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Photons and Planck Radiation Law

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Quantum theory begins with the **Planck radiation law** for thermal radiation at different frequencies,

$$u_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{\exp(\hbar\omega/\tau) - 1}, \tag{1}$$

where u_ω is the spectral density defined as the radiated energy per unit volume per unit frequency range. For a given temperature τ, the spectral density for large frequency is exponentially suppressed, $u_\omega \sim \omega^3 e^{-\hbar\omega/\tau}$, and save us from the famous ultraviolet catastrophe in classical theory.

• photons: quantization of electromagnetic fields

The Planck radiation law can explained by the quantization of electromagnetic fields, i.e. the photons. According to Maxwell equations, the Hamiltonian for electromagnetic fields in vacuum is

$$H = \int d^3\boldsymbol{r} \left[\frac{\epsilon_0}{2} |\boldsymbol{E}(\boldsymbol{r})|^2 + \frac{1}{2\mu_0} |\boldsymbol{B}(\boldsymbol{r})|^2 \right]. \tag{2}$$

It is convenient to decompose the fields into their Fourier components, $\boldsymbol{E}(\boldsymbol{r}) = (1/\sqrt{V}) \sum_{\boldsymbol{k}} \boldsymbol{E}(\boldsymbol{k}) e^{\boldsymbol{k}\cdot\boldsymbol{r}}$ and the same relation for the magnetic field. Because the field is real, its Fourier components are related, $\boldsymbol{E}(-\boldsymbol{k}) = \boldsymbol{E}^*(\boldsymbol{k})$. After some algebra, the above integral can be turned into summation over all possible wave numbers $\boldsymbol{k}$,

$$H = \sum_{\boldsymbol{k}} \left[\frac{\epsilon_0}{2} |\boldsymbol{E}(\boldsymbol{k})|^2 + \frac{1}{2\mu_0} |\boldsymbol{B}(\boldsymbol{k})|^2 \right]. \tag{3}$$

This is still classical – we just look at the same Hamiltonian at a different angle. In quantum theory, the fields $\boldsymbol{E}(\boldsymbol{k})$ and $\boldsymbol{B}(\boldsymbol{k})$ are operators and do not commute with each other, i.e. there is uncertainty relation between them.

Though not completely correct, it is inspiring to compare the above Hamiltonian with the simple harmonic oscillator, $H = \frac{1}{2m} p^2 + \frac{k}{2} x^2$. The similarity is clear. Loosely speaking, $\boldsymbol{E}(\boldsymbol{k})$ and $\boldsymbol{B}(\boldsymbol{k})$ can be viewed as conjugate variables to each other, just like x and p. The analogy turns out to be correct by the more advanced theory named quantum electrodynamics (QED) and the Hamiltonian can be expressed in terms of two pairs of creation/annihilation operators,

$$H = \sum_{\boldsymbol{k}} \hbar\omega_k \left[\left(b^\dagger_{1\boldsymbol{k}} b_{1\boldsymbol{k}} + \frac{1}{2} \right) + \left(b^\dagger_{2\boldsymbol{k}} b_{2\boldsymbol{k}} + \frac{1}{2} \right) \right], \tag{4}$$

where $\omega_k = c|\boldsymbol{k}| = ck$ is the dispersion relation in vacuum. The two distinct sets of creation/annihilation operators arise from two polarizations at each wave number. Because the number operator $b^\dagger_{\lambda\boldsymbol{k}} b_{\lambda\boldsymbol{k}}$ takes on integer values, the energy is quantized in units of $\hbar\omega_k$,

$$\epsilon_{\lambda\boldsymbol{k}} = n_{\lambda\boldsymbol{k}}\, \hbar\omega_k, \tag{5}$$

where $n_{\lambda\boldsymbol{k}} = 0, 1, 2, ...$ are integers. The energy quantization of the electromagnetic fields was first proposed by Einstein with the groundbreaking notion of **photons** (originally named as light quanta).

