\cos\left(\dfrac{n\pi y}{b}\right)\\
E_z = E_0\sin\left(\dfrac{m\pi x}{a}\right)\sin\left(\dfrac{n\pi y}{b}\right) & B_z = 0
\end{cases}$$

$$\omega_{mn} = \pi c\sqrt{(m/a)^2+(n/b)^2}$$

$$v = \frac{c}{\sqrt{1-(\omega_{mn}/c)^2}},\quad v_g = c\sqrt{1-(\omega_{mn}/c)^2}\qquad \frac{\omega_{mn}^{\text{lowest}}(\text{TM})}{\omega_{mn}^{\text{lowest}}(\text{TE})} = \frac{\omega(\text{TM}_{11})}{\omega(\text{TE}_{10})} = \frac{\sqrt{(1/a)^2+(1/b)^2}}{(1/a)} = \sqrt{1+(a/b)^2}$$



Problem 9.37 A microwave antenna radiating at 10 GHz is to be protected from the environment by a plastic shield of dielectric constant 2.5. What is the minimum thickness of this shielding that will allow perfect transmission (assuming normal incidence)? [*Hint:* Use Eq. 9.199.]

Problem 9.36 Light of (angular) frequency ω passes from medium 1, through a slab (thickness d) of medium 2, and into medium 3 (for instance, from water through glass into air, as in Fig. 9.27). Show that the transmission coefficient for normal incidence is given by

$$T^{-1} = \frac{1}{4n_1 n_3}\left[(n_1+n_3)^2 + \frac{(n_1^2-n_2^2)(n_3^2-n_2^2)}{n_2^2}\sin^2\left(\frac{n_2\omega d}{c}\right)\right]. \quad (9.199)$$

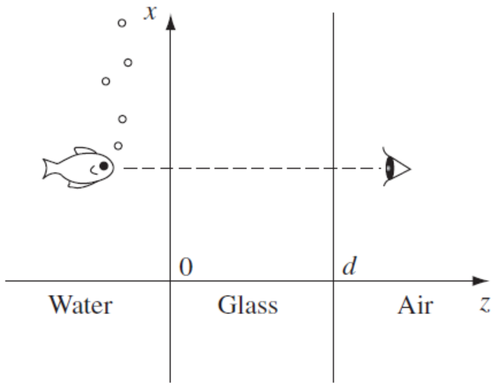


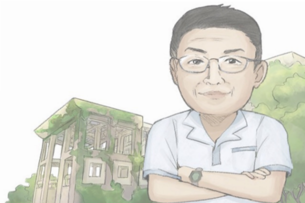
FIGURE 9.27

$$T^{-1} = \frac{1}{4n_1 n_3}\left[(n_1+n_3)^2 + \frac{(n_1^2-n_2^2)(n_3^2-n_2^2)}{n_2^2}\sin^2\left(\frac{n_2\omega d}{c}\right)\right] = 1$$

$$= \frac{1}{4}\left[4 + \frac{(1-n_2^2)(1-n_2^2)}{n_2^2}\sin^2\left(\frac{n_2\omega d}{c}\right)\right]$$

$$= 1 + \frac{(1-n_2^2)(1-n_2^2)}{4n_2^2}\underbrace{\sin^2\left(\frac{n_2\omega d}{c}\right)}_{\text{should be 0}} \Rightarrow \frac{n_2\omega d}{c} = kd = 0, \pi, 2\pi......$$

$$\Rightarrow \text{The minimum (nonzero) thickness } d: \frac{n_2\omega d}{c} = \pi \Rightarrow d = \frac{c\pi}{n_2\omega} = \frac{c\pi}{\sqrt{\mu_r\varepsilon_r}\,2\pi f} \overset{\mu_r = 1}{=} \frac{c\pi}{\sqrt{2.5}\,2\pi(10\times10^9)} \cong 9.49 \text{ mm}$$



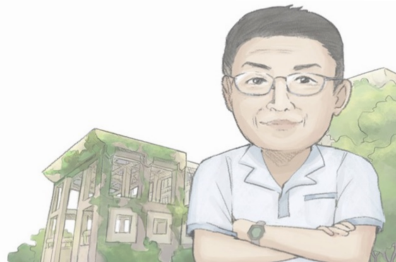
Problem 9.40 Consider the **resonant cavity** produced by closing off the two ends of a rectangular wave guide, at $z = 0$ and at $z = d$, making a perfectly conducting empty box. Show that the resonant frequencies for both TE and TM modes are given by

$$\omega_{lmn} = c\pi\sqrt{(l/d)^2 + (m/a)^2 + (n/b)^2}, \tag{9.204}$$

for integers l, m, and n. Find the associated electric and magnetic fields.

