21.

X : 25 歲男性的身高 (英吋) $\sim N(\mu = 71, \sigma^2 = 6.25)$

$$Z \equiv \frac{X-71}{2.5} \sim N(0,1)$$

(a)

$$P(X > 74) = P(\frac{X-71}{2.5} > \frac{74-71}{2.5}) = P(Z > 1.2) = 0.1151$$

(b)

$$P(X > 77 \mid X \ge 72) = \frac{P(X > 77)}{P(X \ge 72)} = \frac{P(\frac{X-71}{2.5} > \frac{77-71}{2.5})}{P(\frac{X-71}{2.5} \ge \frac{72-71}{2.5})}$$

$$= \frac{P(Z > 2.4)}{P(Z \ge 0.4)} = \frac{0.0082}{0.3446} = 0.0238$$

24.

X : *The lifetimes of* int*eractive computer chip* $\sim N(\mu = 1.4 \times 10^6, \sigma^2 = 9 \times 10^{10})$

$$Z \equiv \frac{X - 1.4 \times 10^6}{3 \times 10^5} \sim N(0,1)$$

$$P(X < 1.8 \times 10^6) = P(Z < \frac{1.8 \times 10^6 - 1.4 \times 10^6}{3 \times 10^5}) = P(Z < 1.33) = 0.9082 \equiv p_1$$

Y : 100 片中，壽命小於 1.8×10^6 的片數 $\sim Bin(n = 100, p_1 = 0.9082)$

$$Z_1 \equiv \frac{Y - 90.82}{\sqrt{90.82 \times 0.0918}} \sim N(0,1)$$

$$P(Y \ge 20) \approx P(\frac{Y - 90.82}{\sqrt{90.82 \times 0.0918}} > \frac{19.5 - 90.82}{\sqrt{90.82 \times 0.0918}}) = P(Z_1 > -24.7001) \approx 1$$

31.

(a)

$X \sim Unif(0, A)$

$$E[|X-a|] = \int_a^A \frac{x-a}{A}dx + \int_0^a \frac{a-x}{A}dx = \frac{A}{2} - \left(a - \frac{a^2}{A}\right) \equiv f(a)$$

$$\frac{d}{da}f(a) = \frac{2a}{A} - 1 \overset{let}{=} 0 \Rightarrow a = \frac{A}{2}$$

(b)

$X \sim Exp(\lambda)$

$$E[|X-a|] = \int_0^a (a-x)\lambda \cdot e^{-\lambda x}dx + \int_a^\infty (x-a)\lambda \cdot e^{-\lambda x}dx = a + \frac{2}{\lambda}e^{-\lambda \cdot a} - \frac{1}{\lambda} \equiv h(a)$$

$$\frac{d}{da}h(a) = 1 - 2e^{-\lambda \cdot a} \overset{let}{=} 0 \Rightarrow a = \frac{\ln 2}{\lambda}$$

32.

X : 修復一台機器所花的時間 $\sim Exp(\lambda = \frac{1}{2})$

(a)

$$P(X>2) = 1 - P(X \le 2) = 1 - \int_0^2 \frac{1}{2} e^{-\frac{1}{2}x} dx = 1 + \left(e^{-\frac{1}{2}\cdot 2} - 1 \right) = e^{-1}$$

(b)

$$P(X>10 \mid X>9) = P(X>1) = 1 - P(X \le 1) = 1 - \int_0^1 \frac{1}{2} e^{-\frac{1}{2}x} dx = 1 + \left(e^{-\frac{1}{2}\cdot 1} - 1 \right) = e^{-\frac{1}{2}}$$

Theoretical Exercises

9.

$Z \sim N(0,1)$

(a)

$$P(Z>x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$= -\int_{-x}^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \qquad (Let\ \ y = -z \Rightarrow dy = -dz)$$

$$= \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy = \int_{-\infty}^{-x} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = P(Z < -x)$$

(b)

$$P(|Z|>x) = P(Z>x \ \ or \ \ Z<-x)$$
$$= P(Z>x) + P(Z<-x) = 2P(Z>x) \qquad \cdots by\ \ (a)$$

(c)

$$P(|Z|<x) = 1 - P(|Z| \ge x) = 1 - P(|Z|>x)$$
$$= 1 - [2 \cdot P(Z>x)] = 1 - 2[1 - P(Z \le x)]$$
$$= 1 - 2[1 - P(Z<x)] = 1 - 2 + 2P(Z<x)$$
$$= 2P(Z<x) - 1$$

13.

(a)

$X \sim Unif(a,b)$

$$F_X(m) = \int_a^m \frac{1}{b-a} dx = \frac{1}{2} \Rightarrow \frac{m-a}{b-a} = \frac{1}{2}$$

$$\Rightarrow m = \frac{b+a}{2}$$

(b)

$X \sim N(\mu, \sigma^2)$

$$F_X(m) = \frac{1}{2} \Rightarrow P(X < m) = \frac{1}{2}$$

$$\Rightarrow P(\frac{X-\mu}{\sigma} < \frac{m-\mu}{\sigma}) = \frac{1}{2} \qquad \left(Z \equiv \frac{X-\mu}{\sigma} \sim N(0,1)\right)$$

$$\Rightarrow P(Z < \frac{m-\mu}{\sigma}) = \frac{1}{2} \qquad \cdots(1)$$

$By \ 9.(b) \Rightarrow P(Z > 0) = P(Z < 0)$

$$and \ \ P(Z > 0) + P(Z < 0) = 1 \quad \Rightarrow P(Z < 0) = \frac{1}{2} \quad \cdots(2)$$

$By (1) \ \& \ (2)$

$$\Rightarrow \frac{m-\mu}{\sigma} = 0 \Rightarrow m = \mu$$

(c)

$X \sim Exp(\lambda)$

$$F_X(m) = \int_0^m \lambda \cdot e^{-\lambda x} dx = \frac{1}{2}$$

$$\Rightarrow 1 - e^{-\lambda \cdot m} = \frac{1}{2} \Rightarrow e^{-\lambda \cdot m} = \frac{1}{2} \Rightarrow m = \frac{\ln 2}{\lambda}$$

15.

$X \sim Exp(\lambda)$

$$Let \ \ Y = cX \Rightarrow X = \frac{Y}{c} \ , \qquad |J| = |\frac{dx}{dy}| = \frac{1}{c}$$

$$h(y) = f(\frac{y}{c}) \cdot |J| = \lambda \cdot e^{-\lambda \cdot \frac{y}{c}} \cdot \frac{1}{c} = \frac{\lambda}{c} \cdot e^{-\frac{\lambda}{c} y} \qquad 0 < y < \infty$$

$$\Rightarrow Y = cX \sim Exp(\frac{\lambda}{c})$$

21.

$$\Gamma(\frac{1}{2}) = \int_0^\infty e^{-x} \cdot x^{-\frac{1}{2}} dx \qquad \left(Let \ \ y = \sqrt{2x} \quad dx = ydy\right)$$

$$= \int_0^\infty e^{-\frac{y^2}{2}} \cdot \sqrt{2} y^{-1} \cdot ydy = \sqrt{2} \cdot \sqrt{2\pi} \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \cdot dy$$

$$= 2\sqrt{\pi} \cdot P(Z > 0) = 2 \cdot \sqrt{\pi} \cdot \frac{1}{2} = \sqrt{\pi} \qquad \left(Z \sim N(0,1)\right)$$