• Planck distribution function

Let us study the thermodynamics of a single mode with frequency ω first, i.e. just one type of photons with energy $n\hbar\omega$, where $n = 0, 1, 2, ...$ is the photon number. The partition is rather straightforward to compute,

$$Z = 1 + e^{-\hbar\omega/\tau} + e^{-2\hbar\omega/\tau} + \cdots = \frac{1}{1 - \exp(-\hbar\omega/\tau)}. \tag{6}$$

The average energy of the photon system in thermal equilibrium is

$$\langle \epsilon \rangle = \tau^2 \frac{\partial \log Z}{\partial \tau} = \frac{\hbar\omega}{\exp(\hbar\omega/\tau) - 1}. \tag{7}$$

In the high temperature limit, $\hbar\omega \ll \tau$, the average energy $\langle \epsilon \rangle \approx \tau$ as described by the equipartition of energy in the classical regime. Because each photon carries energy quantum $\hbar\omega$, the average number of photons in thermal equilibrium is

$$\langle n \rangle = \frac{\langle \epsilon \rangle}{\hbar\omega} = \frac{1}{\exp(\hbar\omega/\tau) - 1}, \tag{8}$$

known as the **Planck distribution function**. At a given temperature τ, the average number of the low-frequency ($\hbar\omega \ll \tau$) photons is huge, $\langle n \rangle \approx$

$\tau/(\hbar\omega) \gg 1$. On the other hand, the average number of the high-frequency ($\hbar\omega \gg \tau$) photons is exponentially suppressed, $\langle n\rangle \approx e^{-\hbar\omega/\tau} \ll 1$. One should not worry about the seemingly divergent photon number for the low frequency, $\langle n\rangle \approx \tau/(\hbar\omega)$, because the corresponding energy quantum $\hbar\omega$ is minuscule and cancels the divergent photon number when computing the average energy.

• mode counting for different wave numbers

To derive the Planck radiation law, we need to count the modes for different wave numbers properly. For electromagnetic fields confined within a perfectly conducting cubic cavity (of length L), the wave number is quantized,

$$\boldsymbol{k}_n = \frac{\pi}{L}(n_x, n_y, n_z). \tag{9}$$

The corresponding frequency from the linear dispersion is

$$\omega_k = c|\boldsymbol{k}_n| = ck_n = \frac{n\pi c}{L}, \tag{10}$$

where $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$. The quantization of the wave number should not be confused with the energy quantization of photons. It arises from the boundary conditions and already shows up even in classical theory.

In the thermodynamic limit $L \to \infty$, the adjacent wave numbers are infinitesimally close to each other. Therefore, one can replace the discrete sum by an integral in the space of wave numbers,

$$\sum_{\boldsymbol{k}}(\cdots) = \sum_{\boldsymbol{n}}(\cdots) = \frac{1}{8}\int_0^\infty 4\pi n^2 dn(\cdots), \tag{11}$$

where $1/8$ arises because only the positive octant of the space is included.

• Planck radiation law for the spectral density

We are now ready to derive the Planck radiation law. The total energy of the photons in the cavity is

$$U = 2 \times \sum_{\boldsymbol{k}}\langle\epsilon_k\rangle = 2 \times \sum_{\boldsymbol{k}} \frac{\hbar\omega_k}{\exp(\hbar\omega_k/\tau) - 1}, \tag{12}$$

where $\omega_k = ck = n\pi c/L$ and the factor of two comes from two polarizations for each wave number. Replacing the sum by the integral leads to

$$\begin{aligned} U &= \pi \int_0^\infty dn\, n^2 \frac{\hbar\omega_k}{\exp(\hbar\omega_k/\tau) - 1}, && (13) \\ &= \frac{V\hbar}{\pi^2 c^3} \int_0^\infty d\omega\, \frac{\omega^3}{\exp(\hbar\omega/\tau) - 1}, && (14) \end{aligned}$$

with the volume $V = L^3$. The spectral density u_ω is defined as the energy per unit volume per unit frequency range,

$$\frac{U}{V} = \int_0^\infty d\omega\, u_\omega. \tag{15}$$

By comparison, the spectral density is given by the Planck radiation law,

$$u_\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{\exp(\hbar\omega/\tau) - 1}. \tag{16}$$

This frequency distribution of thermal radiation is the direct consequence of photons in thermal equilibrium and can be measured experimentally.

Carrying out the integration over frequency, the total energy of the photons in cavity is

$$\frac{U}{V} = \int_0^\infty d\omega\, u_\omega = \frac{\pi^2}{15\hbar^3 c^3}\, \tau^4. \tag{17}$$

The radiant energy per unit volume is proportional to the fourth power of the temperature is known as the Stefan-Boltzmann law of radiation. It is important to emphasize that the proportional constant $\pi^2/(15\hbar^3 c^3)$ contains the Planck constant and cannot be explained by the classical theory at all.