$$\begin{cases} \tilde{\mathbf{E}}(x,y,z,t) = \tilde{\mathbf{E}}_0(x,y)e^{i(kz-\omega t)} \Rightarrow \tilde{\mathbf{E}}_0(x,y,z)e^{-i\omega t} \\ \tilde{\mathbf{B}}(x,y,z,t) = \tilde{\mathbf{B}}_0(x,y)e^{i(kz-\omega t)} \Rightarrow \tilde{\mathbf{B}}_0(x,y,z)e^{-i\omega t} \end{cases}$$

$$\text{Maxwell:}\begin{cases} \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = i\omega B_z & \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} = -\dfrac{i\omega}{c^2}E_z \\ \dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z} = i\omega B_x & \dfrac{\partial B_z}{\partial y} - \dfrac{\partial B_y}{\partial z} = -\dfrac{i\omega}{c^2}E_x \\ \dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} = i\omega B_y & \dfrac{\partial B_x}{\partial z} - \dfrac{\partial B_z}{\partial x} = -\dfrac{i\omega}{c^2}E_y \end{cases} \Rightarrow \begin{cases} E_x = \dfrac{1}{(\omega/c)^2 - k_z^2}\left(\dfrac{\partial^2 E_z}{\partial x \partial z} + i\omega\dfrac{\partial B_z}{\partial y}\right) \\ E_y = \dfrac{1}{(\omega/c)^2 - k_z^2}\left(\dfrac{\partial^2 E_z}{\partial y \partial z} - i\omega\dfrac{\partial B_z}{\partial x}\right) \\ B_x = \dfrac{1}{(\omega/c)^2 - k_z^2}\left(\dfrac{\partial^2 B_z}{\partial x \partial z} - \dfrac{i\omega}{c^2}\dfrac{\partial E_z}{\partial y}\right) \\ B_y = \dfrac{1}{(\omega/c)^2 - k_z^2}\left(\dfrac{\partial^2 B_z}{\partial y \partial z} + \dfrac{i\omega}{c^2}\dfrac{\partial E_z}{\partial x}\right) \end{cases} \Rightarrow \begin{cases} \left[\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} + (\omega/c)^2\right]E_z = 0 \\ \left[\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} + (\omega/c)^2\right]B_z = 0 \end{cases}$$



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for integers l, m, and n. Find the associated electric and magnetic fields.

$$\begin{cases} E_x = \dfrac{1}{(\omega/c)^2 - k_z^2}\left(\dfrac{\partial^2 E_z}{\partial x \partial z} + i\omega \dfrac{\partial B_z}{\partial y}\right) \\ E_y = \dfrac{1}{(\omega/c)^2 - k_z^2}\left(\dfrac{\partial^2 E_z}{\partial y \partial z} - i\omega \dfrac{\partial B_z}{\partial x}\right) \\ B_x = \dfrac{1}{(\omega/c)^2 - k_z^2}\left(\dfrac{\partial^2 B_z}{\partial x \partial z} - \dfrac{i\omega}{c^2}\dfrac{\partial E_z}{\partial y}\right) \\ B_y = \dfrac{1}{(\omega/c)^2 - k_z^2}\left(\dfrac{\partial^2 B_z}{\partial y \partial z} + \dfrac{i\omega}{c^2}\dfrac{\partial E_z}{\partial x}\right) \end{cases}$$

TE_{mnl}

$$\begin{cases} E_x = \dfrac{-i\omega}{(\omega/c)^2 - k_z^2}(k_y) B_0 \cos(k_x x)\sin(k_y y)\sin(k_z z) \\ E_y = \dfrac{i\omega}{(\omega/c)^2 - k_z^2}(k_x) B_0 \sin(k_x x)\cos(k_y y)\sin(k_z z) \\ E_z = 0 \\ B_x = \dfrac{-1}{(\omega/c)^2 - k_z^2}(k_x)(k_z) B_0 \sin(k_x x)\cos(k_y y)\cos(k_z z) \\ B_y = \dfrac{-1}{(\omega/c)^2 - k_z^2}(k_y)(k_z) B_0 \cos(k_x x)\sin(k_y y)\cos(k_z z) \\ B_z = B_0 \cos(k_x x)\cos(k_y y)\sin(k_z z) \end{cases}$$

TM_{mnl}

$$\begin{cases} E_x = \dfrac{-1}{(\omega/c)^2 - k_z^2}(k_x)(k_z) E_0 \cos(k_x x)\sin(k_y y)\sin(k_z z) \\ E_y = \dfrac{-1}{(\omega/c)^2 - k_z^2}(k_y)(k_z) E_0 \sin(k_x x)\cos(k_y y)\sin(k_z z) \\ E_z = E_0 \sin(k_x x)\sin(k_y y)\cos(k_z z) \\ B_x = \dfrac{i\omega/c^2}{(\omega/c)^2 - k_z^2}(k_y)\cos(k_x x)\sin(k_y y)\cos(k_z z) \\ B_y = \dfrac{-i\omega/c^2}{(\omega/c)^2 - k_z^2}(k_x) E_0 \sin(k_x x)\cos(k_y y)\cos(k_z z) \\ B_z = 0 \end{cases}$$

$$\left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2 + k_z^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2 \Rightarrow \omega_{mnl} = c\pi\sqrt{(m/a)^2 + (n/b)^2 + (l/d)^2}$$