2023 Fall PHYS2310 電磁學（Electromagnetism) Quiz #3
[Griffiths Ch. 4]　2023/11/30,　11:00am–12:00am, 教師：張存續

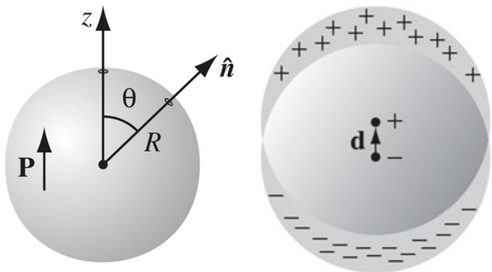
Score

0. 寫上學號:________________及姓名:__________________

1. Consider a homogeneous linear dielectric sphere with a dielectric constant of 10 placed in an otherwise uniform electric field $\mathbf{E}_0$. The total polarization inside the sphere is said to be $\mathbf{P}=P_0\hat{\mathbf{z}}$. [Examples 4.2, 4.3, and 4.7]
   (a) Find the self-field $\mathbf{E}_{in}$ inside the sphere caused by the polarization. (20%)
   (b) The total electric field within the sphere is $\mathbf{E}=\alpha\mathbf{E}_0$, find α. (15%)
   (c) Determine electric potential V outside the sphere in terms of P_0. (15%)

Ans:

Method 1:



Setting $V_{origin}=0$

For $r\to\infty: V(r)=-E_0 r\cos\theta$

$$V(r,\theta)=\begin{cases}\sum A_l r^l P_l(\cos\theta) & (r\le R)\\ -E_0 r\cos\theta+\sum \dfrac{B_l}{r^{l+1}}P_l(\cos\theta) & (r\ge R)\end{cases}$$

$$B.C\begin{cases}1: V_{r\le R}\big|_{r=R}=V_{r\ge R}\big|_{r=R}\\ 2: \varepsilon_0\dfrac{\partial V_{r\ge R}}{\partial r}\bigg|_{r=R}-\varepsilon_0\varepsilon_r\dfrac{\partial V_{r\le R}}{\partial r}\bigg|_{r=R}=\sigma_f=0\\ 3: \dfrac{\partial V_{r\ge R}}{\partial r}\bigg|_{r=R}-\dfrac{\partial V_{r\le R}}{\partial r}\bigg|_{r=R}=-\dfrac{\sigma_b}{\varepsilon_0}=-\dfrac{P_0}{\varepsilon_0}\cos\theta\end{cases}\Rightarrow\begin{cases}1:\begin{cases}A_0=\dfrac{B_0}{R} & l=0\\ A_1R\cos\theta=-E_0R\cos\theta+\dfrac{B_1}{R^2}\cos\theta & l=1\\ A_lR^lP_l(\cos\theta)=\dfrac{B_l}{R^{l+1}}P_l(\cos\theta) & l>1\end{cases}\\ 2:\begin{cases}0=\varepsilon_0\dfrac{-B_0}{R^2} & l=0\\ \varepsilon_0\varepsilon_rA_1\cos\theta=-\varepsilon_0E_0\cos\theta+\varepsilon_0(-2)\dfrac{B_1}{R^3}\cos\theta & l=1\\ \varepsilon_0\varepsilon_r lA_lR^{l-1}P_l(\cos\theta)=\varepsilon_0(-l-1)\dfrac{B_l}{R^{l+2}}P_l(\cos\theta) & l>1\end{cases}\\ 3: A_1\cos\theta-(-1)E_0\cos\theta-(-2)\dfrac{B_1}{R^3}\cos\theta=-\dfrac{P_0}{\varepsilon_0}\cos\theta \quad l=1\end{cases}$$

$$\Rightarrow\begin{cases}A_0=B_0=A_{l>1}=B_{l>1}=0\\ A_1=-\dfrac{3\varepsilon_0}{\varepsilon_0\varepsilon_r+2\varepsilon_0}E_0=-\dfrac{3}{\varepsilon_r+2}E_0=-\dfrac{1}{4}\\ B_1=\dfrac{\varepsilon_r-1}{\varepsilon_r+2}E_0R^3=\dfrac{3}{4}E_0R^3\\ E_0=\dfrac{4}{9}\dfrac{P_0}{\varepsilon_0}\end{cases}\Rightarrow\boxed{V(r,\theta)=\begin{cases}-\dfrac{1}{4}E_0r\cos\theta & (r\le R)\\ -E_0r\cos\theta+\dfrac{3}{4}E_0\dfrac{R^3}{r^2}\cos\theta & (r\ge R)\end{cases}}$$ ......The final, the total potential.

$$(r\le R): V(r,\theta)=-\frac{1}{4}E_0r\cos\theta\Rightarrow\mathbf{E}^{tot}_{r\le R}=\mathbf{E}_0+\mathbf{E}_{in}\Rightarrow\mathbf{E}_{in}=-\frac{3}{4}\mathbf{E}_0=\boxed{-\frac{\mathbf{P}}{3\varepsilon_0}}$$ ......(a)

$$=\frac{1}{4}E_0\hat{\mathbf{z}}\Rightarrow\boxed{\alpha=\frac{1}{4}}$$ ......(b)

$$\boxed{(r\ge R): V(r,\theta)=-E_0r\cos\theta+\frac{3}{4}E_0\frac{R^3}{r^2}\cos\theta=-E_0r\cos\theta+\frac{P_0}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta=-\frac{4}{9}\frac{P_0}{\varepsilon_0}r\cos\theta+\frac{P_0}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta}$$ ......(c)