• Einstein's A and B

Einstein cooked up a smart way to derive the Planck radiation law by detailed balance in thermal equilibrium. Consider two energy levels for electrons with energies E_a and E_b and $E_a - E_b = \hbar\omega > 0$. The occupation numbers for the energy levels are denoted by n_a, n_b respectively. When an electron hops from the higher energy level to the lower one, it emits a photon with energy $\hbar\omega = E_a - E_b$. On the other hand, an electron can also absorb a photon and jump from the lower energy level to the higher one. Einstein assumed that there are two kinds of emission precesses: the spontaneous emissions characterized by a constant A and the stimulated emissions characterized by Bu_ω, where B is another constant. The appearance of u_ω is reasonable because the strength of stimulation should be proportional to the energy density of the corresponding frequency, i.e. the spectral density u_ω. On the other hand, absorptions only occur when photons are around. That is to say, there is only stimulated absorption. Einstein assumed that the stimulated absorptions are also characterized by the same factor Bu_ω, as in the stimulated emissions. The above arguments lead to the following rate equations for the occupation numbers:

$$\begin{aligned} \frac{dn_a}{dt} &= -(A + Bu_\omega)n_a + Bu_\omega n_b, \\ \frac{dn_b}{dt} &= (A + Bu_\omega)n_a - Bu_\omega n_b. \end{aligned} \tag{18}$$

In thermal equilibrium, the occupation numbers do not change with time. The detailed balance between the transfer rates implies the relation to hold,

$$(A + Bu_\omega)n_a = Bu_\omega n_b. \tag{19}$$

Note that the occupation number are related by the Boltzmann factor in thermal equilibrium,

$$\frac{n_a}{n_b} = e^{-\hbar\omega/\tau}. \tag{20}$$

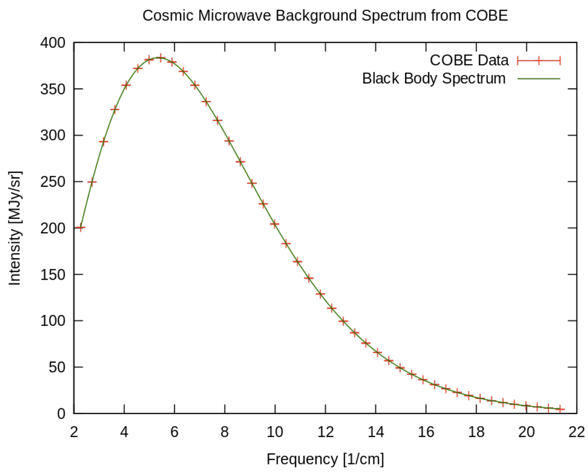
It is then straightforward to solve for the spectral density,

$$u_\omega = \frac{A}{B} \frac{1}{\exp(\hbar\omega/\tau) - 1}. \tag{21}$$

This simple theory is consistent with the Planck radiation law with the ratio A/B needed to be computed from the microscopic quantum theory.

• cosmic microwave background

It is quite a surprise that our universe is filled with almost uniform and isotropic black-body radiation at about 2.73 K. The existence of this radiation is a strong support for big bag theory, which predicts that the universe is expanding and thus cooling down with time. According to the big bang theory, the temperature of the early universe is quite high and the matter exists primarily in the form of plasma (charged electrons and protons) which interacts strongly with photons. It is this reasonable to assume that photons reach thermal equilibrium through interactions with charged particles.



As the universe cools down to about 3000 K, neutral hydrogen atoms are stable and only interact with photons at the frequencies of specific hydrogen spectral lines. The photons are effectively decoupled from the matter with spectral density captured by the Planck radiation law. Because there is no hear transfer into the equilibrated photon gas, the entropy remains constant. Starting from the differential relation $\tau d\sigma = dU$ at constant volume,

$$d\sigma = \frac{1}{\tau} dU = \frac{4\pi^2}{15\hbar^3 c^3} V \tau^2 d\tau. \tag{22}$$

Integrate the above relation to obtain the entropy for the photon gas,

$$\sigma = \left(\frac{4\pi^2}{45\hbar^3 c^3} \right) V \tau^3. \tag{23}$$

As the universe expands, the volume V increases and the temperature drops as $1/V^{1/3}$. The measured Planck radiation law at 2.73 K is the cooled equilibrated photon gas, originally at the decoupling temperature around 3000 K, after isentropic expansion.

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