

0. 寫上學號：_____及姓名：_____

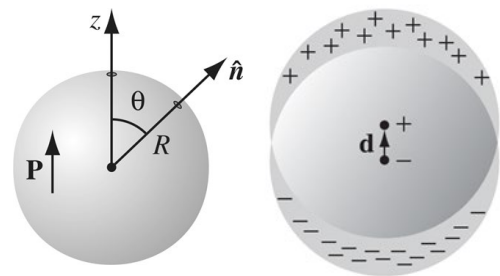
1. Consider a homogeneous linear dielectric sphere with a dielectric constant of 10 placed in an otherwise uniform electric field $\mathbf{E}_0$. The total polarization inside the sphere is said to be $\mathbf{P} = P_0 \hat{\mathbf{z}}$. [Examples 4.2, 4.3, and 4.7]
- (a) Find the self-field $\mathbf{E}_{in}$ inside the sphere caused by the polarization. (20%)
- (b) The total electric field within the sphere is $\mathbf{E} = \alpha \mathbf{E}_0$, find α . (15%)
- (c) Determine electric potential V outside the sphere in terms of P_0 . (15%)

Ans:

Method 1:

Setting $V_{origin} = 0$ For $r \rightarrow \infty: V(r) = -E_0 r \cos \theta$

$$V(r, \theta) = \begin{cases} \sum A_l r^l P_l(\cos \theta) & (r \leq R) \\ -E_0 r \cos \theta + \sum \frac{B_l}{r^{l+1}} P_l(\cos \theta) & (r \geq R) \end{cases}$$



$$B.C \begin{cases} 1: V_{r \leq R}|_{r=R} = V_{r \geq R}|_{r=R} \\ 2: \epsilon_0 \frac{\partial V_{r \geq R}}{\partial r} \Big|_{r=R} - \epsilon_0 \epsilon_r \frac{\partial V_{r \leq R}}{\partial r} \Big|_{r=R} = \sigma_f = 0 \\ 3: \frac{\partial V_{r \geq R}}{\partial r} \Big|_{r=R} - \frac{\partial V_{r \leq R}}{\partial r} \Big|_{r=R} = -\frac{\sigma_b}{\epsilon_0} = -\frac{P_0}{\epsilon_0} \cos \theta \end{cases} \Rightarrow \begin{cases} 1: \begin{cases} A_0 = \frac{B_0}{R} & l=0 \\ A_l R \cos \theta = -E_0 R \cos \theta + \frac{B_l}{R^2} \cos \theta & l=1 \\ A_l R^l P_l(\cos \theta) = \frac{B_l}{R^{l+1}} P_l(\cos \theta) & l>1 \end{cases} \\ 2: \begin{cases} 0 = \epsilon_0 \frac{-B_0}{R^2} & l=0 \\ \epsilon_0 \epsilon_r A_l \cos \theta = -\epsilon_0 E_0 \cos \theta + \epsilon_0 (-2) \frac{B_l}{R^3} \cos \theta & l=1 \\ \epsilon_0 \epsilon_r l A_l R^{l-1} P_l(\cos \theta) = \epsilon_0 (-l-1) \frac{B_l}{R^{l+2}} P_l(\cos \theta) & l>1 \end{cases} \\ 3: A_l \cos \theta - (-1) E_0 \cos \theta - (-2) \frac{B_l}{R^3} \cos \theta = -\frac{P_0}{\epsilon_0} \cos \theta & l=1 \end{cases}$$

$$\Rightarrow \begin{cases} A_0 = B_0 = A_{l>1} = B_{l>1} = 0 \\ A_1 = -\frac{3\epsilon_0}{\epsilon_0 \epsilon_r + 2\epsilon_0} E_0 = -\frac{3}{\epsilon_r + 2} E_0 = -\frac{1}{4} E_0 \\ B_1 = \frac{\epsilon_r - 1}{\epsilon_r + 2} E_0 R^3 = \frac{3}{4} E_0 R^3 \\ E_0 = \frac{4 P_0}{9 \epsilon_0} \end{cases} \Rightarrow V(r, \theta) = \begin{cases} -\frac{1}{4} E_0 r \cos \theta & (r \leq R) \\ -E_0 r \cos \theta + \frac{3}{4} E_0 \frac{R^3}{r^2} \cos \theta & (r \geq R) \end{cases} \dots \text{The final, the total potential.}$$

$$(r \leq R): V(r, \theta) = -\frac{1}{4} E_0 r \cos \theta \Rightarrow \mathbf{E}_{r \leq R}^{tot} = \mathbf{E}_0 + \mathbf{E}_{in} \Rightarrow \mathbf{E}_{in} = -\frac{3}{4} \mathbf{E}_0 = \boxed{-\frac{\mathbf{P}}{3\epsilon_0}} \dots (a)$$

$$= \frac{1}{4} E_0 \hat{\mathbf{z}} \Rightarrow \boxed{\alpha = \frac{1}{4}} \dots (b)$$

$$(r \geq R): V(r, \theta) = -E_0 r \cos \theta + \frac{3}{4} E_0 \frac{R^3}{r^2} \cos \theta = -E_0 r \cos \theta + \frac{P_0}{3\epsilon_0} \frac{R^3}{r^2} \cos \theta = -\frac{4}{9} \frac{P_0}{\epsilon_0} r \cos \theta + \frac{P_0}{3\epsilon_0} \frac{R^3}{r^2} \cos \theta \dots (c)$$