Method 2

$$\mathbf{E}_{in} = \mathbf{E}_+ + \mathbf{E}_- = \frac{1}{4\pi\varepsilon_0}\frac{Q_+}{r_+^2}\hat{\mathbf{r}}_+ + \frac{1}{4\pi\varepsilon_0}\frac{Q_-}{r_-^2}\hat{\mathbf{r}}_- = \frac{1}{4\pi\varepsilon_0}\frac{\frac{4\pi r_+^3}{3}\rho_+}{r_+^2}\hat{\mathbf{r}}_+ + \frac{1}{4\pi\varepsilon_0}\frac{\frac{4\pi r_-^3}{3}\rho_-}{r_-^2}\hat{\mathbf{r}}_-$$

Take $r_+ \approx r_-$, $\mathbf{r}_+ - \mathbf{r}_- = -\mathbf{d}$

$$\mathbf{E}_{in} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho}{3\varepsilon_0}(\mathbf{r}_+ - \mathbf{r}_-) = -\frac{\rho}{3\varepsilon_0}\mathbf{d} = \boxed{-\frac{\mathbf{P}}{3\varepsilon_0}} \text{......(a)}$$

$$\mathbf{E}_{r\le R}^{tot} = \mathbf{E}_0 + \mathbf{E}_{in} = \mathbf{E}_0 - \frac{\mathbf{P}}{3\varepsilon_0} = \mathbf{E}_0 - \frac{\varepsilon_0\chi_e\mathbf{E}_{r\le R}^{tot}}{3\varepsilon_0} \Rightarrow \mathbf{E}_{r\le R}^{tot} = \frac{1}{1+\chi_e/3}\mathbf{E}_0 = \frac{1}{4}\mathbf{E}_0 \Rightarrow \boxed{\alpha = \frac{1}{4}} \text{......(b)}$$

$$V_{r\ge R} = -E_0 r\cos\theta + \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2} = -E_0 r\cos\theta + \frac{R^3\mathbf{P}\cdot\hat{\mathbf{r}}}{3\varepsilon_0 r^2} = \boxed{-E_0 r\cos\theta + \frac{P_0 R^3\cos\theta}{3\varepsilon_0 r^2}} \text{......(c)}$$

Partial Image Charge (Example 4.8)

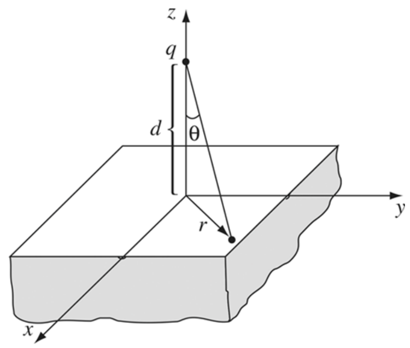
Suppose the entire region below the plane $z = 0$ is filled with uniform linear dielectric material of susceptibility χ_e. Consider a point charge q situated at distance d above the origin.

(a) Write down the electric potential V above the plane ($z > 0$) and below the plane ($z < 0$) using the image charge q_b. (20%)

(b) Find q_b using the boundary condition $D^{\perp}_{above} - D^{\perp}_{below} = 0$. (15%)

(c) Calculate the force on the point charge q. (15%)

Ans:



(a)

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{q_b}{\sqrt{x^2+y^2+(z+d)^2}}\right] \qquad z>0$$

$$V = \frac{1}{4\pi\varepsilon_0}\left[\frac{(q+q_b)}{\sqrt{x^2+y^2+(z-d)^2}}\right] \qquad z<0$$

(b)

$$D^{\perp}_{above} - D^{\perp}_{below} = 0 \;\Rightarrow\; \varepsilon_0 E^{\perp}_{above} = \varepsilon_0\varepsilon_r E^{\perp}_{below} = (1+\chi_e)\varepsilon_0 E^{\perp}_{below} \text{ when } z=0$$

$$\Rightarrow\; -\varepsilon_0 \left.\frac{\partial V_{above}}{\partial z}\right|_{z=0^+} = -(1+\chi_e)\varepsilon_0 \left.\frac{\partial V_{below}}{\partial z}\right|_{z=0^-}$$

$$\frac{(q-q_b)d}{(x^2+y^2+d^2)^{3/2}} = \frac{(1+\chi_e)(q+q_b)d}{(x^2+y^2+d^2)^{3/2}} \;\Rightarrow\; q_b = \frac{-\chi_e}{\chi_e+2}q$$

(c) The surface bound charge on the xy plane is of opposite sign to q , so the force will be attractive.

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{qq_b}{(2d)^2}\hat{\mathbf{z}} = -\frac{1}{4\pi\epsilon_0}\left(\frac{\chi_e}{\chi_e+2}\right)\frac{q^2}{4d^2}\hat{\mathbf{z}}. \qquad (4.54)$$