

0. 寫上學號：_____及姓名：_____即得 10%

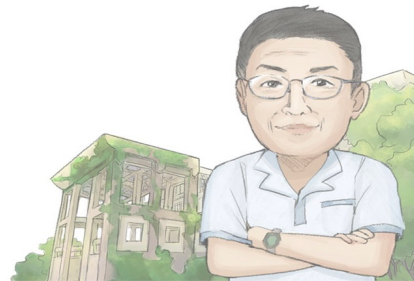
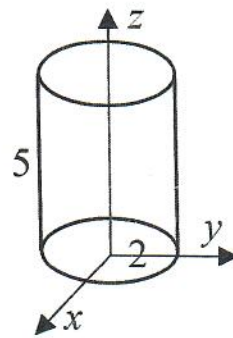
1. Let $\vec{r}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z) , and let r be its length. Show that
- (a) $\nabla(1/r) = ?$ (20%)
 - (b) $\nabla \times \nabla(1/r) = ?$ (20%)

Ans:

2. The divergence theorem

- (a) Find the divergence of the function
 $\mathbf{v} = s(2 + \sin^2 \phi) \hat{s} + s \sin \phi \cos \phi \hat{\phi} + 3z \hat{z}$ (25%)
- (b) Test the divergence of this function, using the cylinder (radius 2, height 5) shown in the figure. (25%)

Ans:



向量請以粗體 **A** 表示，或者戴箭頭 $\vec{A}$ 或者 both $\vec{A}$ ，單位向量請戴尖帽子 $\hat{A}$
基本上法向量 $\hat{n}$ 不同於 $\hat{r}$ ，並且有時候， $\hat{r}$ 終究不等同球座標的單位向量 $\hat{r}$ ，請
同學們注意，習慣請儘早培養

隨著以後題目複雜度的提升，有時候一份答案的完整回答會分成許多部份的計算或者是證明，所以請養成習慣 Highlight 自己計算過程中符合題目要求的答案
別隨意更改題目定義，麻煩請跟助教或者是老師確認

注意積分撰寫的格式

注意積分因子的寫法

1.

a.20%

$$\begin{aligned}\vec{r} &= (x, y, z) - (x', y', z') & r &= \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \\ \nabla \left(\frac{1}{r} \right) &= -\frac{1}{r^2} \nabla r = -\frac{1}{r^2} (\hat{x} \partial_x + \hat{y} \partial_y + \hat{z} \partial_z) \left(\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2} \right) \\ &= -\frac{1}{r^2} \frac{2(x-x')\hat{x} + 2(y-y')\hat{y} + 2(z-z')\hat{z}}{2\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \\ &= -\frac{1}{r^2} \frac{1}{r} \vec{r} = -\frac{\vec{r}}{r^3}\end{aligned}$$

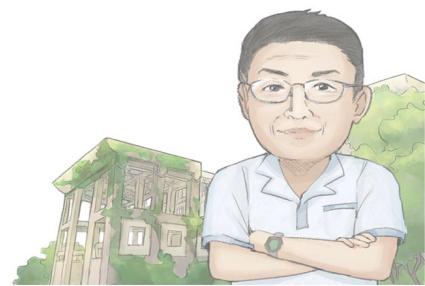
b.20%

$$\begin{aligned}\nabla \times \nabla \left(\frac{1}{r} \right) &= 0 \\ \nabla \times \left(-\frac{\vec{r}}{r^2} \right) &= - \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ \frac{x-x'}{r^3} & \frac{y-y'}{r^3} & \frac{z-z'}{r^3} \end{vmatrix} \\ &= - \left\{ \hat{x} \left[\partial_y \left(\frac{z-z'}{r^3} \right) - \partial_z \left(\frac{y-y'}{r^3} \right) \right] - \hat{y} \left[\partial_x \left(\frac{z-z'}{r^3} \right) - \partial_z \left(\frac{x-x'}{r^3} \right) \right] + \hat{z} \left[\partial_x \left(\frac{y-y'}{r^3} \right) - \partial_y \left(\frac{x-x'}{r^3} \right) \right] \right\} \\ &= - \left\{ \hat{x} \left[\frac{-2(z-z')\partial_y(r)}{r^4} - \frac{-2(y-y')\partial_z(r)}{r^4} \right] - \hat{y} \left[\frac{-2(z-z')\partial_x(r)}{r^4} - \frac{-2(x-x')\partial_z(r)}{r^4} \right] \right. \\ &\quad \left. + \hat{z} \left[\frac{-2(y-y')\partial_x(r)}{r^4} - \frac{-2(x-x')\partial_y(r)}{r^4} \right] \right\} \\ &= - \left\{ \hat{x} \left[\frac{-2(z-z') \times \frac{-2(y-y')}{2r}}{r^4} - \frac{-2(y-y') \frac{-2(z-z')}{2r}}{r^4} \right] - \hat{y} \left[\frac{-2(z-z') \frac{-2(x-x')}{2r}}{r^4} - \frac{-2(x-x') \frac{-2(z-z')}{2r}}{r^4} \right] \right. \\ &\quad \left. + \hat{z} \left[\frac{-2(y-y') \frac{-2(x-x')}{2r}}{r^4} - \frac{-2(x-x') \frac{-2(y-y')}{2r}}{r^4} \right] \right\} \\ &= 0\end{aligned}$$

2.

a.25%

$$\nabla \cdot \mathbf{v} = \frac{1}{s} \frac{\partial}{\partial s} (sv_s) + \frac{1}{s} \frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 2(2 + \sin^2 \phi) + (\cos^2 \phi - \sin^2 \phi) + 3 = 8$$



b.25%

$$\int_V \nabla \cdot \mathbf{v} d\tau = \oint_A \mathbf{v} \cdot d\mathbf{a} \dots 5\%$$

$$\int_V \nabla \cdot \mathbf{v} d\tau = \int_0^2 ds \int_0^{2\pi} \int_0^5 \textcolor{red}{sd\phi} dz (8) = \left(\frac{1}{2} \times 2^2 \times 2\pi \times 5 \right) \times 8 = 160\pi \dots 10\%$$

$$\begin{aligned} \oint_A \mathbf{v} \cdot d\mathbf{a} &= \overbrace{\int_0^5 dz \int_0^{2\pi} sd\phi \left[s(2 + \sin^2 \phi) \right]}^{\text{side face}} \bigg|_{s=2} + \overbrace{\int_0^2 ds \int_0^{2\pi} sd\phi (3z)}^{\text{top face}} \bigg|_{z=5} - \overbrace{\int_0^2 ds \int_0^{2\pi} sd\phi (3z)}^{\text{bottom face}} \bigg|_{z=0} \\ &= \left(80\pi + 20 \underbrace{\int_0^{2\pi} \sin^2 \phi d\phi}_0 \right) + 60\pi - 0\pi = (80\pi + 20\pi) + 60\pi - 0\pi = 160\pi \dots 10\% \\ &\quad \sin^2 \phi = \frac{1}{2}(1 - \cos 2\phi) \end{aligned}$$