Method 2

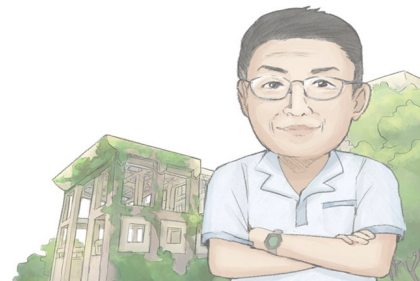
$$\mathbf{E}_{in} = \mathbf{E}_+ + \mathbf{E}_- = \frac{1}{4\pi\epsilon_0} \frac{Q_+}{r_+^2} \hat{\mathbf{r}}_+ + \frac{1}{4\pi\epsilon_0} \frac{Q_-}{r_-^2} \hat{\mathbf{r}}_- = \frac{1}{4\pi\epsilon_0} \frac{\frac{4\pi r_+^3}{3} \rho_+}{r_+^2} \hat{\mathbf{r}}_+ + \frac{1}{4\pi\epsilon_0} \frac{\frac{4\pi r_-^3}{3} \rho_-}{r_-^2} \hat{\mathbf{r}}_-$$

Take $r_+ \approx r_-$, $\mathbf{r}_+ - \mathbf{r}_- = -\mathbf{d}$

$$\mathbf{E}_{in} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho}{3\epsilon_0} (\mathbf{r}_+ - \mathbf{r}_-) = -\frac{\rho}{3\epsilon_0} \mathbf{d} = \boxed{-\frac{\mathbf{P}}{3\epsilon_0}} \dots\dots (a)$$

$$\mathbf{E}_{r \leq R}^{tot} = \mathbf{E}_0 + \mathbf{E}_{in} = \mathbf{E}_0 - \frac{\mathbf{P}}{3\epsilon_0} = \mathbf{E}_0 - \frac{\epsilon_0 \chi_e \mathbf{E}_{r \leq R}^{tot}}{3\epsilon_0} \Rightarrow \mathbf{E}_{r \leq R}^{tot} = \frac{1}{1 + \chi_e/3} \mathbf{E}_0 = \frac{1}{4} \mathbf{E}_0 \Rightarrow \boxed{\alpha = \frac{1}{4}} \dots\dots (b)$$

$$V_{r \geq R} = -E_0 r \cos \theta + \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} = -E_0 r \cos \theta + \frac{R^3 \mathbf{P} \cdot \hat{\mathbf{r}}}{3\epsilon_0 r^2} = \boxed{-E_0 r \cos \theta + \frac{P_0 R^3 \cos \theta}{3\epsilon_0 r^2}} \dots\dots (c)$$



Partial Image Charge (Example 4.8)

Suppose the entire region below the plane $z = 0$ is filled with uniform linear dielectric material of susceptibility χ_e . Consider a point charge q situated at distance d above the origin.

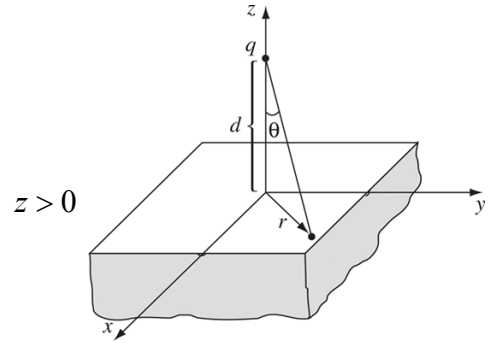
- Write down the electric potential V above the plane ($z > 0$) and below the plane ($z < 0$) using the image charge q_b . (20%)
- Find q_b using the boundary condition $D_{above}^\perp - D_{below}^\perp = 0$. (15%)
- Calculate the force on the point charge q . (15%)

Ans:

(a)

$$V = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{q_b}{\sqrt{x^2 + y^2 + (z+d)^2}} \right] \quad z > 0$$

$$V = \frac{1}{4\pi\epsilon_0} \left[\frac{(q+q_b)}{\sqrt{x^2 + y^2 + (z-d)^2}} \right] \quad z < 0$$



(b)

$$D_{above}^\perp - D_{below}^\perp = 0 \Rightarrow \epsilon_0 E_{above}^\perp = \epsilon_0 \epsilon_r E_{below}^\perp = (1 + \chi_e) \epsilon_0 E_{below}^\perp \text{ when } z = 0$$

$$\Rightarrow -\epsilon_0 \frac{\partial V_{above}}{\partial z} \Big|_{z=0^+} = -(1 + \chi_e) \epsilon_0 \frac{\partial V_{below}}{\partial z} \Big|_{z=0^-}$$

$$\frac{(q - q_b)d}{(x^2 + y^2 + d^2)^{3/2}} = \frac{(1 + \chi_e)(q + q_b)d}{(x^2 + y^2 + d^2)^{3/2}} \Rightarrow q_b = \frac{-\chi_e}{\chi_e + 2} q$$

- The surface bound charge on the xy plane is of opposite sign to q , so the force will be attractive.

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{qq_b}{(2d)^2} \hat{\mathbf{z}} = -\frac{1}{4\pi\epsilon_0} \left(\frac{\chi_e}{\chi_e + 2} \right) \frac{q^2}{4d^2} \hat{\mathbf{z}}. \quad (4.54)$$

